Lecture 18?

- INDUCTION PROOFS

- hw PB3 due Friday
- exam 2
- no HW
- optional class Wed
- TA candidates
  CS1800 Spring 22
  Khoury-admin portal
  (Khoury account)
  reg
  this week!
- project → teammates

INDU

$$\sum_{k=1}^{n} k^3 = 1^3+2^3+3^3+\dots+n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

weak ind step $n \to n+1$

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2 \Rightarrow \sum_{k=1}^{n+1} k^3 = \left(\frac{(n+1)(n+2)}{2}\right)^2$$

old customer

new customer

proof:

$$\sum_{k=1}^{n+1} k^3 = \sum_{k=1}^{n} k^3 + (n+1)^3 \overset{IH}{=} \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3$$

$$= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 \quad ? \quad \text{vs} \quad \left(\frac{(n+1)(n+2)}{2}\right)^2$$

$\times 4$

$\div (n+1)^2$

$n^2 + (n+1)\cdot 4 \quad ? \quad (n+2)^2$

$n^2+4n+4 \quad ? \quad n^2+2\cdot 2\cdot n+2^2$

+ BASE CASE $n=1$

IND 12 Binomial Th by induction over $n \to n+1$ $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$

ind step $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \overset{IH}{\Longrightarrow} (x+y)^{n+1} = \boxed{\sum_{k=0}^{n+1} \binom{n+1}{k} x^k y^{n+1-k}}$

old customer new customer

proof: $(x+y)^{n+1} = (x+y)^n (x+y) \overset{IH}{=} (x+y) \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} =$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{k+1} y^{n-k} + \sum_{k=0}^{n} \binom{n}{k} x^k y^{n+1-k}$$

separate $k=0$

$$= x^{n+1} + \sum_{k=1}^{n} \binom{n}{k-1} x^k y^{n+1-k} + \sum_{k=1}^{n} \binom{n}{k} x^k y^{n+1-k} + y^{n+1}$$

$k=n$ separate

indexed from $1:n$ instead of $0:n-1$

$k=1$: $\binom{n}{0} x^1 y^n$ | $k=n$: $\binom{n}{n-1} x^n y^1$

same prev $k=n-1$

$$= \underbrace{x^{n+1}}_{k=n+1} + \sum_{k=1}^{n} \underbrace{\left[\binom{n}{k-1} + \binom{n}{k}\right]}_{\checkmark\ k=1:n} x^k y^{n+1-k} + \underbrace{y^{n+1}}_{k=0}$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} x^k y^{n+1-k}$$

