Machine Learning Homework Solution

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Question:
X and Y are two sets containing finite number of points. C_1 and C_2 are convex hulls of X and Y, respectively. Prove that if C_1 and C_2 don't intersect, there must exist a hyper plane that separates C_1 and C_2.
Answer:
First we need a few lemmas:
Lemma 1: In a triangle ABC (Figure 1), if $\angle A > \angle B$, then $d(B, C) > d(A, C)$, where d is the Euclidean distance.

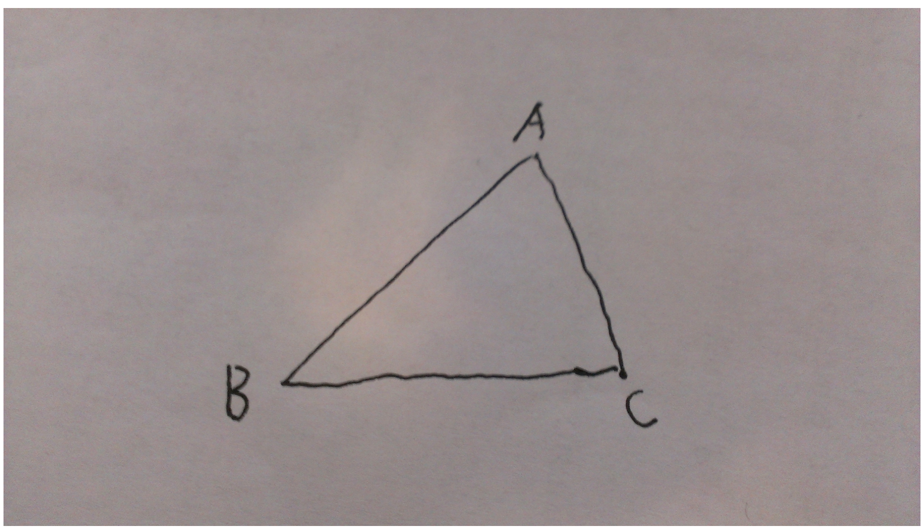

Figure 1: Lemma 1

Proof of the Lemma 1: $\frac{d(B,C)}{d(A,C)} = \frac{sin(\angle A)}{sin(\angle B)}$.
Lemma 2: A convex hull generated by a finite number of points is a closed set.
Proof of the Lemma 2: Omitted.
Proof the the main results:
Since C_1 and C_2 don't intersect and they are closed sets, there exist $A \in C_1$, $B \in C_2$ such that

$$d(A, B) = \min_{M \in C_1, N \in C_2} d(M, N) = \inf_{M \in C_1, N \in C_2} d(M, N) = d(C_1, C_2) > 0$$

Define the hyper plane P to be the unique plane which contains the middle point of A and B and is perpendicular to the line AB. It's easy to verify that for any point G on the plane P, $d(G, A) = d(G, B)$. For any point G on the A side of the plane, $d(G, A) < d(G, B)$. For any point G on the B side of the plane, $d(G, A) > d(G, B)$. We now prove that the hyper plane P defined this way separates C_1 and C_2. It suffices to show that $\forall C \in C_2$, C must lie on the B side of the the plane. The other direction of the proof can be shown similarly.

Assume to the contrary that C lies on the A side of the plane or on the plane. Then $d(A, C) \leq d(B, C)$. There are two major cases:

- case 1: A, B, C are in a line. There are two sub-cases.
  - case 1.1: C lies on the line segment AB (Figure 2). Clearly, $d(A, C) < d(A, B)$. This contradicts $d(A, B) = \min_{M \in C_1, N \in C_2} d(M, N)$.

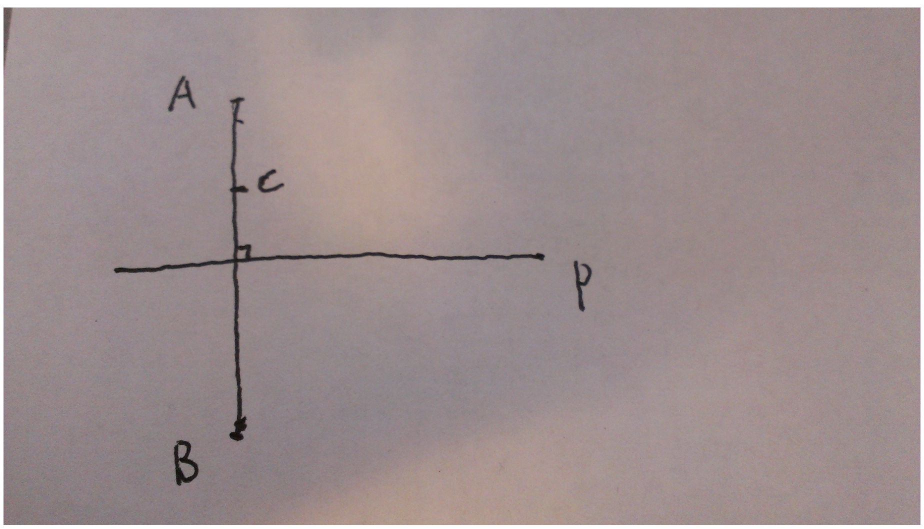

Figure 2: case 1.1

  - case 1.2: A lies on the line segment BC (Figure 3). By the definition of convex hull, $A \in C_2$. This contradicts the fact that C_1 and C_2 don't intersect.
- case 2: A, B, C are not in a line. Consider the triangle ABC. We know $d(A, C) \leq d(B, C)$. By Lemma 1, $\angle B \leq \angle A$. Also it is always the case that $\angle A + \angle B + \angle C = \pi$ and $\angle C > 0$. Therefore, $\angle B < \pi/2$. Let's consider two sub-cases.
  - case 2.1: $\angle C \geq \pi/2$ (Figure 4). Therefore $\angle C > \angle B$. By Lemma 1, $d(A, B) > d(A, C)$. This contradicts $d(A, B) = \min_{M \in C_1, N \in C_2} d(M, N)$.

