System Specification, Verification and Synthesis (SSVS) – CS 4830/7485, Fall 2019

15: Formal Verification: CTL Model Checking

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Solving the general model-checking problem

We know how to model check LTL and CTL formulas of the form

$$\mathbf{G}\psi \qquad \text{or} \qquad \mathbf{AG}\psi$$

where ψ is a propositional formula: we do this by reachability analysis.

But how can we model-check **arbitrary** LTL and CTL formulas?

We first look at CTL. Then at LTL.

CTL Model-Checking

Recall: the model-checking problem for CTL

Given:

- the implementation: a transition system (Kripke structure) $M = (\mathsf{AP}, S, S_0, L, R)$
- the specification: a CTL formula ϕ

check where M satisfies ϕ:

$$M \overset{?}{\models} \phi$$

i.e., check whether for **every initial state of** M **satisfies** ϕ:

$$\forall s \in S_0 : s \models \phi \quad ?$$

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We will assume that M is finite and has no deadlock states.
What if M has deadlocks? $\to$ **Homework.**

CTL model-checking: basic idea

1. Compute $[\![\phi]\!]$: the set of all states satisfying ϕ.
   (Note that $[\![\phi]\!]$ may contain unreachable states. That's OK.)
2. Check that $S_0 \subseteq [\![\phi]\!]$: every initial state satisfies ϕ.

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We will compute $[\![\phi]\!]$ recursively based on the syntax of ϕ:

1. Compute $[\![\psi]\!]$ for every **subformula** ψ of ϕ: **bottom-up** on the syntax tree of ϕ.
2. Combine the results to obtain $[\![\phi]\!]$.

Computing $[\![\phi]\!]$

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1. For atomic proposition $p \in \mathsf{AP}$: $[\![p]\!] = \{s \in S \mid p \in L(s)\}$
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Recall that:

$$\mathbf{pre}(X) = \{s \in S \mid \exists s' \in X : s \longrightarrow s'\}$$

that is, $\mathbf{pre}(X)$ is the set of 1-step **predecessors** of states in X.

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We will see later how to compute $\mathbf{pre}(X)$ symbolically.

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This is like the symbolic reachability algorithm, except we are going backwards.

Will the iteration terminate? Why?

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Because the state-space is finite.

Fixpoints

Fixpoints on monotonic functions on powersets

Let $F : 2^S \to 2^S$ be a function from sets of states to sets of states.

A **fixpoint** of F is a set of states $X \subseteq S$, such that

$$F(X) = X$$

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Then F has a **least fixpoint** X^*, meaning that:

- X^* is a fixpoint of F: $F(X^*) = X^*$
- X^* is the least fixpoint of F: for any X, if $F(X) = X$ then $X^* \subseteq X$.

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X^* is often denoted $\mathbf{lfp}F$ or μF.

Computing least fixpoints iteratively

X^* can be computed by starting from the empty set and applying F repeatedly:

$$\emptyset \subseteq F(\emptyset) \subseteq F(F(\emptyset)) \subseteq F^3(\emptyset) \subseteq \cdots \subseteq F^n(\emptyset) \subseteq \cdots$$

(When) does this iteration terminate?

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(When) does this iteration terminate?

It terminates when $F^{n+1}(\emptyset) = F^n(\emptyset)$.

When S is finite, this is bound to happen.
In fact, $F^{n+1}(\emptyset) = F^n(\emptyset)$ for some $n \leq |S|$.

Theorem

Let S be a finite set. Let $n = |S|$. Let $F : 2^S \to 2^S$ be a monotonic function on the powerset of S (i.e., $X_1 \subseteq X_2 \Rightarrow F(X_1) \subseteq F(X_2)$). Then:

1. *F has a least fixpoint X^*.*
2. *$X^* = F^n(\emptyset)$.*

Proof.

$\emptyset \subseteq F(\emptyset)$, since the emptyset is a subset of any other set. By monotonicity of F, $F(\emptyset) \subseteq F(F(\emptyset)) = F^2(\emptyset)$. Continuing the same way, we can prove by induction that $F^i(\emptyset) \subseteq F^{i+1}(\emptyset)$ for all $i = 0, 1, 2, \ldots$. Since S is finite, for some $i \leq n$, it must be that $F^i(\emptyset) = F^{i+1}(\emptyset)$. Therefore, for all $i \leq j \leq n$, it must be that $F^j(\emptyset) = F^{j+1}(\emptyset)$. Thus, it must also be that $F^n(\emptyset) = F^{n+1}(\emptyset)$. Let $X^* = F^n(\emptyset)$. By construction, $F(X^*) = F^{n+1}(\emptyset) = F^n(\emptyset) = X^*$, i.e., X^* is a fixpoint of F.

