Lecture 19 : 7 lectures left

3 before THXGV

- Induction (3)
- Graphs
- Graphs + Skiplists
  + Trees
  + BST

  hon PB4

4 AFTER THXGV

- Graphs + Algorithms
- Algorithms
- Algorithms + Run Time
- RSA Cryptography (Number Theory part 4)

This week: project check.

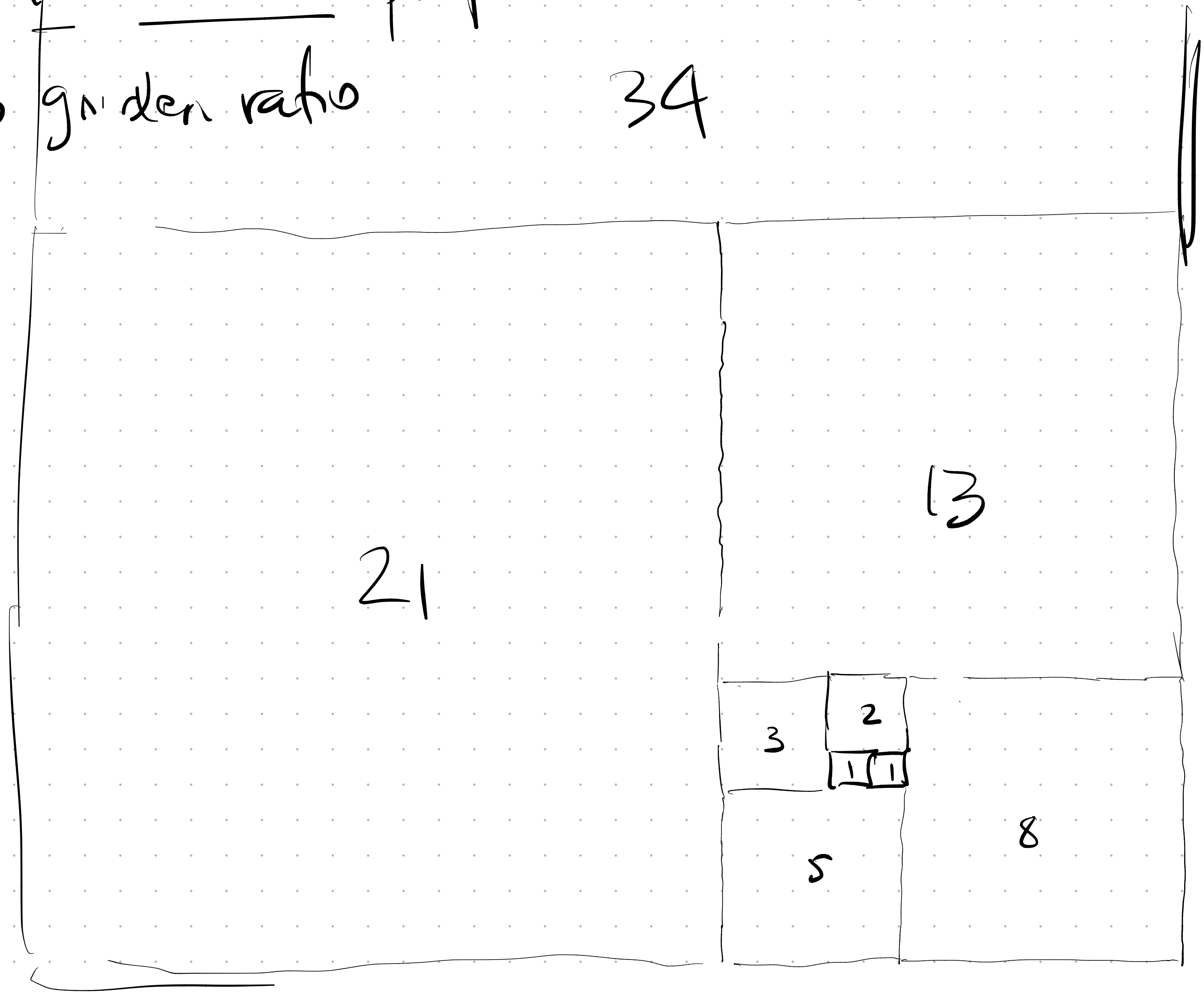
$$F_n = \frac{\varphi^n - \bar{\varphi}^n}{\varphi - \bar{\varphi}}$$

