Homework 2: Logic and Proofs

Instructions:

- We expect that you will study with friends and often work out problem solutions together, but *you must write up your own solutions, in your own words.* **Cheating will not be tolerated**.
- *To get full credit*, **show INTERMEDIATE steps** leading to your answers, throughout.

Problem 1 Logic statements

Abraham Lincoln is supposed to have said "You may fool all the people some of the time, you can even fool some of the people all of the time, but you cannot fool all of the people all the time."
You are given 2 sets, 1 predicate and 1 constant:
sets: P for people and T for time.
predicate: CanFool(p_1, p_2, t) - p_1 can fool p_2 at time t
constant: $u \in P$ representing you.
Do not create your own sets, predicates or constants. Assume that "some" means "at least one".

i. Express the following in English: $_1 : \forall_{t \in T}\ \exists_{p \in P}$ CanFool(u, p, t).

ii. Express in predicate logic the sentiment S_2: "you can fool some of the people all of the time." (Hint: the intent of this sentiment is to assert the existence of a very gullible person - one who can be fooled all the time.)

iii. Describe a situation where S_1 is true but not S_2.

iv. Express in predicate logic the sentiment S_3: "you can fool all of the people some of the time." (Hint: the intent of this sentiment is to assert the existence of a time when the population, en masse, can be fooled.)

v. Express in predicate logic the sentiment S_4: "you can not fool all of the people all of the time"

vi. Prove that $\neg S_4 \implies (S_2 \wedge S_3)$. An "informal" proof in English is sufficient.

Problem 2 Logic Puzzle

A teacher writes six words on a board: `cat dog has max dim tag`
She picks one of the words, and gives the three letters of that word to three students, Albert, Bernard and Cheryl. Then she asks, "Albert, do you know the word?" Albert replies yes. She asks, "Bernard, do you know the word?" He considers Albert's answer and then replies yes. Then she asks Cheryl the same question. She thinks and then replies yes. What is the word?
Explain Cheryl's reasoning

Problem 3 : Tarski and First-Order Logic

In figure 1 we have a Tarski world, the domains for all variables consists of the objects (individual shapes) in the Tarski world. Objects are either circles or triangles. Every object has a shading, either fully shaded or clear (no shading). Finally, an object can be above some other object. We can define these properties as predicates:

- shaded(x) = x is shaded
- triangle(x) = x is a triangle
- circle(x) = x is a circle
- above(x, y) = x is on a row strictly above y

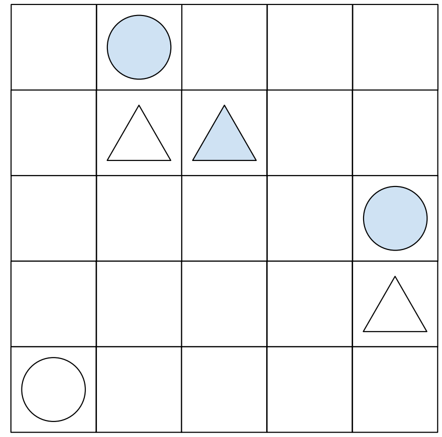

Figure 1: A Tarski World, a tool for understanding logical statements invented by Jon Barwise and John Etchemendy, named after Alfred Tarski

Using these predicates and our logical symbols, we can write statements about this Tarski World into formal logic. For instance, we can translate the statement

"Every shape is a circle or a triangle"

into

$$\forall x,\ \text{circle}(x) \vee \text{triangle}(x)$$

i. (1 point each) State whether the following statements are true or false in this Tarski world.

- Every circle is shaded.
- There is a triangle that is above a circle that is not shaded.

ii. (2 points each) Write the following statements using formal logic

- All triangles are shaded.
- There is a circle that is both shaded and above every triangle.

iii. (2 points each) Negate the following statements in formal logic. Distribute all negations in your solution.

- $\forall x,\ \text{circle}(x) \wedge \neg\ \text{shaded}(x)$
- $\forall x\ \exists y,\ \text{shaded}(x) \implies ((\text{triangle}(y) \wedge \text{shaded}(y)) \wedge \neg\ \text{above}(y, x))$

Problem 4 Sequence Limit

We say that an infinite sequence of real numbers $(a_i) = (a_1, a_2, a_3...)$ has a limit if and only if the following is true:

$$\exists x \in \mathbb{R}, \forall \epsilon > 0, \exists n_0, \forall n \geq n_0, |a_n - x| \leq \epsilon$$

If true, the sequence limit is x, and we say "converges to x", and write $\lim_{n\to\infty} a_n = x$.

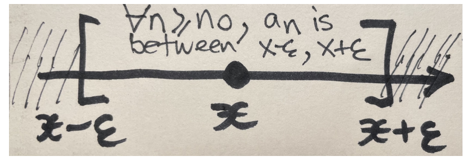

Figure 2: All sequence values after a point fall within ϵ of limit x

i. Give an example of an infinite sequence with all values in a fixed range that has no limit. Explain why it has no limit, specifically why the logical formula given is false.

ii. Write the logic for statement " if the sequence limit exists, it is unique ". Something like $\big[\exists x, \forall \epsilon > 0, ...\big] \wedge \big[\exists y, \forall \epsilon > 0, ...\big] \Rightarrow ...$

Then prove that statement by contradiction.

iii. Prove that if two sequences have the same limit $\lim_{n\to\infty} a_n = x = \lim_{n\to\infty} b_n$ then the interleaved sequence $C_n = (a_1, b_1, a_2, b_2, a_3, b_3, ...)$ also converges to that limit x.

iv. Prove that if two sequences have the same limit $\lim_{n\to\infty} a_n = x = \lim_{n\to\infty} b_n$, and with a third sequence (c_i) satisfy $\forall i \geq n_0, a_i \leq c_i \leq b_i$ then (c_i) also has limit x.

v. Prove that if two sequences have different limits $\lim_{n\to\infty} a_n = x \neq y = \lim_{n\to\infty} b_n$ then the interleaved sequence $C_n = (a_1, b_1, a_2, b_2, a_3, b_3, ...)$ has no limit.
Hint: Prove by contradiction. What value of ϵ would lead to the values being unable to be close to both of the original limits at once (similar to part ii)?

vi. Give an example of a sequence of rational numbers with limit an irrational number.

vii. ★ Given an infinite sequence (x_n) of real numbers in range [0 100], prove that there is a subsequence of (x_n) that is convergent (even if x_n is not).
Definition: A subsequence is a new infinite sequence obtained from the original by picking indices in order but not consecutive; for example $(x_3, x_7, x_{13}, x_{55}, x_{710}, x_{124301}, ...)$

Problem 5 What infinite sets are countable (enumerable)?
An infinite set S is countable if there is an (infinite) sequence indexed by natural integers $(x_1, x_2, ..., x_{100}, ...x_{1000}, ...)$ containining all elements in S, possibly with repetitions.

(A): The integers set $\mathbb{Z}$ is countable. Describe an infinite sequence that contains all integers.

(B) Rational Numbers set $\mathbb{Q}$, that is integer fractions, is countable. Describe an infinite sequence that contains all positive rationals. Hint: think of positive rationals as a table with rows the integers numerators, and columns integers denominators.
Then argue that if positive rationals are countable, then all rationals are also countable.

(C)★ Real Numbers set $\mathbb{R}$ is not countable. For any given infinite sequence (x_i) of real numbers, show how to find a real value v that is not in the sequence.

(D)★ Explain why your argument for part (C) on reals set cannot be used for part (B) on rationals set.

(E)★★ (optional, no credit) Assuming sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are defined, give a formal definition for the reals set $\mathbb{R}$.