We next show that X^* is the least fixpoint of F. Suppose X is another fixpoint of F, i.e., $F(X) = X$. $\emptyset \subseteq X$, since the emptyset is a subset of any set. By monotonicity of F, $F(\emptyset) \subseteq F(X)$, and since $F(X) = X$, $F(\emptyset) \subseteq X$. Continuing the same way, we can prove by induction that $F^i(\emptyset) \subseteq X$ for all $i = 0, 1, 2, \ldots$. Thus, $X^* = F^n(\emptyset) \subseteq X$. □

CTL Model-Checking continued

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But also:

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This looks like a fixpoint equation! What is the function F?

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Is F monotonic?

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$$F(X) = [\![\phi]\!] \cup \mathbf{pre}(X)$$

Is F monotonic? Yes: follows from the fact that $\mathbf{pre}$ is monotonic.

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To compute the least fixpoint of F, we need to compute the sequence:

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X_0 & = & \emptyset & & \\
X_1 & = & F(X_0) & = & [\![\phi]\!] \cup \mathbf{pre}(\emptyset) \\
 & & & = & [\![\phi]\!] \qquad \text{Why?} \\
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\ldots & & & &
\end{array}$$

so that

$$[\![\mathbf{EF}\phi]\!] = [\![\phi]\!] \cup \mathbf{pre}([\![\phi]\!]) \cup \mathbf{pre}(\mathbf{pre}([\![\phi]\!])) \cup \cdots = \mathbf{lfp} F$$

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Lambda notation for the fixpoint: $\mathbf{lfp}X.[\![\phi]\!] \cup \mathbf{pre}(X)$

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6. $[\![\mathbf{E}(\phi_1\,\mathbf{U}\,\phi_2)]\!] = ???$

Computing $[\![\mathbf{E}(\phi_1 \, \mathbf{U} \, \phi_2)]\!]$

$$\mathbf{E}(\phi_1 \, \mathbf{U} \, \phi_2) \quad \Leftrightarrow \quad \phi_2 \vee (\phi_1 \wedge \mathbf{EX} \, \mathbf{E}(\phi_1 \, \mathbf{U} \, \phi_2))$$

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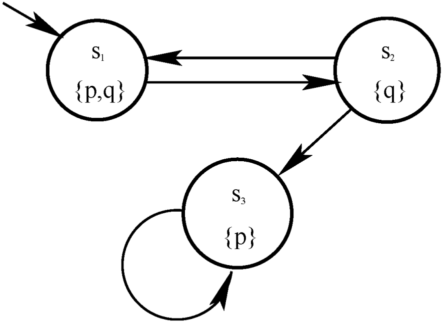
Iterative computation:

$$\begin{array}{lclcl} X_0 & = & \emptyset & & \\ X_1 & = & F(X_0) & = & [\![\phi_2]\!] \cup ([\![\phi_1]\!] \cap \mathbf{pre}(\emptyset)) \\ & & & = & [\![\phi_2]\!] \\ X_2 & = & F(X_1) & = & [\![\phi_2]\!] \cup ([\![\phi_1]\!] \cap \mathbf{pre}([\![\phi_2]\!])) \\ \ldots & & & & \end{array}$$

Computing $[\![\mathbf{E}(\phi_1 \, \mathbf{U} \, \phi_2)]\!]$: example



- **Homework**: model-check $\mathbf{E}(p \, \mathbf{U} \, q)$.

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Computing $\llbracket \phi \rrbracket$ (continued)

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Is there a more direct way to compute $\llbracket \mathbf{AX}\phi_1 \rrbracket$ and $\llbracket \mathbf{AG}\phi_1 \rrbracket$?

Computing $[\![\mathbf{AG}\phi]\!]$

$$\mathbf{AG}\phi \quad \Leftrightarrow \quad \phi \wedge \mathbf{AX}\,\mathbf{AG}\phi$$

therefore (?)

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What is the least fixpoint here?

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therefore (?)

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What is the least fixpoint here?

Iterative computation:

$$\begin{aligned}
X_0 &= \emptyset \\
X_1 &= F(X_0) = [\![\phi]\!] \cap \overline{\mathbf{pre}(\overline{\emptyset})} \\
&= [\![\phi]\!] \cap \overline{\mathbf{pre}(S)} \\
&= [\![\phi]\!] \cap \overline{S} \qquad \text{Why?} \\
&= [\![\phi]\!] \cap \emptyset \\
&= \emptyset
\end{aligned}$$

Oops. What has gone wrong?

Computing $[\![\mathbf{AG}\phi]\!]$

$$\mathbf{AG}\phi \quad \Leftrightarrow \quad \phi \wedge \mathbf{AX}\,\mathbf{AG}\phi$$

tells us that $[\![\mathbf{AG}\phi]\!]$ is **a** fixpoint of the function

$$F(X) = [\![\phi]\!] \cap \overline{\mathbf{pre}(\overline{X})}$$

but it does not tell us **which one**.

F may have **more than one fixpoints**: e.g., S or $\emptyset$.

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but it does not tell us **which one**.

F may have **more than one fixpoints**: e.g., S or $\emptyset$.

In this case the least fixpoint is $\emptyset$.
What we want instead is the greatest fixpoint.