$\varphi = \frac{1+\sqrt{5}}{2}$ $\bar{\varphi} = \frac{1-\sqrt{5}}{2}$

$\varphi^2 = 1 + \varphi$; $\bar{\varphi}^2 = 1 + \bar{\varphi}$

Fnacci properties 0, 1, 1, 2, 3, 5, 8, 13, 21, 34

- golden ratio



- Exercise $F_{n+1} = \varphi \cdot F_n + \bar{\varphi}^n$ (- induction - easier w/out induction)

IND 18

old cust

- $\varphi^n = F_n \cdot \varphi + F_{n-1} \xrightarrow{\text{ind step}} \varphi^{n+1} = F_{n+1} \cdot \varphi + F_n$

new cust

proof: $\varphi^{n+1} = \varphi^n \cdot \varphi \overset{IH}{=} (F_n \cdot \varphi + F_{n-1})\varphi =$

$\varphi^2 = 1 + \varphi$

$= F_n \varphi^2 + F_{n-1}\varphi = F_n (1+\varphi) + F_{n-1} \cdot \varphi =$

$= F_n + F_n\varphi + F_{n-1}\varphi = \varphi(F_n + F_{n-1}) + F_n =$

$= \varphi \cdot F_{n+1} + F_n$ ✓

base case? exercise

IND 19 Count S_n = # strings of n bits without consecutive 1s?

cannot be 1

_ _ _ 1 0 _ _ _ _

n bit

base cases

$S_1 = F_3$

$S_2 = F_4$

n small (say n=10) => cases

Solution: S_n follows fibonacci: $S_n = F_{n+2}$

ind step $\left.\begin{matrix} S_n = F_{n+2} \\ S_{n-1} = F_{n+1} \end{matrix}\right\} \Rightarrow S_{n+1} = F_{n+3}$

STRONG INDUCTION

proof

S_{n+1}: # valid strings of n+1 bits: 2 cases (disjoint)

count = S_n

- $\underline{0}\ \underbrace{\_\ \_\ \_\ \_\ \_\ \_}_{n \text{ bits}}$

count = S_{n-1}

- $\underline{1}\ \underline{0}\ \underbrace{\_\ \_\ \_\ \_\ \_}_{n-1 \text{ bits}}$

$S_{n+1} = S_n + S_{n-1}$ => Fibonacci

- **Exercise** $F_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k}$ "diagonals of Pascal Δ"

- **Exercise** $S_n = \#$ ordered decompositions of n into sums of 1 and 2. $\boxed{S_n = F_{n+1}}$

ex $n=5$:

$1+1+1+1+1$ $\quad$ $2+2+1$

$1+1+1+2$ $\quad$ $2+1+2$

$1+1+2+1$ $\quad$ $1+2+2$

$1+2+1+1$

$2+1+1+1$

IND20

$$r^0 + r^1 + \cdots + r^n = \sum_{k=0}^{n} r^k = \frac{r^{n+1} - 1}{r - 1} \qquad r \neq 1$$

base case? $n = 0$, $n = 1$

IND STEP $\Rightarrow$ $$\sum_{k=0}^{n+1} r^k = \frac{r^{n+2} - 1}{r - 1}$$

proof

$$\sum_{k=0}^{n+1} r^k = \sum_{k=0}^{n} r^k + r^{n+1} =$$

$$\stackrel{IH}{=} \frac{r^{n+1} - 1}{r - 1} + r^{n+1} \underset{\text{WANT}}{\overset{?}{=}} \frac{r^{n+2} - 1}{r - 1} \quad \Big| \times (r-1)$$

$$r^{n+1} - 1 + (r-1) r^{n+1} \underset{\text{want}}{\overset{?}{=}} r^{n+2} - 1$$

$$\cancel{r^{n+1}} - 1 + r^{n+2} - \cancel{r^{n+1}} \underset{\text{want}}{\overset{?}{=}} r^{n+2} - 1$$

