IND 12 Binomial Th by induction over $n \to n+1$ $\quad \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$

ind step $\quad (x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \xRightarrow{IH} (x+y)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} x^k y^{n+1-k}$

old customer $\qquad$ new customer

proof: $(x+y)^{n+1} = (x+y)^n (x+y) \overset{IH}{=} (x+y) \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} =$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{k+1} y^{n-k} + \sum_{k=0}^{n} \binom{n}{k} x^k y^{n+1-k}$$

$$= \underbrace{x^{n+1}}_{k=n \text{ separate}} + \sum_{k=1}^{n} \binom{n}{k-1} x^k y^{n+1-k} + \sum_{k=1}^{n} \binom{n}{k} x^k y^{n+1-k} + \underbrace{y^{n+1}}_{\text{separate } k=0}$$

indexed from 1:n instead of 0:n-1

$k=1$: $\binom{n}{0} x^1 y^n$ | $k=n$: $\binom{n}{n-1} x^n y^1$

same prev $k=n-1$

$$= \underbrace{x^{n+1}}_{k=n+1} + \sum_{k=1}^{n} \underbrace{\left[\binom{n}{k-1} + \binom{n}{k}\right]}_{k=1:n} x^k y^{n+1-k} + \underbrace{y^{n+1}}_{k=0}$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} x^k y^{n+1-k}$$

✓

Base case $n=1$

$$(x+y)^1 = \sum_{k=0}^{1} \binom{1}{k} x^k y^{n-k}$$

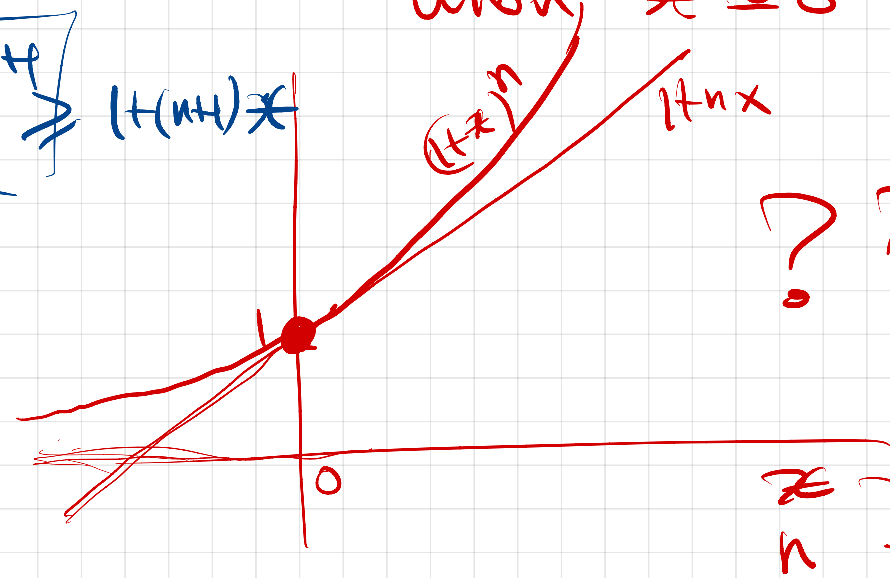
IND 13 $\quad x > -1 \quad n \geq 0$ integer $\Rightarrow (1+x)^n \geq 1 + nx$

$x+1 > 0$

$x \in \mathbb{R}$

useful approx $(1+x)^n \simeq 1+nx$ when $x \simeq 0$

ind step $n \to n+1$ $\quad (1+x)^n \geq 1+nx \Rightarrow \boxed{(1+x)^{n+1} \geq 1+(n+1)x}$



?? x? n?

proof

$$(1+x)^{n+1} = (1+x)^n \cdot (1+x)$$

$$\overset{IH}{\geq} (1+nx)(1+x) = 1 + nx + x + nx^2$$

$1+x>0$

$$= 1+(n+1)x + \underbrace{nx^2}_{\geq 0} \geq 1+(n+1)x \checkmark$$

"YB" application

wanted last time: $\underbrace{\left(1+\frac{1}{n}\right)^n}_{a_n} < \underbrace{\left(1+\frac{1}{n+1}\right)^{n+1}}_{a_{n+1}}$ (a_n) mon increasing

Binomial Th $x=1$ $y=\frac{1}{n}$

$$(x+y)^n = \left(1+\frac{1}{n}\right)^n = \underset{k=0}{1} + \sum_{k=1}^{n} \binom{n}{k} \frac{1}{n^k} = 1 + \sum_{k=1}^{n} \frac{n!}{k!(n-k)!} \cdot \frac{1}{n^k}$$