Greatest Fixpoints

Greatest fixpoints

Let $F : 2^S \rightarrow 2^S$ be a monotonic function from sets of states to sets of states.

Then F has a **greatest fixpoint** X^*:

- X^* is a fixpoint of F: $F(X^*) = X^*$
- X^* is the greatest fixpoint of F: for any X, if $F(X) = X$ then $X^* \supseteq X$.

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X^* can be computed by starting from the set S and applying F repeatedly:

$$S \supseteq F(S) \supseteq F(F(S)) \supseteq F^3(S) \supseteq \cdots \supseteq F^n(S) \supseteq \cdots$$

As with least fixpoints, for finite S the above terminates after at most $n = |S|$ steps.

CTL Model-Checking continued

Computing $[\![\mathbf{AG}\phi]\!]$

$$[\![\mathbf{AG}\phi]\!] = \mathbf{gfp}X.[\![\phi]\!] \cap \overline{\mathbf{pre}(\overline{X})}$$

Iterative computation:

$$\begin{array}{lllll}
X_0 & = & S & & \\
X_1 & = & F(X_0) & = & [\![\phi]\!] \cap \overline{\mathbf{pre}(\overline{S})} \\
 & & & = & [\![\phi]\!] \cap \overline{\mathbf{pre}(\emptyset)} \\
 & & & = & [\![\phi]\!] \cap \overline{\emptyset} \\
 & & & = & [\![\phi]\!] \cap S \\
 & & & = & [\![\phi]\!] \\
X_2 & = & F(X_1) & = & [\![\phi]\!] \cap \overline{\mathbf{pre}(\overline{[\![\phi]\!]})} \\
\ldots & & & &
\end{array}$$

The $\overline{\mathbf{pre}(\bar{\cdot})}$ operator

What does $\overline{\mathbf{pre}(\overline{X})}$ really compute?

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In other words, the set of all states whose 1-step successors are all in X.

In other words, since we assumed no deadlocks, the set of all states which will **inevitably** move into X in 1-step.

We define:

$$\mathbf{inev}(X) = \overline{\mathbf{pre}(\overline{X})}$$

$\mathbf{inev}(X)$ is often denoted $\widetilde{\mathbf{pre}}(X)$.

Computing $[\![\phi]\!]$ (continued)

1. For atomic proposition $p \in$ AP: $[\![p]\!] = \{s \in S \mid p \in L(s)\}$
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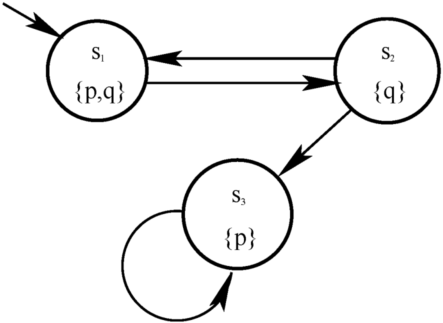
8. $[\![\mathbf{AG}\phi_1]\!] = \mathbf{gfp}X.[\![\phi_1]\!] \cap \mathbf{inev}(X)$

More CTL model-checking examples



Homework:

- Let's model-check $\mathbf{E}(q\,\mathbf{U}\,\mathbf{AG}p)$.
- What about $\mathbf{E}(p\,\mathbf{U}\,\mathbf{AG}q)$?

Computing $[\![\phi]\!]$ (continued)

1. For atomic proposition $p \in$ AP: $[\![p]\!] = \{s \in S \mid p \in L(s)\}$
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9. $[\![\mathbf{EG}\phi_1]\!] = ???$
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$$\mathbf{EG}\phi \quad \Leftrightarrow \quad \phi \wedge \mathbf{EX}\,\mathbf{EG}\phi$$

Fixpoint equation:

$$F(X) =$$

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Why not take the greatest fixpoint here? **Homework.**

Computing $[\![\phi]\!]$ – final version!

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Symbolic CTL model-checking

The definitions of $[\![\phi]\!]$ directly suggest symbolic implementations.

It suffices to be able to compute $\mathbf{pre}$ and $\mathbf{inev}$ symbolically.

Recall:

$$\mathbf{pre}(X) = \{s \in S \mid \exists s' \in X : s \longrightarrow s'\}$$

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How can we implement these operators symbolically?
Hint: recall $\mathbf{succ}$.

Symbolic CTL model-checking

Symbolic **post** :

$$\mathbf{succ}(\phi) \;=\; \big(\exists x : \phi(x) \wedge \mathit{Trans}(x, x')\big)[x' \leadsto x]$$

Symbolic **pre** :

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Symbolic **inev** :

$$\mathbf{syminev}(\phi) \quad =$$

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$$\mathbf{pred}(\phi) = \exists x' : \phi(x') \wedge \mathit{Trans}(x, x')$$

Symbolic $\mathbf{inev}$:

$$\mathbf{syminev}(\phi) = \forall x' : \mathit{Trans}(x, x') \rightarrow \phi(x')$$

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