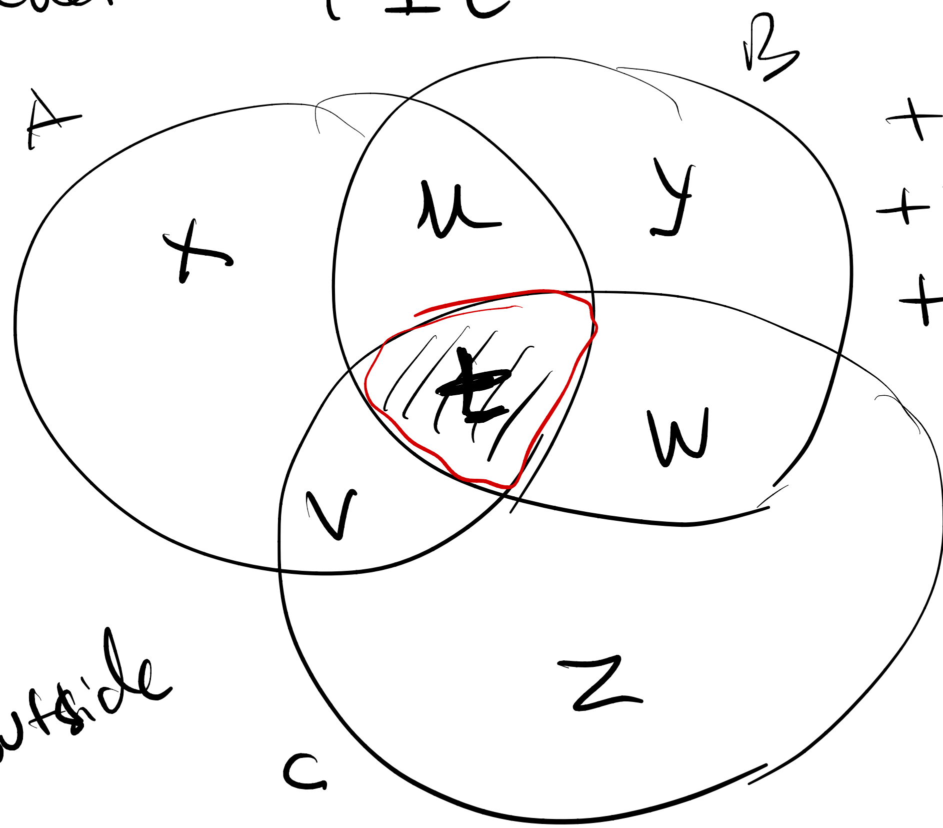
# Lecture 7 : Counting - part 1

- Sum Rule / Union / Partition / Venn Diagrams
- Set indexing
- PigeonHole principle
- Product Rule → PowerSet
- Sequences with repetition
- Sequences without repetition : permutations, $P(n,k)$
- Sets/subsets without repetition : $\binom{n}{k}$ "n choose k"