✓

IND 21 $\quad n > m$

example $n=5 \quad m=2$

$$\sqrt{2\sqrt{3\sqrt{4\sqrt{5}}}} < 3$$

new inst

$$\underbrace{\sqrt{m\sqrt{(m+1)\sqrt{(m+2)\sqrt{\cdots\sqrt{n}}}}}}_{R_m} < m+1$$

IND STEP backwards $m+1 \rightarrow m$

old inst

$$\underbrace{\sqrt{(m+1)\sqrt{(m+2)\sqrt{\cdots\sqrt{n}}}}}_{R_{m+1}} < m+2$$

proof $\quad R_m = \sqrt{m \cdot R_{m+1}} \overset{IH}{\le} \sqrt{m\,(m+2)} \overset{?}{<} m+1$ want

$$m^2+2m \overset{?}{<} m^2+2m+1 \quad \checkmark$$

want

IND22

$$3^{n+1} \mid 2^{3^n}+1 \xRightarrow{\text{IND STEP}} 3^{n+2} \mid \underbrace{2^{3^{n+1}}+1}_{\text{new custom}}$$

IH: $\boxed{2^{3^n}+1 = 3^{n+1}\cdot k} \Rightarrow$ - - - -

$$2^{3^n} = 3^{n+1}\cdot k - 1$$

proof

$$2^{3^{n+1}}+1 = 2^{3^n\cdot 3}+1 = \left(2^{3^n}\right)^3+1$$

$$\stackrel{IH}{=} \left(3^{n+1}\cdot k - 1\right)^3 + 1$$

$$\boxed{(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3}$$

$$\boxed{a = 3^{n+1}\cdot k \qquad b = 1}$$

$$= \left(3^{n+1}\cdot k\right)^3 - 3\left(3^{n+1}\cdot k\right)^2 + 3\cdot 3^{n+1}\cdot k \cancel{-1} \cancel{+1}$$

$$= 3^{3(n+1)}\cdot k^3 - 3^{(n+1)\cdot 2+1}\cdot k + 3^{n+2}\cdot k$$

$$= 3^{n+2}\left(\text{something}\right) \Rightarrow 3^{n+2} \mid 2^{3^{n+1}}+1 \ \checkmark$$

base case: $n=1$ $\quad 9 \mid 2^3+1$ ✓

IND23

$a_i > 0$ reals

quad mean $\geq$ arith mean $\geq$ geom mean $\geq$ harmonic mean

$$\sqrt{\frac{\sum a_i^2}{n}} \geq \frac{\sum a_i}{n} \geq \sqrt[n]{\prod a_i} \geq \frac{n}{\sum \frac{1}{a_i}}$$

$$\sum_{i=1}^{n} a_i = a_1 + a_2 \ldots + a_n \qquad \prod_{i=1}^{n} a_i = a_1 \cdot a_2 \cdot a_3 \ldots \cdot a_n$$

sum — product

$(\ )^2, \times n^2$

- easy to prove with convexity arguments

First ineq by induction $n \to n+1$

$$\boxed{n \sum_{i=1}^{n} a_i^2 \geq \left(\sum_{i=1}^{n} a_i\right)^2} \Rightarrow (n+1)\sum_{i=1}^{n+1} a_i^2 \geq \left(\sum_{i=1}^{n+1} a_i\right)^2$$

proof:

$$(n+1)\sum_{i=1}^{n+1} a_i^2 = n\sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} a_i^2 + \boxed{(n+1)a_{n+1}^2}$$

$$= n\sum_{i=1}^{n} a_i^2 + a_{n+1}^2 + \sum_{i=1}^{n}(a_i^2 + a_{n+1}^2) \overset{?}{\geq} \left(\sum_{i=1}^{n} a_i + a_{n+1}\right)^2$$

$$n\sum_{i=1}^{n} a_i^2 + \cancel{a_{n+1}^2} + \sum_{i=1}^{n}(a_i^2 + a_{n+1}^2) \overset{?}{\geq} \left(\sum_{i=1}^{n} a_i\right)^2 + \cancel{a_{n+1}^2} + 2\left(\sum_{i=1}^{n} a_i\right)a_{n+1}$$