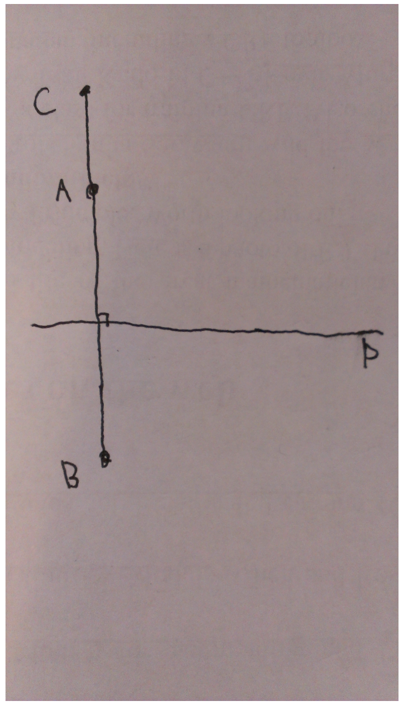

Figure 3: case 1.2

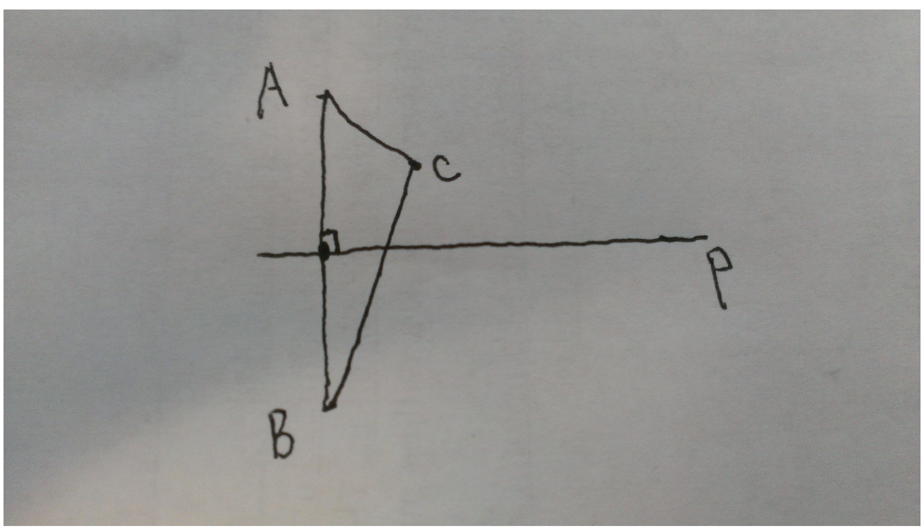

Figure 4: case 2.1

– case 2.2: $\angle C < \pi/2$ (Figure 5). Since $\angle B < \pi/2$, there exists a point D on the line segment BC such that $AD \perp BC$. Therefore, $d(A, D) < d(A, B)$. According to the definition of convex hull, $D \in C_2$. This contradicts $d(A, B) = \min_{M \in C_1, N \in C_2} d(M, N)$.

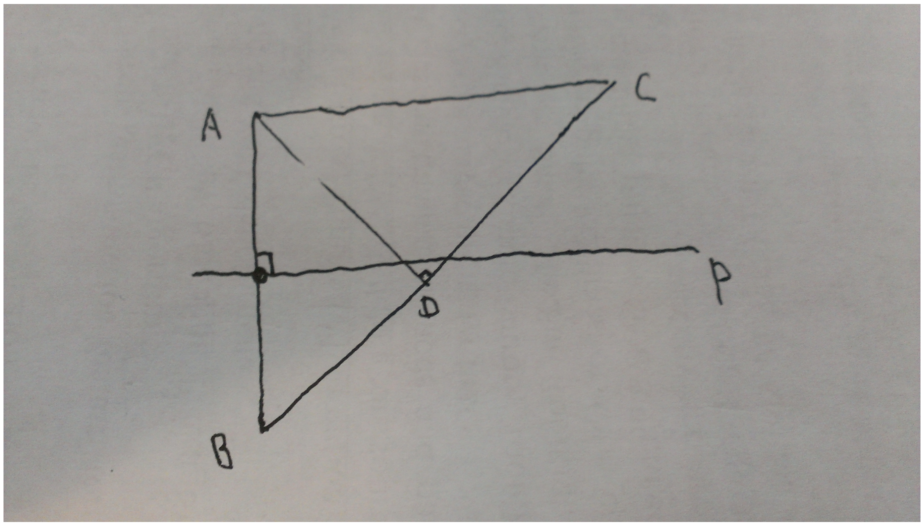

Figure 5: case 2.2