# General : PIE




m = outside

$|A \cup B \cup C| = x+u+y+v+t+w+z$

$+|A| \rightarrow x+\cancel{u}+\cancel{t}+v$

$+|B| \rightarrow u+y+\cancel{t}+\cancel{w}$

$+|C| \rightarrow z+\cancel{t}+\cancel{v}+w$

$-|A \cap B| \rightarrow -\cancel{u}-\cancel{t}$

$-|B \cap C| \rightarrow -\cancel{t}-\cancel{w}$

$-|C \cap A| \rightarrow -\cancel{t}-\cancel{v}$

$+|A \cap B \cap C| \rightarrow +t$

$x+v+u+y+z+w+t$

Partition : Split everything into <u>disjoint</u> parts

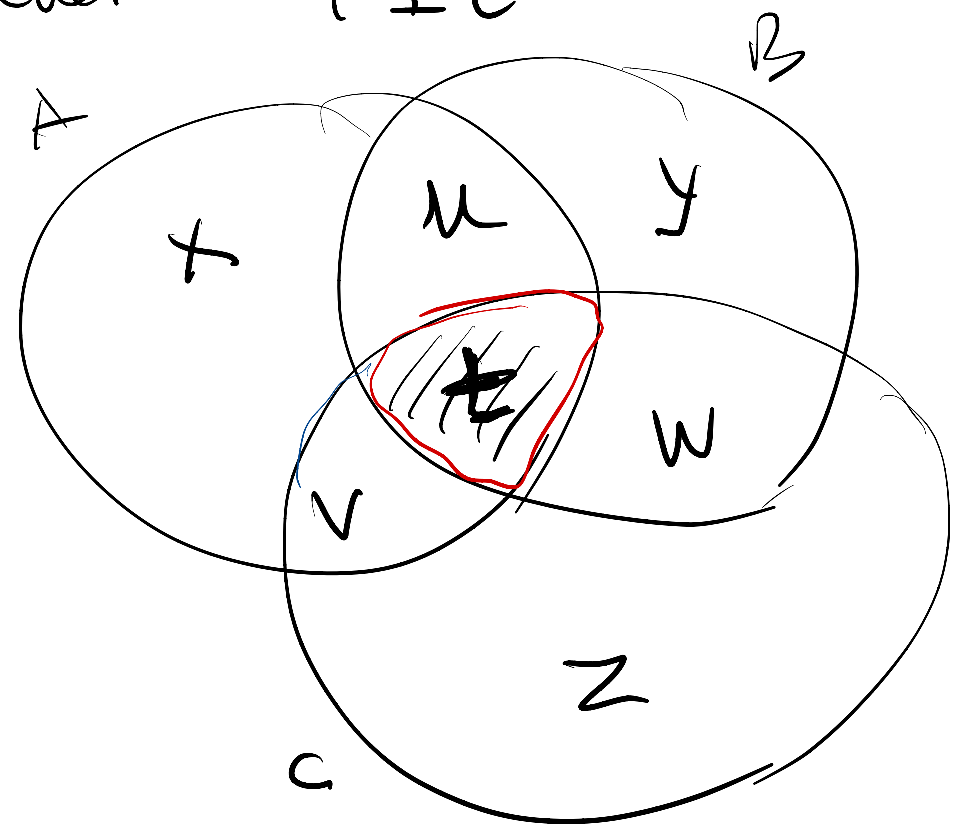
$x = A \setminus B \setminus C$

$u = A \cap B \setminus C$

$m = \overline{A \cup B \cup C}$

$\vdots$

# General: PIE




**PB1**

$$(A \cap B) \cup C \overset{?}{=} A \cap (B \cup C)$$

$$u + t + \cancel{t} + \cancel{t} + v + w + z = u + v + t$$

$$u + t + v + w + z$$

NO

**PB2** $(A \setminus C) - B \overset{?}{=} A - (C - B)$

$$x \overset{?}{=} x + u + t$$

NO

**PB3** $(A \cup B) - C \overset{?}{=} (A \cup B \cup C) - C$

$$x + u + y \overset{\text{YES}}{=} x + u + y$$

**PB4**

$$|A \cup B \cup C| = |A| + |B| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$x + u + y = x + \cancel{u} + \cancel{t} + \cancel{v} - \cancel{u} - \cancel{t} - \cancel{v} - \cancel{t} - \cancel{t} - \cancel{w} + \cancel{t}$$

$$u + y + \cancel{t} + \cancel{w}$$

YES

$$|(A\cup B)\setminus C| \overset{?}{=} |A|+|B|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C|$$

Set algebra:

$$|(A\cup B)\setminus C| = |A\cup B\cup C\setminus C| = |\underline{A\cup B\cup C}|-|C| =$$

subset of first term

$$|A|+|B|+\cancel{|C|}$$
$$-|A\cap B|-|A\cap C|-|B\cap C|$$
$$+|A\cap B\cap C| - \cancel{|C|}$$


Sum Rule  A, B, C, ... disjoint ($\cap = \emptyset$)
(partition)