$x=1$ $y=\frac{1}{n+1}$

$$(x+y)^{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1} = \underset{k=0}{1} + \sum_{k=1}^{n} \binom{n+1}{k} \frac{1}{(n+1)^k} + \underset{k=n+1}{\frac{1}{(n+1)^{n+1}}}$$

$$= 1 + \sum_{k=1}^{n} \frac{(n+1)!}{k!(n+1-k)!} \frac{1}{(n+1)^k} + \frac{1}{(n+1)^{n+1}}$$

want:

$$\cancel{1}+\sum_{k=1}^{n} \frac{n!}{k!(n-k+1)!} \cdot \frac{n-k+1}{n^k} \overset{?}{\leq} \cancel{1}+\sum_{k=1}^{n} \frac{n!}{k!(n-k+1)!} \cdot \frac{n+1}{(n+1)^k} + \frac{1}{(n+1)^{n+1}}$$

extra

Sufficient to

$$\sum_{k=1}^{n} \frac{n!}{k!\,(n-k+1)!} \cdot \frac{n-k+1}{n^k} \overset{?}{\leq} \sum_{k=1}^{n} \frac{n!}{k!\,(n-k+1)!} \cdot \frac{n+1}{(n+1)^k}$$

$$\sum_{k=1}^{n} \frac{n!}{k!\,(n-k+1)!} \left(\frac{n-k+1}{n^k} - \frac{n+1}{(n+1)^k} \right) \leq 0$$

Lucky? Sufficient

$$\frac{n-k+1}{n^k} \overset{?}{\leq} \frac{n+1}{(n+1)^k}$$

$$\left(\frac{n+1}{n}\right)^k \overset{?}{\leq} \frac{n+1}{n-k+1}$$

$$\left(\frac{n}{n+1}\right)^k \overset{?}{\geq} \frac{n-k+1}{n+1}$$

$$\left(1-\frac{1}{n+1}\right)^k \overset{?}{\geq} 1-k\cdot\frac{1}{n+1}$$

proved

$$(1+x)^k \geq 1+kx$$

$$x=\frac{-1}{n+1} \Rightarrow \left(1-\frac{1}{n+1}\right)^k \geq 1-k\cdot\frac{1}{n+1}$$

$$x > -1$$

IND 14

Fermat's Little Th

p prime $a \neq 0 \mod p$ then $a^{p-1} = 1 \mod p$

(ind $a \geq 0$ then $\boxed{a^p = a}^{IH} \mod p \ \forall a$)

induction by $a \to a+1$ $a \neq 0 \mod p$

$a+1 \neq 0 \mod p$

$$a^{p-1} = 1 \mod p \Rightarrow (a+1)^{p-1} = 1 \mod p$$

prod P new customer

$$(a+1)^p = \sum_{k=0}^{p} a^k \cdot \binom{p}{k} = 1 + a^p + \sum_{k=1}^{p-1} \binom{p}{k} a^k$$

