Polynomials and Fast Fourier Transform (FFT)

Polynomials

$A(x) = \sum_{i=0}^{n-1} a_i x^i$ a polynomial of degree n-1

Evaluate at a point x = b in time ?

Polynomials

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Evaluate at a point x = b in time O(n): Horner's rule:
Compute a_{n-1} x,

$a_{n-2} + a_{n-1}x^2$,

$a_{n-3} + a_{n-2}\, x + a_{n-1}\, x^3$

…

Each step O(1) operations, multiply by and add coefficient.

There are ≤ n steps. ➔ O(n) time

Summing Polynomials

$\sum_{i=0}^{n-1} a_i x^i$ a polynomial of degree n-1

$\sum_{i=0}^{n-1} b_i x^i$ a polynomial of degree n-1

$\sum_{i=0}^{n-1} c_i x^i$ the sum polynomial of degree n-1

$c_i = a_i + b_i$

Time O(n)

How to multiply polynomials?

$\sum_{i=0}^{n-1} a_i x^i$ a polynomial of degree n-1

$\sum_{i=0}^{n-1} b_i x^i$ a polynomial of degree n-1

$\sum_{i=0}^{2n-2} c_i x^i$ the product polynomial of degree n-1

$c_i = \sum_{j \leq i} a_j b_{i-j}$

Trivial algorithm: time $O(n^2)$
FFT gives time $O(n \log n)$

Polynomial representations

Coefficient: $(a_0, a_1, a_2, \ldots a_{n-1})$

Point-value: have points $x_0, x_1, \ldots x_{n-1}$ in mind

Represent polynomials $A(X)$ by pairs

$\{ (x_0, y_0), (x_1, y_1), \ldots \}$ $\qquad A(x_i) = y_i$

To multiply in point-value, just need $O(n)$ operations.

Approach to polynomial multiplication:

A, B given as coefficient representation

1) Convert A, B to point-value representation

2) Multiply C = AB in point-value representation

3) Convert C back to coefficient representation

2) done esily in time O(n)

FFT allows to do 1) and 3) in time O(n log n).

Note: For C we need 2n-1 points; we'll just think "n"

From coefficient to point-value:

$$\begin{vmatrix} y_0 \\ y_1 \\ \cdots \\ \cdots \\ \cdots \\ y_{n-1} \end{vmatrix} = \begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1} \end{pmatrix} \begin{vmatrix} a_0 \\ a_1 \\ \cdots \\ \cdots \\ \cdots \\ a_{n-1} \end{vmatrix}$$

From point-value representation, note above matrix is invertible (if points distinct)

Alternatively, Lagrange's formula

We need to evaluate A at points $x_1 \ldots x_n$ in time $O(n \log n)$

Idea: divide and conquer:

$A(x) = A^0 (x^2) + x A^1 (x^2)$
where A^0 has the even-degree terms, A^1 the odd

Example:

$A = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$

$A^0 (x) = a_0 + a_2 x^2 + a_4 x^4$
$A^1 (x) = a_1 x + a_3 x^3 + a_5 x^5$

How is this useful?

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If my points are x_1 , x_2 , $x_{n/2}$, $-x_1$, $-x_2$, $-x_{n/2}$

I just need the evaluations of A^0 , A^1 at points x_1^2 , x_2^2, $x_{n/2}^2$

$T(n) \leq 2\, T(n/2) + O(n)$, with solution $O(n \log n)$. Are we done?