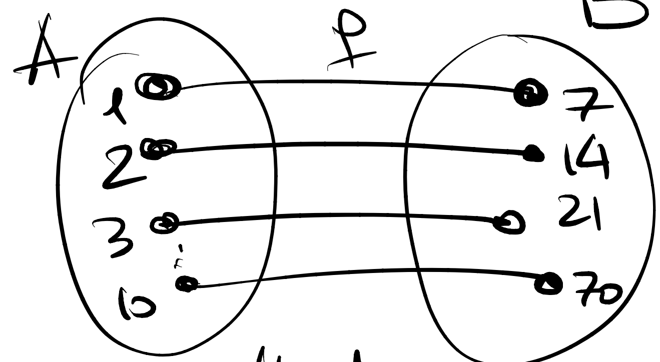
$$|A\cup B\cup C\ldots| = |A|+|B|+|C|+\ldots$$

- counting objects in set S:
  - partition $S = S_1\cup S_2 \ldots S_n$ ($S_i\cap S_j=\emptyset$)
  - count each part $|S_i|$
  - sum up $|S_1|+|S_2|+\ldots+|S_n|$

Counting technique: indexing / mapping

f = map

map = one-to-one function (bijection)




example

$A = \{1, 2, 3, \dots 10\}$

$B = \{x \in \mathbb{N};\ 2 \leq x \leq 72,\ x = 7k\}$ multiple of 7

$= \{7, 14, 21 \dots 70\}$

$f: A \to B$ bijection (one-to-one) $f(x) = 7 \cdot x$

x in A, 7x in B

called indexing if

$A = \{1, 2, 3, \dots, n\}$

$A = \{1 : n\}$

Th $\exists f: A \to B$ one-to-one $\Rightarrow |A| = |B|$

$\mathbb{Z}_n$ = remainders at integer-division with $n = \{0, 1, 2, \ldots, n-1\}$

$\mathbb{Z}_{10} = \{0, 1, 2, \ldots 9\}$

$\mathbb{Z}_2 = \{0, 1\}$

$\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$.

$$\left\{ \text{all pairs } (x \in \mathbb{Z}_2,\ y \in \mathbb{Z}_5) \right\}$$

$$\mathbb{Z}_{10} \overset{f}{\longleftrightarrow} \mathbb{Z}_2 \times \mathbb{Z}_5$$

$$f(x) \longleftrightarrow (x \bmod 2,\ x \bmod 5)$$

$$x = 3 \longleftrightarrow (1, 3)$$

$$x = 7 \longleftrightarrow (1, 2)$$

$$x = 4 \longleftrightarrow (0, 4)$$

$X =$ set    example $X = \{a, b, c, d\}$    $a \in X$

$A \subset \mathcal{P}(X)$   $A = \{$ subsets of $X$ <u>include</u> "a" $\}$

$\{a\}$ $\{a,b\}$ $\{a,b,c\}$ - - -

$A \cup B = \mathcal{P}(X)$

$B \subset \mathcal{P}(X)$   $B = \{$ subsets of $X$ <u>do not</u> include "a" $\}$

$\emptyset$, $\{b\}$, $\{c\}$ $\{b,c\}$ $\{b,d\}$ $\{d\}$ . . .

$|A| = |B|$ ?

$$A \xleftrightarrow[\text{map}]{f} B$$

one-to-one

$f$ one-to-one

$$T \in A \xleftrightarrow{f} S \text{ in } B$$

$\boxed{T = \text{subset of } X \ \ a \in T}$    $\boxed{S \text{ subset of } X \ \ a \notin S}$

$f(T) = T \setminus \{a\}$ ✓ $\Rightarrow |A| = |B| = \dfrac{|\mathcal{P}(X)|}{2}$

* $X$ set $a \in X$ $b \in X$ $a \neq b$. $X = \{a, b, c, d, e\}$

$A$ = set of $\left\{ \begin{array}{l} \text{all subsets of } X \\ \text{contain } a \quad a \in T \end{array} \right\}$

$\{a\}$ $\{a,b\}$ $\{a,c\}$ $\{a,b,c\}$ - - -

$B = \left\{ \begin{array}{l} \text{all subsets of } X \\ S \text{ that contain "} b \text{"} \end{array} \right\}$

$\{b\}$ $\{b,a\}$ $\{b,c\}$ $\{b,d\}$ $\{b,c,d\}$ $\{b,a,c\}$ - - -

$A \cap B \neq \emptyset$? $\{a,b\} \in A \cap B$

$\{a,b,c\} \in A \cap B$

$A \xleftrightarrow{f \text{ one to one?}} B$ $\underline{\text{Yes}} \Longrightarrow |A| = |B|$