IH

$$\overset{IH}{\underset{\mod p}{=}} 1 + a + p(\text{something})$$

exercise $p \mid \binom{p}{k}$ $\forall\ 1 \leq k \leq p-1$

$$= 1 + a \mod p$$

$$(a+1)^{p-1} =$$

if $\exists (1+a)^{-1}$: $(a+1)^p (a+1)^{-1} = (1+a)(1+a)^{-1} = 1 \mod p$

IND 15

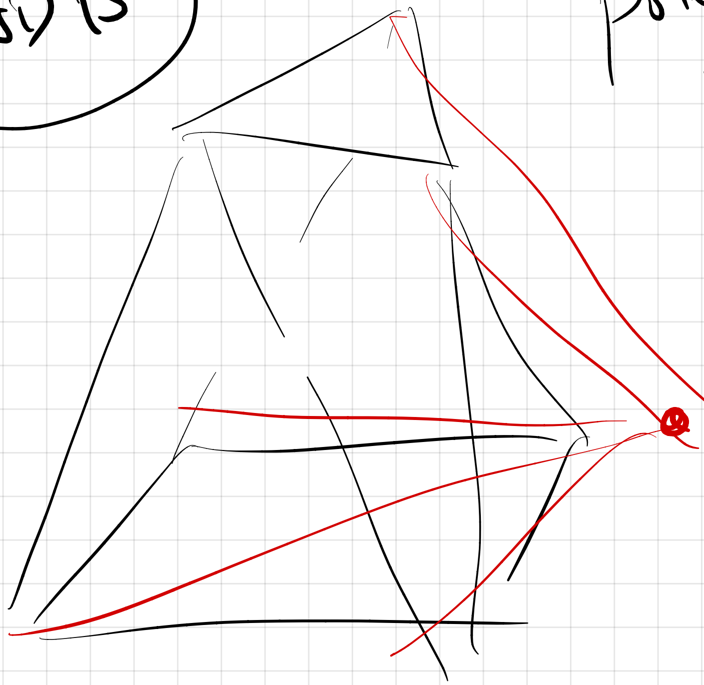
polyhedra CONVEX

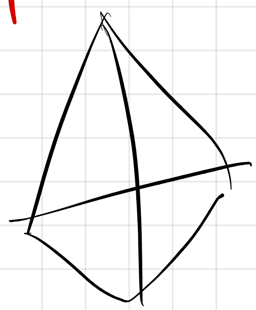
F = faces

E = edges

V = vertices

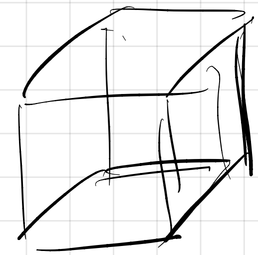
$$F + V = E + 2$$





V = 4
F = 4
E = 6

✓



V = 8
F = 6
E = 12

✓

Exercise

induction by V = # vertices. $V \to V+1$

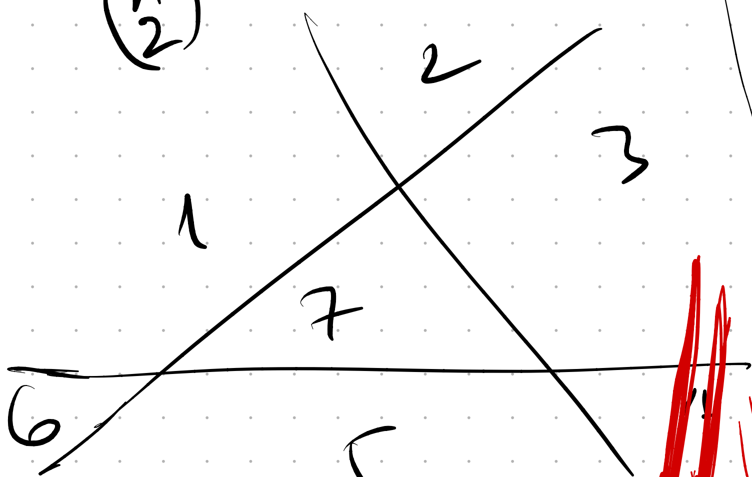
IND 24 n lines ∩ every pair distinct points