$(a+b)^2 = a^2+b^2+2ab$

$\xrightarrow{IH}$

$$\sum_{i=1}^{n}(a_i^2 + a_{n+1}^2) \overset{?}{\geq} \sum_{i=1}^{n} 2\cdot a_i \cdot a_{n+1}$$

$$\sum_{i=1}^{n}(a_i^2 + a_{n+1}^2 - 2a_i a_{n+1}) \overset{?}{\geq} 0$$

$$\sum_{i=1}^{n}(a_i - a_{n+1})^2 \overset{?}{\geq} 0$$ true ✓

sum $(\;)^2 \geq 0$

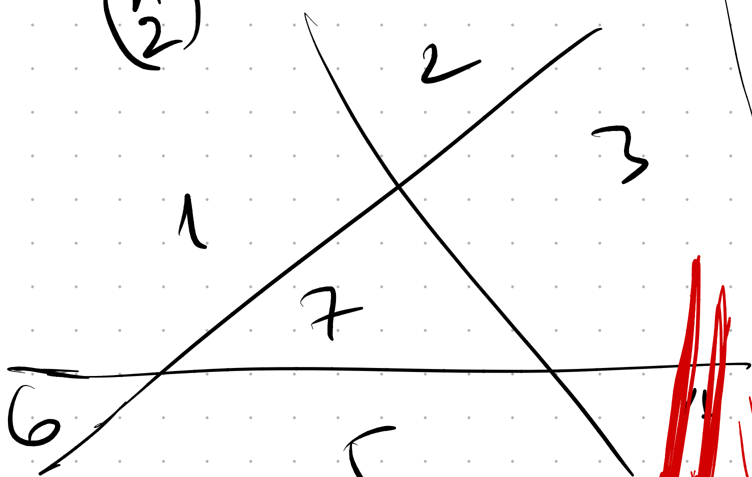
IND 24 n lines ∩ every pair distinct points