<u>exercise</u>

$\mathbb{N} = \{0, 1, 2, 3, 4 \ldots\}$ infinite, countable

$\mathbb{Z} = \{\ldots -3, -2, -1, 0, +1, +2, +3 \ldots\}$

$\mathbb{N} \subset \mathbb{Z}$

$\mathbb{Z} \setminus \mathbb{N} \neq \emptyset$ for example $-1, -2,$

$\mathbb{N} \longleftrightarrow \mathbb{Z} \longrightarrow \boxed{|\mathbb{N}| = |\mathbb{Z}|}$ Countable infinite

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\mathbb{N}$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | ... |
| $\mathbb{Z}$ | 0 | +1 | −1 | +2 | −2 | +3 | −3 | +4 | −4 | |

$x=2k \quad f(2k) = -k$

$x=2k+1 \quad f(2k+1) = k-1$

# Pigeon Hole principle

non-math version

- $n$ items placed in $n-1$ boxes (spots) $\Longrightarrow \exists$ at least one box with 2 items or more
- $n$ items placed on $k$ boxes $\Longrightarrow \exists$ at least one box with $\lceil \frac{n}{k} \rceil$ items

math-version

$x_1, x_2, x_3 \dots x_n \in \mathbb{R}$ $\quad \mu = \frac{x_1 + x_2 \dots + x_n}{n} = E[x]$

- at least one of them $x_i \geq \mu$ → prove by contradiction
- at least one of them $x_j \leq \mu$

assume $x_i < \mu \ \forall i$

$\sum x_i < n \cdot \mu$

$\sum x_i < \sum x_i$ ! contradiction

$\Longrightarrow \exists i \ x_i \geq \mu$

$n = 25$ students
$k = 3$ classrooms $\Rightarrow \exists$ one classroom with $\geq \lceil \frac{25}{3} \rceil = 9$

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10 people $x_1, x_2 \ldots x_{10}$ salaries avg 80,000 / sum of salaries is 800,000

$$\frac{x_1 + x_2 + \ldots + x_{10}}{10} = 80,000$$

$\Rightarrow$ at least one ($\exists i$) $\underline{x_i \geq 80,000}$

Technique for counting sets: product rule
(hats × pants × jackets)

$A = \{a, b, c\}$ $B = \{1, 2\}$ $C = \{\alpha, \beta, \gamma\}$

want triplet $(\underline{x \in A}, \underline{y \in B}, \underline{z \in C})$

- any combination works

$(a, 1, \beta) \neq (1, \beta, a)$

$\#$ triplets : $|A| \cdot |B| \cdot |C|$

$3 \cdot 2 \cdot 3$

is $(1, a, \alpha)$ triplet?
NO

triplet = sequence

$A \times B \times C = \left\{ \begin{array}{lll} (a, 1, \alpha), & (a, 1, \beta), & (a, 1, \gamma) \\ (a, 2, \alpha), & (a, 2, \beta), & (a, 2, \gamma) \\ \vdots & & \\ (c, 2, \alpha), & (c, 2, \beta), & (c, 2, \gamma) \end{array} \right\}$

- with repetition: $S = \{A, B, C, D\}$

sequence of 5 letters (rep. allowed): A B A A C

A A C D C

A B D | A B C D | ___ | ___ | A B C D

4 possi | 4 possib | | | 4 possib

- any combination (sequence) valid

$$\#\text{sequences} = 4 \times 4 \times 4 \times 4 \times 4 = 4^5$$

- License plates 8 spots

___ ___ ___ ___ ___ ___ ___ ___

26 36 36 ... $36^8$

$S = \{$ Letters (cap) + all digits $\}$ $\Longrightarrow 36$

• $\boxed{\text{no repetitions}}$ $S = \{a, b, c, d, e\}$

$\Rightarrow$ max # spots $= |S|$

5 spots (longest sequence)

$\boxed{\text{permutation (of all elements in S)}}$ $= 5!$

$\underline{\quad}$ $\underline{\quad}$ $\underline{\quad}$ $\underline{\quad}$ $\underline{\quad}$