no 2 lines parallel
no 3 lines concurrent

#intersections $\binom{n}{2}$

regions $r_n = \frac{n^2+n+2}{n}$

$n=3 \quad r_3 = \frac{9+3+2}{2} = 7$ example

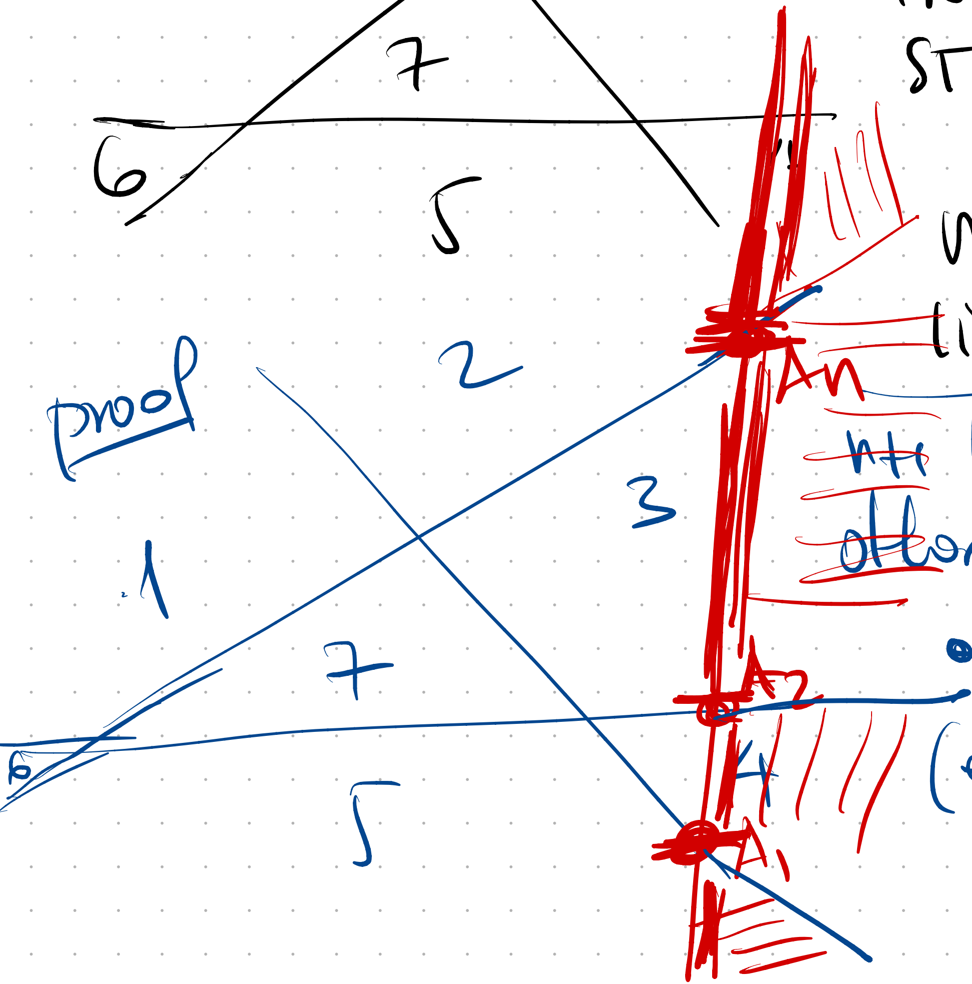


IND STEP

$$r_{n+1} = \frac{(n+1)^2+(n+1)+2}{2}$$

n+1 lines regions

proof



~~n+1~~ line (red) has ∩ with the ~~other~~ n lines: $A_1 A_2 \dots A_n \Rightarrow$ k+1 segments

• creates n+1 new regions (every segment splits an exist region in two)

segment k+1

$$r_{n+1} = r_n + n+1 \overset{IH}{=} \frac{n^2+n+2}{2} + n+1$$

$$= \frac{n^2+n+2+2(n+1)}{2} \underset{\text{want}}{\overset{?}{=}} \frac{(n+1)^2+(n+1)+2}{2} \quad \Big| \cdot 2$$

$$\cancel{n^2}+n+2+\cancel{2n}+2 \underset{\text{want}}{\overset{?}{=}} \cancel{n^2}+\cancel{2n}+1+n+1+2$$