No 2 lines parallel
No 3 lines concurrent

#intersections $\binom{n}{2}$

regions $r_n = \dfrac{n^2+n+2}{n}$

$n=3 \quad r_3 = \dfrac{9+3+2}{2} = 7$ example

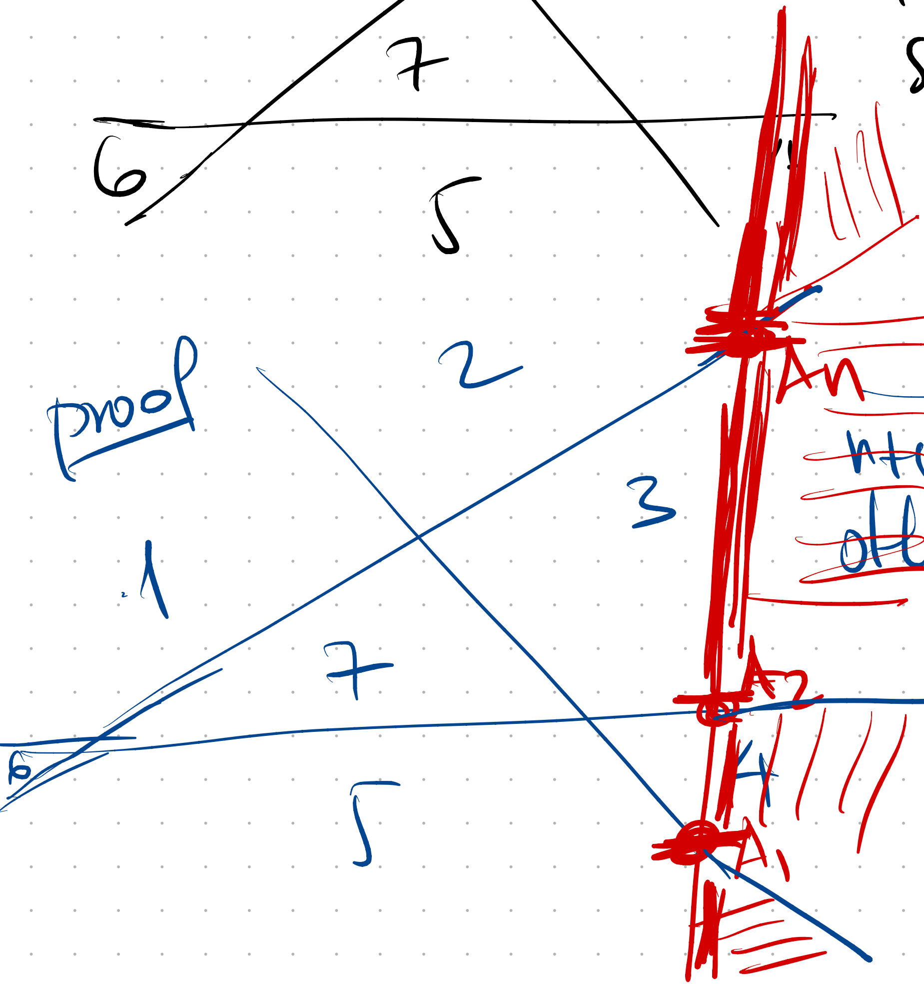


IND STEP

$r_{n+1} = \dfrac{(n+1)^2+(n+1)+2}{2}$

n+1 lines regions

proof



n+1 line (red) has ∩ with the other n lines : $A_1 A_2 \ldots A_n \Rightarrow$ K+1 segments

• creates n+1 new regions (every segment splits an exist region in two)

segment K+1

$$r_{n+1} = r_n + n+1 \overset{\text{IH}}{=} \frac{n^2+n+2}{2} + n+1$$

$$= \frac{n^2+n+2+2(n+1)}{2} \underset{\text{want}}{\overset{?}{=}} \frac{(n+1)^2+(n+1)+2}{2} \quad \Big| \cdot 2$$

$$\cancel{n^2}+n+2+\cancel{2n}+2 \underset{\text{want}}{\overset{?}{=}} \cancel{n^2}+\cancel{2n}+1+n+1+2$$