5 4 3 2 1 120

(one is used) (two are used) the only one left

no repetitions

a b c d e
b c d a e
e d b c a
⋮

5-plets
sequence of length 5

no repetitions $S = \{a, b, c, d, e\}$ $n = 5$

sequences of length $K = 3$ $K \leq n$

$5 \cdot 4 \cdot 3$

$\underline{\quad}$ $\underline{\quad}$ $\underline{\quad}$

5 4 3

$n$ $n-1$ $n-2$

$\boxed{\begin{matrix}(a\ b\ c) \\ (b\ a\ c)\end{matrix}}$ diff

(c a b)

(d a c)

(e d b)

$$\frac{5!}{2!}$$

$$\frac{5!}{(5-3)!}$$

general $S = \{1, 2, \ldots n\}$

want sequence of length $K \leq n$

$n \times (n-1) \times \ldots \times (n-k+1)$

$\underline{\quad} \times \underline{\quad} \times \underline{\quad} \times \underline{\quad} \times \underline{\quad}$

spot 1 spot 2 spot k

$n$ $n-1$ $n-2$ $n-k+1$

(one used)

choose seq of k IN ORDER out of $n$ $= P(n,k) = \dfrac{n!}{(n-k)!} = \dfrac{n \cdot (n-1) \cdots (n-k+1)(n-k) \cdots 1}{(n-k)(n-k-1) \cdots 1}$

- no repetition. no order

$n$ items: $1, 2, \ldots, n$

$K$ spots

$$\underset{\text{spot}1}{n} \times \underset{\text{spot}2}{(n-1)} \times \underset{\text{spot}3}{(n-2)} \times \_\_\_ \; \_\_\_ \times \underset{\text{spot}K}{(n-k+1)}$$

remaining $n-K$

write it down (output) AS A SET

$n=5 \quad K=3$

Sequen

| | | | |
|---|---|---|---|
| a | b | c | d, e |
| b | a | c | d, e |
| c | a | b | |
| a | c | b | |
| b | c | a | |
| c | b | a | |

$6 = \frac{5!}{2!}$

$\frac{n!}{(n-k)!}$

SETS

one set $\{a, b, c\}$

$\{b, a, c\}$

$\{c, a, b\}$

$\{a, c, b\}$

$\{b, c, a\}$

$\{c, b, a\}$

$K!$ occurences of set $\{a, b, c\}$

# different sets $\times K! = $ # diff seq

$P(n, k) = \frac{n!}{(n-k)!}$

# diff sets $= \frac{n!}{(n-k)!\,k!}$

choose k items (no order, no repetition) out of n

"n choose k" $= \boxed{\binom{n}{k}} = C(n,k) = nCk$

choose a subset of size k out of a set of n elements

set = {1, 2, 3, ... n}

$\binom{n}{k}$ = # different subsets of size k (out of n)

$$2^n = \underbrace{\binom{n}{0}}_{\#\text{subsets size}=0} + \underbrace{\binom{n}{1}}_{\#\text{subs. size }1} + \underbrace{\binom{n}{2}}_{\#\text{subsets size }2} + \underbrace{\binom{n}{3}}_{\#\text{subsets size }3} + \cdots + \underbrace{\binom{n}{n-1}}_{\#\text{subsets size }n-1} + \underbrace{\binom{n}{n}}_{\#\text{subsets size}=n}$$

$\binom{n}{0}$: $\emptyset$

$\binom{n}{1}$: {1}, {2}, {3}, ..., {n}

$\binom{n}{2}$: {1, 2}, {1, 3}, ..., {n-1, n}

$\binom{n}{3}$: {1, 2, 3}, {1, 2, 4}, ..., {n-2, n-1, n}

$\binom{n}{n-1}$: {1, 2, ... n-1}, ..., {2, 3, ... n}

$\binom{n}{n}$: 1: {1:n}

Th $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

$$\sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} = 2^n$$

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choosing k-set

$\equiv$ choosing remaining n-k set

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\frac{n}{k!\,(n-k)!} = \frac{n!}{(n-k)!\,(n-(n-k))!}$$