$$n+4 \overset{?}{=} n+4$$

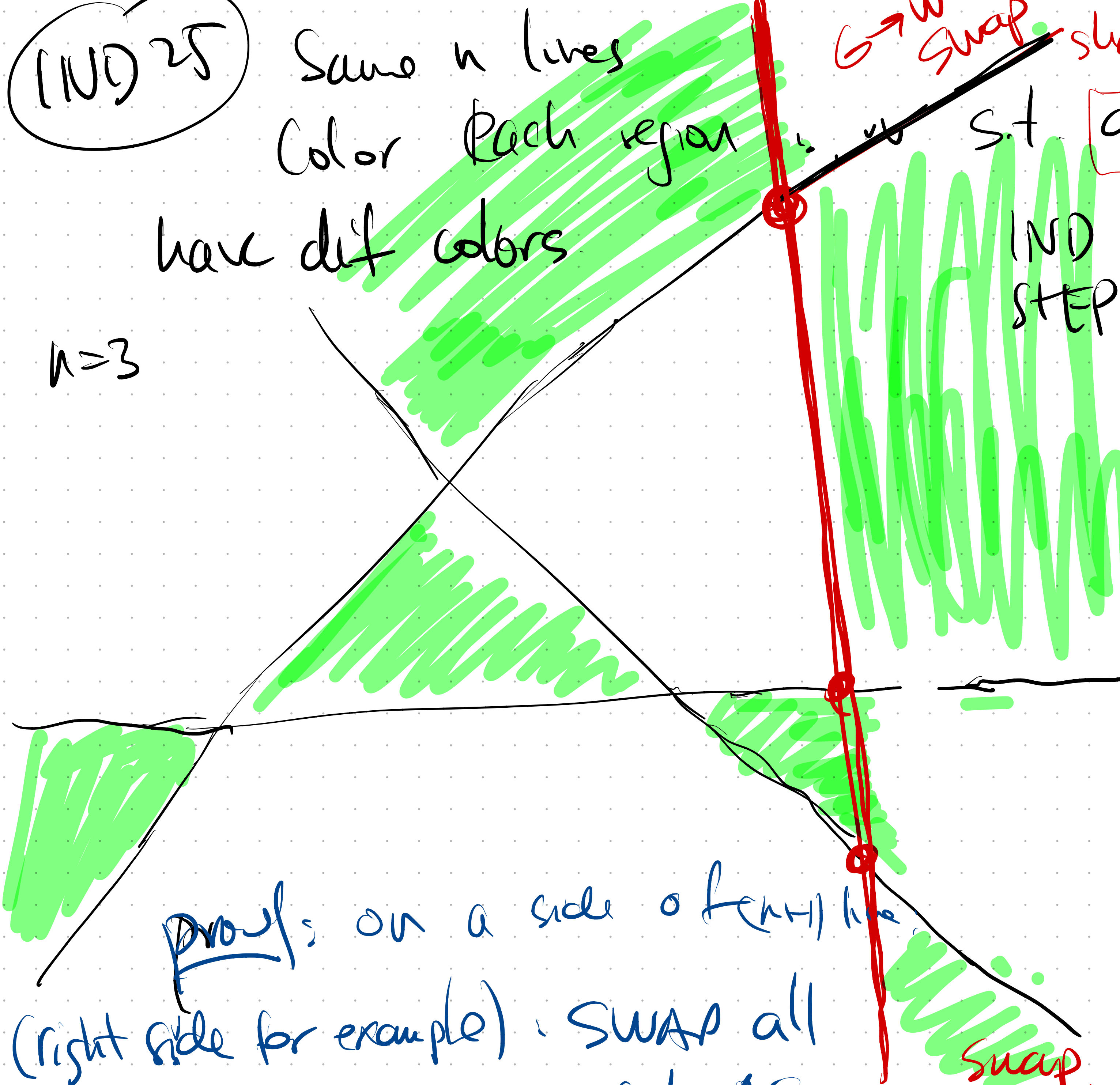
$$\checkmark$$

IND 25

Same n lines

Color each region ... s.t $\boxed{\text{adj regions}}$ have dif colors

n=3



IND STEP

n lines 2 colors

⇓

n+1 lines 2 colors

proof: on a side of (n+1) line (right side for example): SWAP all COLORS

ADJ: REGIONS r_1, r_2

- Left side: same as before

- Right side: both r_1, r_2 swapped so still dif

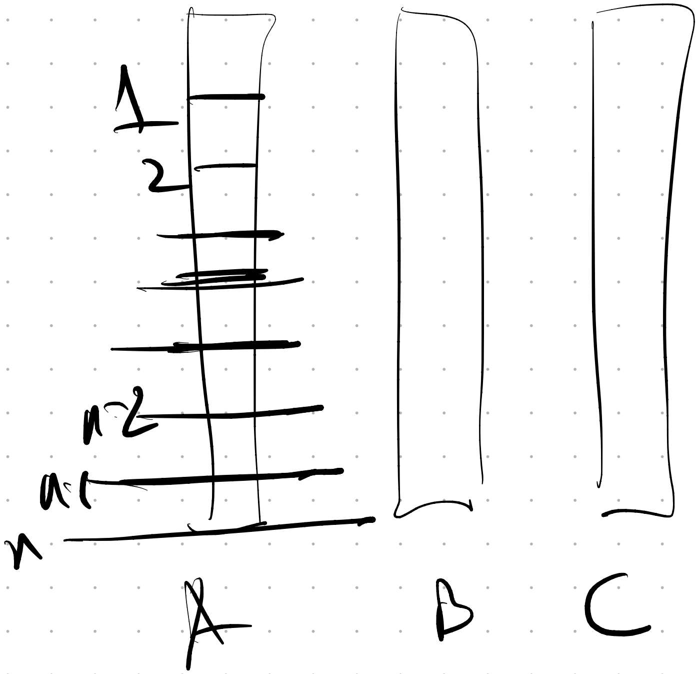
  $\Rightarrow$ same color

- across line $n+1$: $r_1 + r_2$ before swap

  same region

  but now RSide swap $\Rightarrow$ r_1, r_2 different colors

IND26 Towers of Hanoi



n discs sizes $1, 2, \dots n$

any disc can stay on top of bigger discs (or on bottom)