✓ Base case $n=1$

$$(x+y)^1 = \sum_{k=0}^{1} \binom{1}{k} x^k y^{n-k}$$

INDIB $\quad x > -1 \quad n \geq 0 \Rightarrow (1+x)^n \geq 1 + nx$

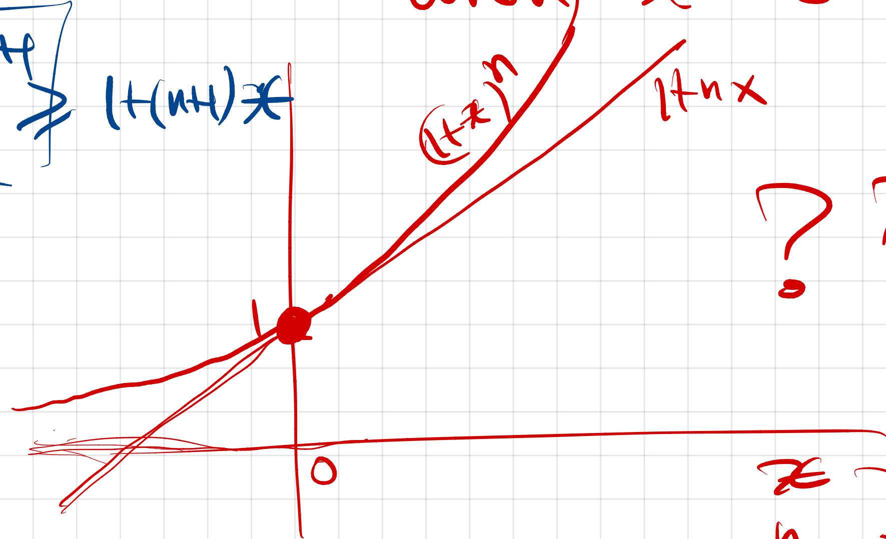
$x+1 > 0 \quad$ integer

$x \in \mathbb{R}$

useful approx $(1+x)^n \simeq 1+nx$ when $x \simeq 0$

ind step $n \to n+1$

$$(1+x)^n \geq 1+nx \Rightarrow \boxed{(1+x)^{n+1} \geq 1+(n+1)x}$$



??

proof

$$(1+x)^{n+1} = (1+x)^n \cdot (1+x)$$

$$\underset{1+x>0}{\overset{IH}{\geq}} (1+nx)(1+x) = 1+nx+x+nx^2$$

$$= 1+(n+1)x + \underbrace{nx^2}_{\geq 0} \geq 1+(n+1)x \checkmark$$

"YB" application

wanted last time: $\underbrace{\left(1+\frac{1}{n}\right)^n}_{a_n} < \underbrace{\left(1+\frac{1}{n+1}\right)^{n+1}}_{a_{n+1}}$ (a_n) mon increasing

Binomial Th $x=1$ $y=\frac{1}{n}$

$$(x+y)^n = \left(1+\frac{1}{n}\right)^n = \underset{k=0}{1} + \sum_{k=1}^{n} \binom{n}{k} \frac{1}{n^k} = 1 + \sum_{k=1}^{n} \frac{n!}{k!(n-k)!} \cdot \frac{1}{n^k}$$

$x=1$ $y=\frac{1}{n+1}$

$$(x+y)^{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1} = \underset{k=0}{1} + \sum_{k=1}^{n} \binom{n+1}{k} \frac{1}{(n+1)^k} + \underset{k=n+1}{\frac{1}{(n+1)^{n+1}}}$$

$$= 1 + \sum_{k=1}^{n} \frac{(n+1)!}{k!(n+1-k)!} \frac{1}{(n+1)^k} + \frac{1}{(n+1)^{n+1}}$$

want:

$$\cancel{1} + \sum_{k=1}^{n} \frac{n!}{k!(n-k+1)!} \cdot \frac{n-k+1}{n^k} \overset{?}{\leq} \cancel{1} + \sum_{k=1}^{n} \frac{n!}{k!(n-k+1)!} \cdot \frac{n+1}{(n+1)^k} + \frac{1}{(n+1)^{n+1}}$$

extra

Sufficient to

$$\sum_{k=1}^{n} \frac{n!}{k!\,(n-k+1)!} \cdot \frac{n-k+1}{n^k} \overset{?}{\leq} \sum_{k=1}^{n} \frac{n!}{k!\,(n-k+1)!} \cdot \frac{n+1}{(n+1)^k}$$

$$\sum_{k=1}^{n} \frac{n!}{k!\,(n-k+1)!} \left(\frac{n-k+1}{n^k} - \frac{n+1}{(n+1)^k} \right) \leq 0$$

Lucky? Sufficient

$$\frac{n-k+1}{n^k} \overset{?}{\leq} \frac{n+1}{(n+1)^k}$$

$$\left(\frac{n+1}{n}\right)^k \overset{?}{\leq} \frac{n+1}{n-k+1}$$

$$\left(\frac{n}{n+1}\right)^k \overset{?}{\geq} \frac{n-k+1}{n+1}$$

$$\left(1-\frac{1}{n+1}\right)^k \overset{?}{\geq} 1 - k \cdot \frac{1}{n+1}$$

proved

$$(1+x)^k \geq 1+kx$$

$$x = -\frac{1}{n+1} \Rightarrow \left(1-\frac{1}{n+1}\right)^k \geq 1 - k \cdot \frac{1}{n+1}$$

$$x > -1$$

IND 14

p prime $\quad a \neq 0 \mod p$ then $a^{p-1} = 1 \mod p$

(ind $a=0$ then $\boxed{a^p = a}^{IH} \mod p \ \forall a$)