$$n+4 \overset{?}{=} n+4$$

✓

(IND 25) Same n lines

Color each region s.t. adj regions have dif colors

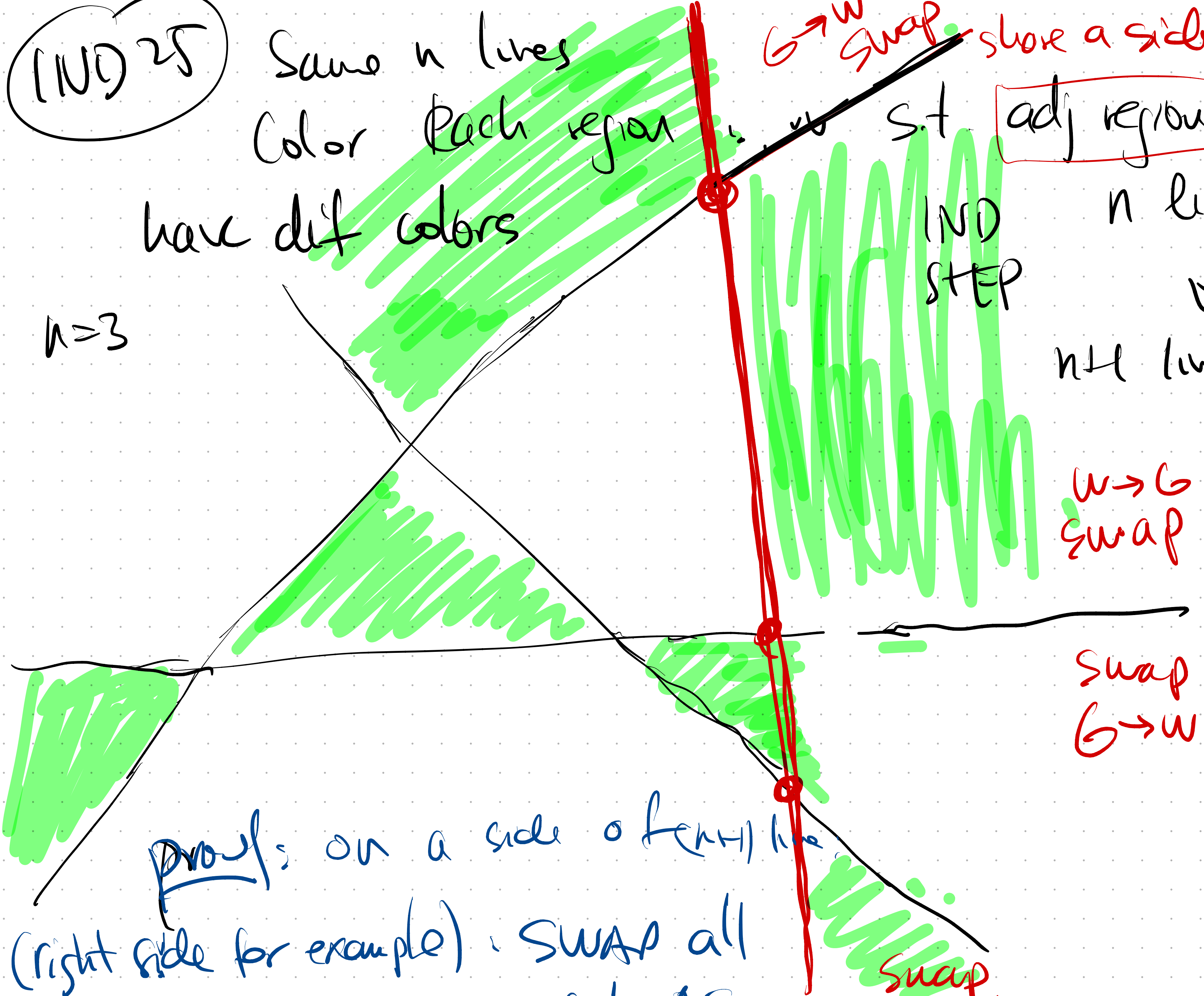
n lines 2 colors

$\Downarrow$

n+1 lines 2 colors

IND STEP

n=3



proof: on a side of (n+1) line (right side for example): SWAP all COLORS

ADJ: REGIONS r_1, r_2

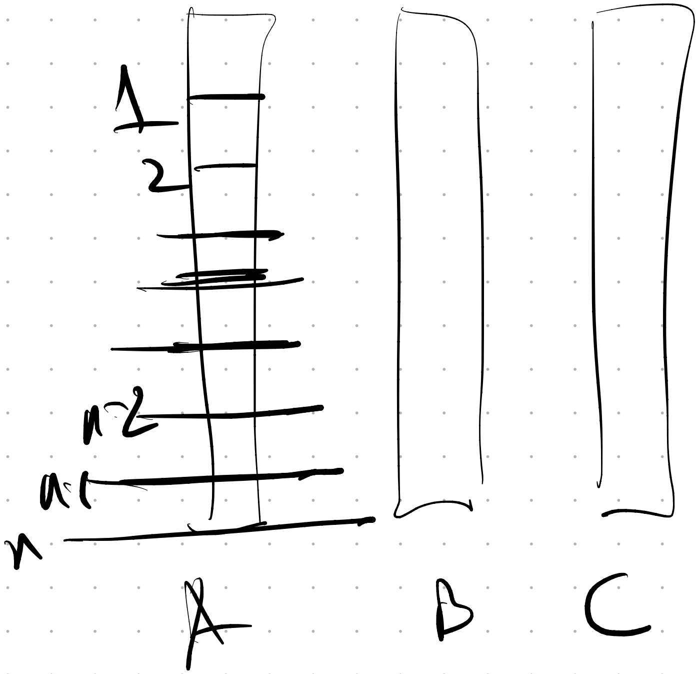
- Left side: same as before

- Right side: both r_1, r_2 swapped so still dif $\Rightarrow$ same color

- across line $n+1$: $r_1 + r_2$ before swap same region

but now RSide swap $\Rightarrow$ r_1, r_2 different colors

IND26 Towers of Hanoi



n discs sizes $1, 2, \ldots n$

any disc can stay on top of bigger discs (or on bottom)