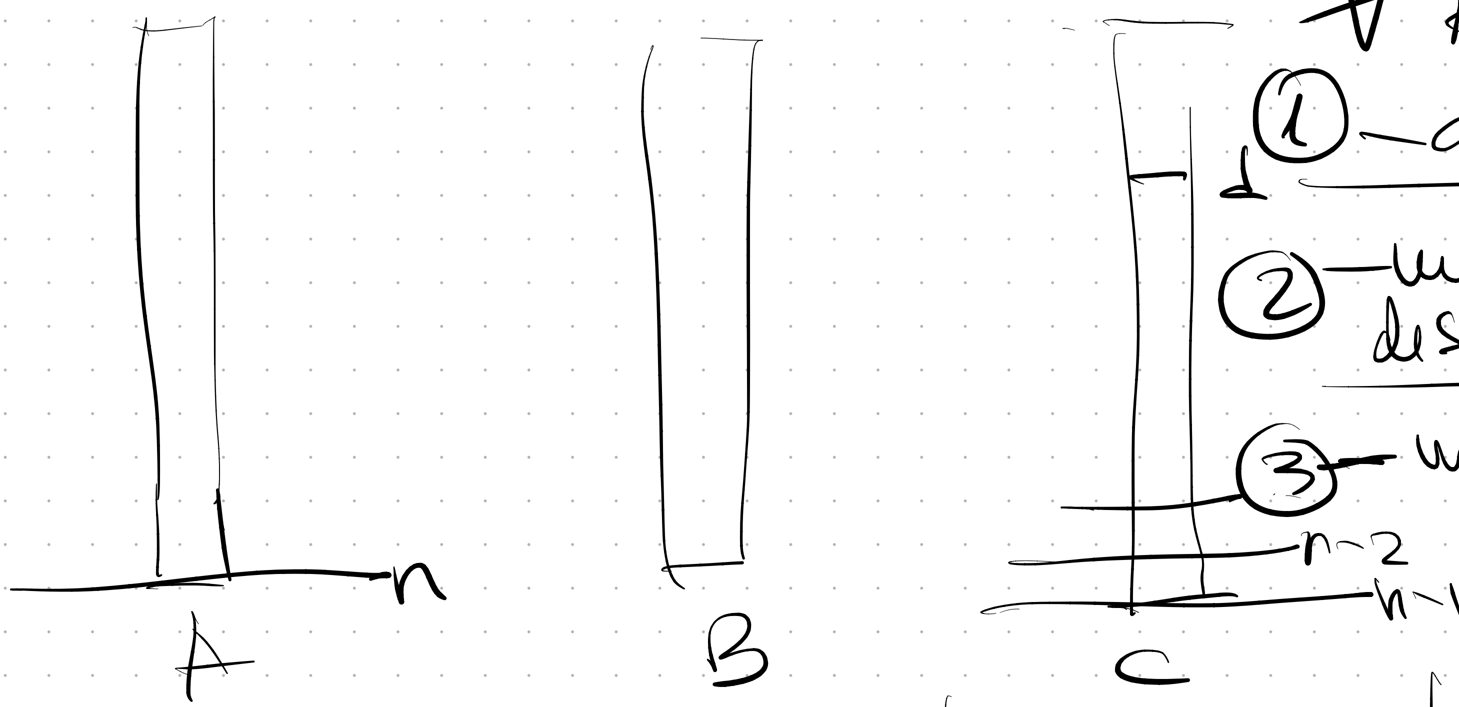
- move all discs one by one from $A \to B$

Prove that $\#\text{moves}_{\text{required}}(n) \geq 2^n - 1$

ind step $\text{moves}(n-1) \geq 2^{n-1} - 1 \Rightarrow \text{moves}(n) \geq 2^n - 1$

<u>structural property</u>: un-avoidable state (invariant)

at some point state must be:



∀ Alg:

1. — get to this state
2. — move disc n A→B
3. — move all other discs C→B

$$\text{Alg: moves}(n) \underset{IH}{\geq} \left|\begin{array}{c|c|c} n-1 \text{ discs } A\to C & n\ A\to B & n-1 \text{ discs } C\to B \\ 2^{n-1}-1 & 1 & 2^{n-1}-1 \\ 1 & 2 & 3 \end{array}\right.$$

$$\text{moves}(n) \geq \underbrace{2^{n-1}-1}_{\text{moves}(n-1) \text{ by IH}} + 1 + \underbrace{2^{n-1}-1}_{\text{moves}(n-1) \text{ by IH}} = 2^n - 1 \checkmark$$