Fermat's Little Th

induction by $a \to a+1$ $\qquad a \neq 0 \mod p$

$a+1 \neq 0 \mod p$

$$a^{p-1} = 1 \mod p \implies (a+1)^{p-1} = 1 \mod p$$

prod p new customer

$$(a+1)^p = \sum_{k=0}^{p} a^k \cdot \binom{p}{k} = 1 + a^p + \sum_{k=1}^{p-1} \binom{p}{k} a^k$$

IH

$$\overset{IH}{\underset{\mod p}{=}} 1 + a + p(\text{something})$$

exercise $p \mid \binom{p}{k}$ $\forall\, 1 \le k \le p-1$

$$= 1+a \mod p$$

$(a+1)^{p-1} =$

if $\exists (1+a)^{-1}$: $(a+1)^p (a+1)^{-1} = (1+a)(1+a)^{-1} = 1 \mod p$

IND 15

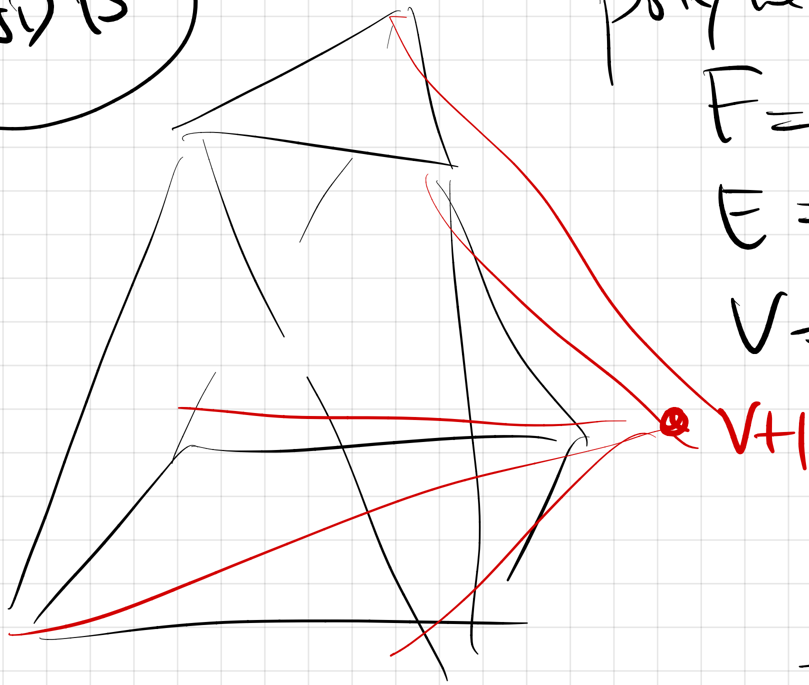
polyhedra CONVEX

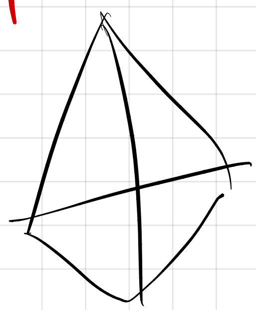
F = faces

E = edges

V = vertices

$$F + V = E + 2$$



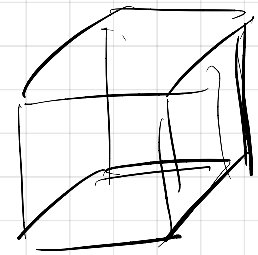


V = 4

F = 4

E = 6

✓



V = 8

F = 6

E = 12

✓

Exercise

induction by V = # vertices. $V \to V+1$

$\boxed{\text{IND 16}}$

Fibonacci

F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}
0,	1,	1,	2,	3,	5,	8,	13,	21,	34,	55 ...