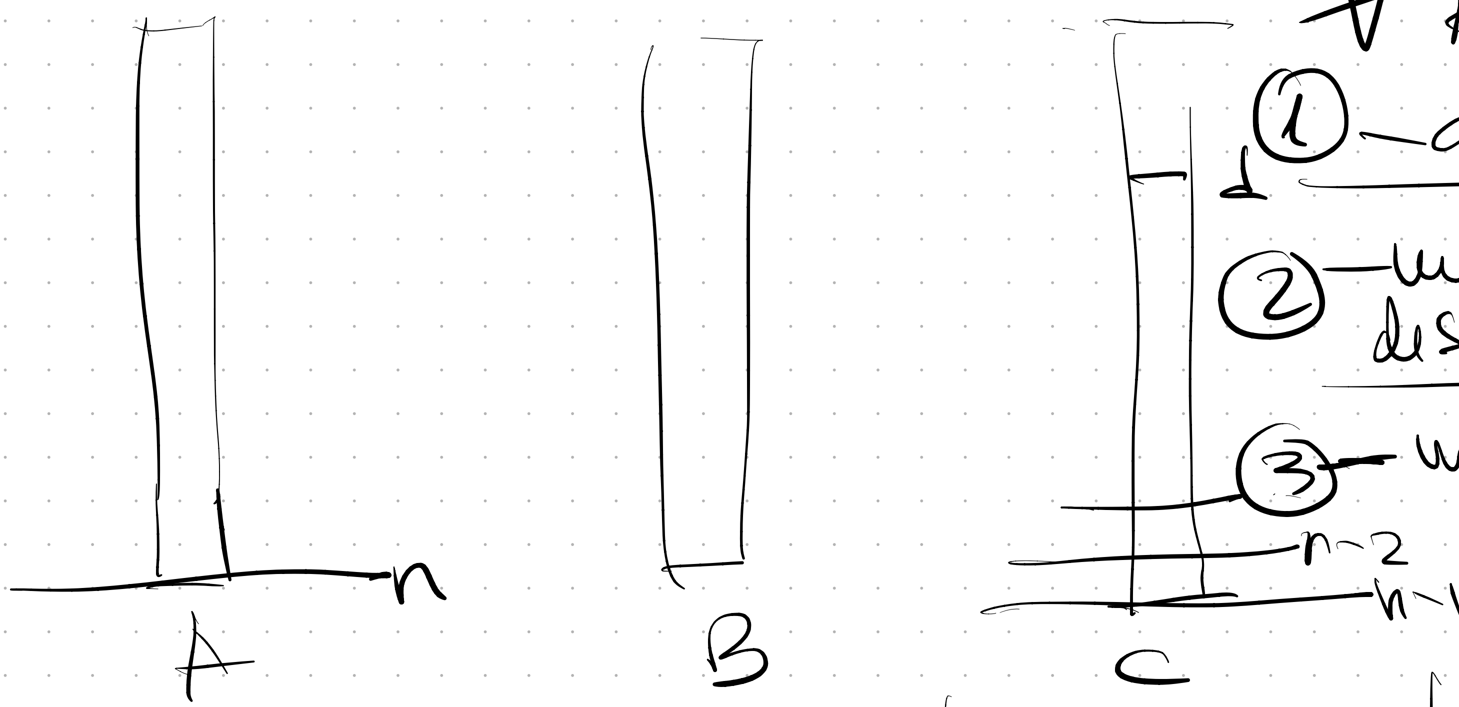
- move all discs one by one from $A \to B$

Prove that $\#\text{moves}_{\text{required}}(n) \geq 2^n - 1$

ind step $\text{moves}(n-1) \geq 2^{n-1} - 1 \Rightarrow \text{moves}(n) \geq 2^n - 1$

<u>structural property</u>: un-avoidable state (invariant)

at some point state must be:



∀ Alg:

1. — get to this state
2. — move disc n $A \to B$
3. — move all other discs $C \to B$

Alg: $\text{moves}(n) \geq$

$n-1$ discs $A \to C$	n $A \to B$	$n-1$ discs $C \to B$
$2^{n-1}-1$	1	$2^{n-1}-1$
1	2	3

IH

$$\text{moves}(n) \geq \underbrace{2^{n-1}-1}_{\text{moves}(n-1) \text{ by IH}} + 1 + \underbrace{2^{n-1}-1}_{\text{moves}(n-1) \text{ by IH}} = 2^n - 1$$

✓