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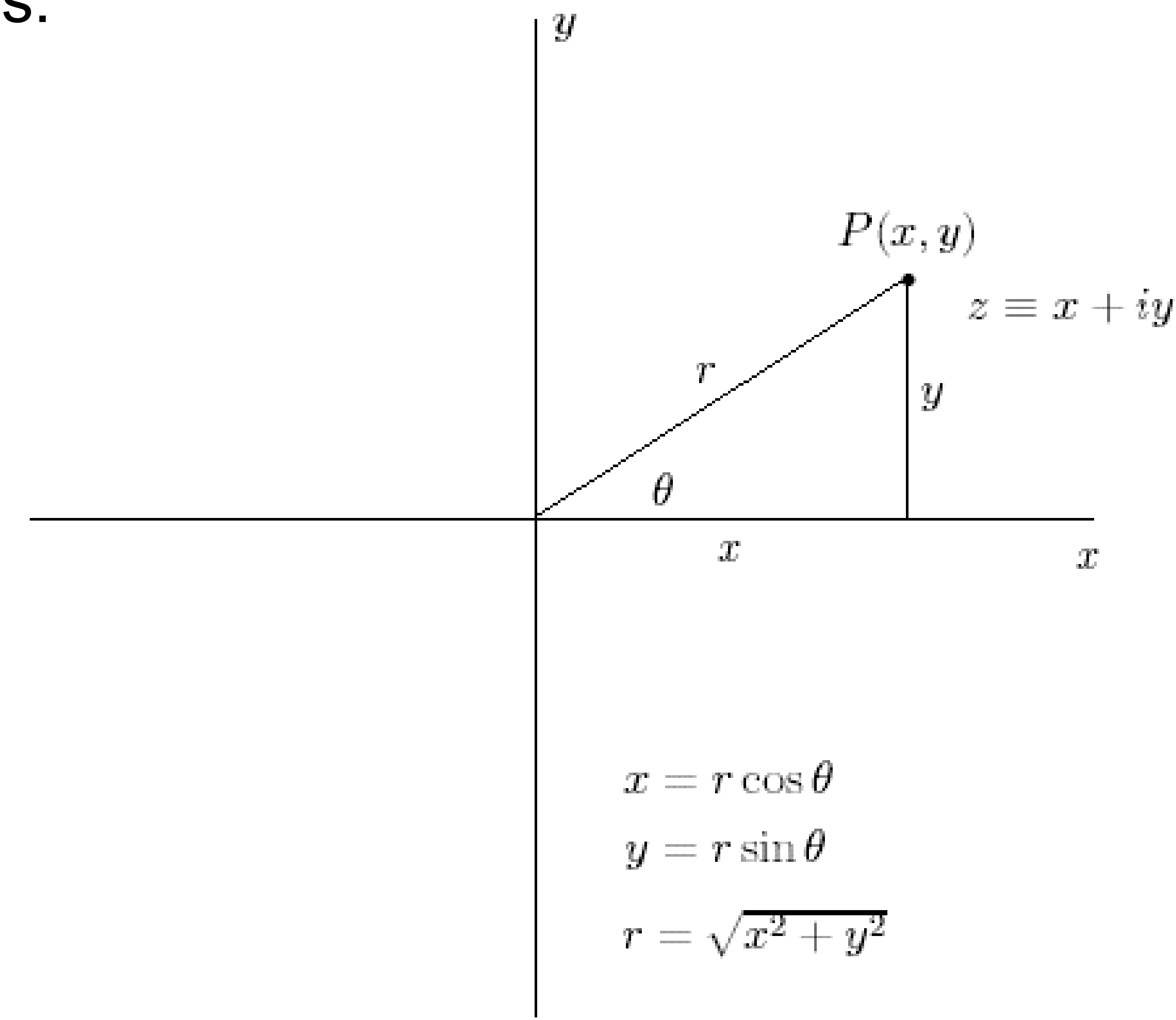
Need points which can be iteratively decomposed in + and -

Complex numbers:



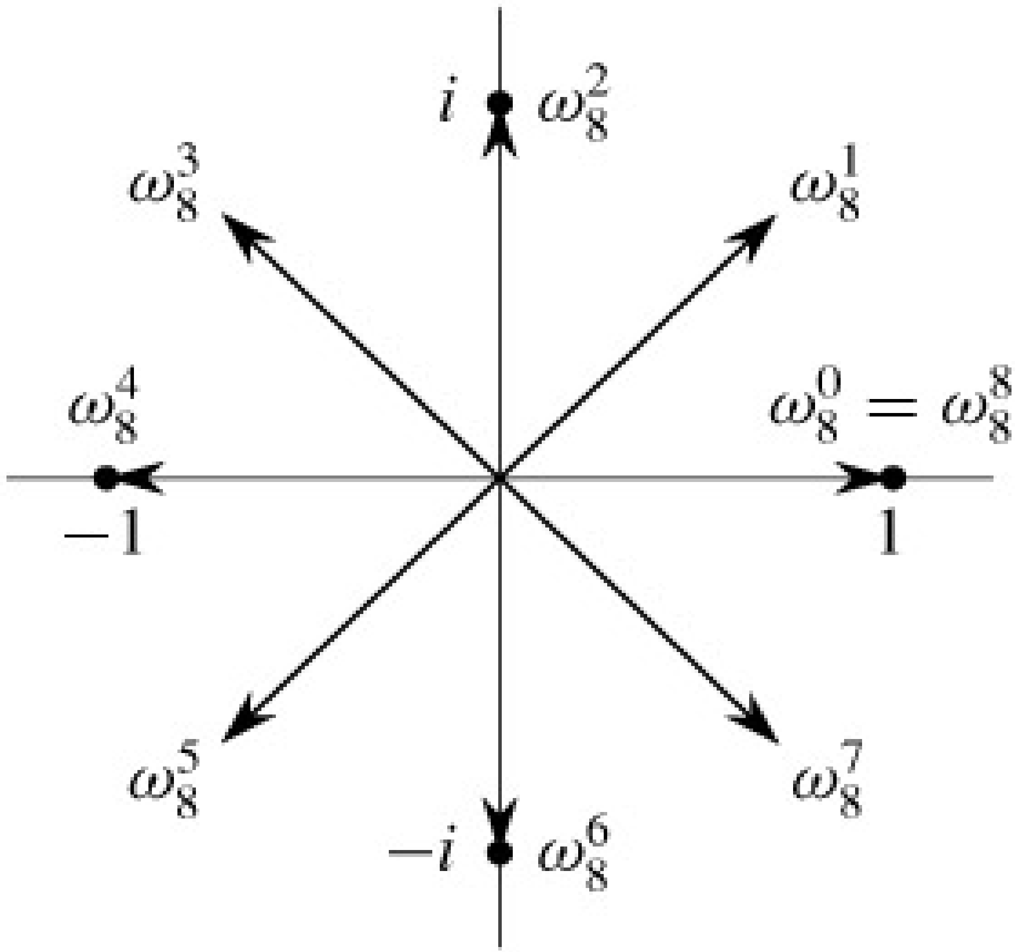
$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$r = \sqrt{x^2 + y^2}$$

ω_n = n-th primitive root of unity

$\omega_n^0, \ldots, \omega_n^{n-1}$
n-th roots of unity



We evaluate polynomial A of degree n-1 at roots of unity $\omega_n^0, \ldots, \omega_n^{n-1}$

Fact: The n squares of the n-th roots of unity are:
first the n/2 n/2-th roots of unity,
then again the n/2 n/2-th roots of unity.

➔ from coefficient to point-value in O(n log n) (complex) steps

Summary:

We need to evaluate A at n-th roots of unity $\omega_n^0, \ldots, \omega_n^{n-1}$

Divide: $A(x) = A^0(x^2) + x\,A^1(x^2)$

where A^0 has the even-degree terms, A^1 the odd

Conquer: Evaluate A^0, A^1 at n/2-th roots $\omega_{n/2}^0, \ldots, \omega_{n/2}^{n/2-1}$

This yields evaluation vectors y^0, y^1

Combine: $z := 1 = \omega_n^0$

for (k = 0, k < n, k++)

  $y[k] = y^0[k \text{ modulo } n/2] + z\, y^1[k \text{ modulo } n/2]$

  $z = z \cdot \omega_n$

$T(n) \leq 2\,T(n/2) + O(n)$, with solution $O(n \log n)$.

It only remains to go from point-value to coefficient represent.

$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n & \omega_n^2 & \omega_n^3 & \cdots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \omega_n^6 & \cdots & \omega_n^{2(n-1)} \\ 1 & \omega_n^3 & \omega_n^6 & \omega_n^9 & \cdots & \omega_n^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \omega_n^{3(n-1)} & \cdots & \omega_n^{(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \end{pmatrix}$$

F

We need to invert F

It only remains to go from point-value to coefficient represent.

$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n & \omega_n^2 & \omega_n^3 & \cdots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \omega_n^6 & \cdots & \omega_n^{2(n-1)} \\ 1 & \omega_n^3 & \omega_n^6 & \omega_n^9 & \cdots & \omega_n^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \omega_n^{3(n-1)} & \cdots & \omega_n^{(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \end{pmatrix}$$

Fact: $(F^{-1})_{j,k} = \omega_n^{-jk} \;/\; n$ Note $j,k \in \{0,1,\ldots, n-1\}$

To compute inverse, use FFT with ω^{-1} instead of ω, then divide by n.