$F_0 = 0,\ F_1 = 1$ base

$$\boxed{F_{n+1} = F_n + F_{n-1}} \quad \forall n \geq 1$$

recursive definition

guess : $F_n \simeq a^n$ exponential

if true, $a^{n+1} = a^n + a^{n-1} \quad \Big| \div a^{n-1}$

$$\boxed{a^2 = a + 1}$$

quad roots $a = \boxed{\frac{1+\sqrt{5}}{2}}$ or $\boxed{\frac{1-\sqrt{5}}{2}}$

φ Golden Ratio.

$\bar{\varphi}$ or ϕ conjugate

$$\varphi^2 = \varphi + 1 \Rightarrow \varphi^{n+1} = \varphi^n + \varphi^{n-1}$$

$$\bar{\varphi}^2 = \bar{\varphi} + 1 \Rightarrow \bar{\varphi}^{n+1} = \bar{\varphi}^n + \bar{\varphi}^{n-1}$$

Theorem

Fib close form

$$\boxed{F_n = \frac{\varphi^n - \bar\varphi^n}{\varphi - \bar\varphi}} = \frac{\left(\frac{1+\sqrt5}{2}\right)^n - \left(\frac{1-\sqrt5}{2}\right)^n}{\sqrt5} \simeq \frac{\varphi^n}{\varphi - \bar\varphi} \quad \text{approx}$$

proof by induction

step

$$F_n = \frac{\varphi^n - \bar\varphi^n}{\varphi - \bar\varphi}$$

$$F_{n-1} = \frac{\varphi^{n-1} - \bar\varphi^{n-1}}{\varphi - \bar\varphi}$$

$$\Rightarrow F_{n+1} = \frac{\varphi^{n+1} - \bar\varphi^{n+1}}{\varphi - \bar\varphi}$$

STRONG IND

proof

"new webinar"

$$F_{n+1} = F_n + F_{n-1} \overset{IH}{=} \frac{\varphi^n - \bar\varphi^n}{\varphi - \bar\varphi} + \frac{\varphi^{n-1} - \bar\varphi^{n-1}}{\varphi - \bar\varphi} =$$

$$= \frac{\overbrace{\varphi^n + \varphi^{n-1}}^{\varphi^{n+1}} - \overbrace{(\bar\varphi^n + \bar\varphi^{n-1})}^{\bar\varphi^{n+1}}}{\varphi - \bar\varphi} = \frac{\varphi^{n+1} - \bar\varphi^{n+1}}{\varphi - \bar\varphi} \checkmark$$

base case:

$$F_1 = \frac{\varphi^1 - \bar\varphi^1}{\varphi - \bar\varphi} = 1 \qquad F_0 = \frac{\varphi^0 - \bar\varphi^0}{\varphi - \bar\varphi} = \frac{0}{\varphi - \bar\varphi} = 0$$

IND 17 Fibonacci F_n

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

2x2 matrix

Induction Step.

$$\underbrace{\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}}_{\substack{2\times 2 \\ A_{n+1}}} = \begin{array}{c} \\ 1 \\ 2 \end{array} \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{2\times 2} \cdot \begin{array}{cc} 1 & 2 \\ \multicolumn{2}{c}{\underbrace{\begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix}}_{2\times 2}} \end{array} \; A_n$$

$$\begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 1\cdot F_n + 1\cdot F_{n-1} = F_{n+1} & 1\cdot F_{n-1} + 1\cdot F_{n-2} = F_n \\ 2 & 1\cdot F_n + 0\cdot F_{n-1} = F_n & 1\cdot F_{n-1} + 0\cdot F_{n-2} = F_{n-1} \end{array}$$

2x2

$$A_n = \begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix}$$

$$\boxed{A_{n+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \cdot A_n}$$

already proved the induction step

$$A_{n+1} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{n}$$

· base A_1

$$\boxed{A_n = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \cdot \quad \text{Base}}$$

use repeated sq.
compute F_n in $\simeq \log(n)$ time.