Set = collection of items / objects / elements

OUT OF ORDER

$\{1,2,4,3,6,5,10,9,8,7\}$

enumeration $U = \{1,2,3,4,5,6,7,8,9,10\}$ $|U| = 10$

set notation / builder / property

$U = \{x \mid x \in \mathbb{Z},\ x > 0,\ x \leq 10\}$

$= \{x \in \mathbb{Z} \mid 0 < x \leq 10\}$

$|A| = 5$
$|B| = 5$

$A \subset U$ subset
$B \subset U$ subset
$\cup$

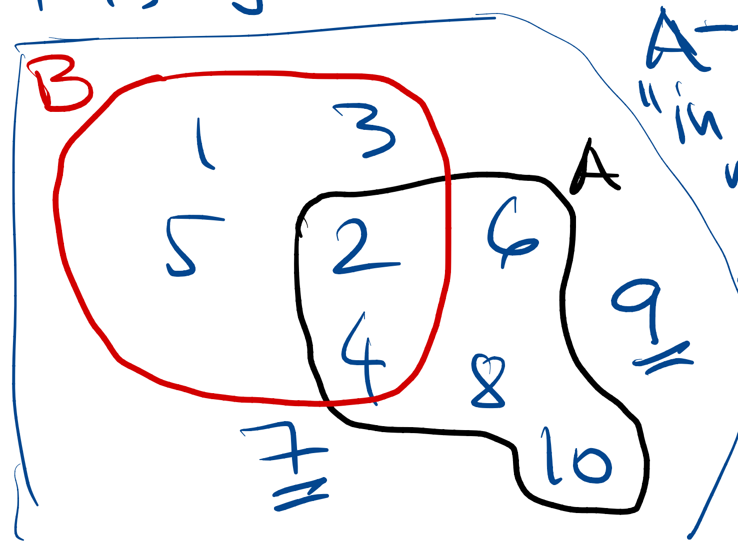
$A =$ "evens in U" $= \{2,4,6,8,10\}$

$B =$ "≤ 5 in U" $= \{1,2,3,4,5\}$

Venn Diagram

$A \cap B =$ "common elem to A and B"
$= \{2,4\}$

$A \cup B =$ "elements in A or B",
$\{1,2,3,4,5,6,8,10\}$

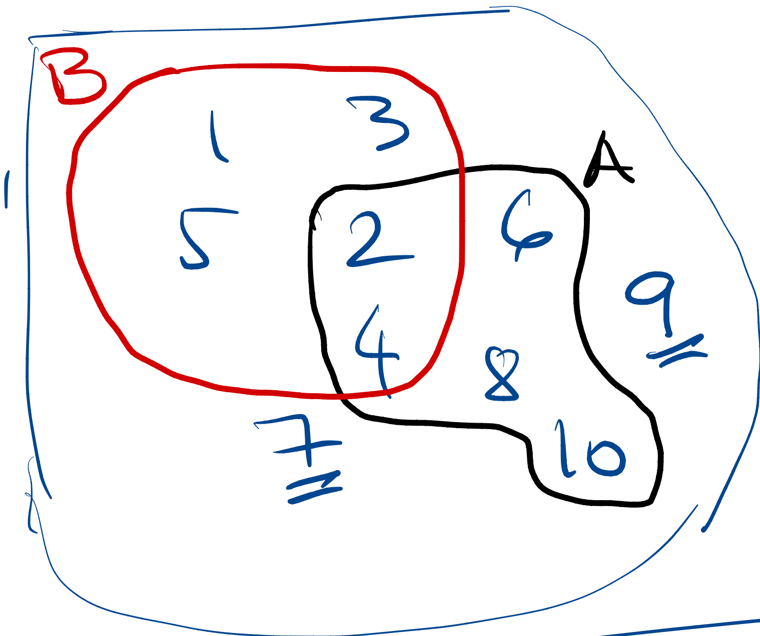


$A - B = A \setminus B$
"in A and not B"
$\{6,8,10\}$

$B - A =$
$\{1,3,5\}$

$A - B =$ "elem in A" and not in B

$= A \cap \text{not } B = A \cap \overline{B}$



symmetric difference $\simeq$ XOR

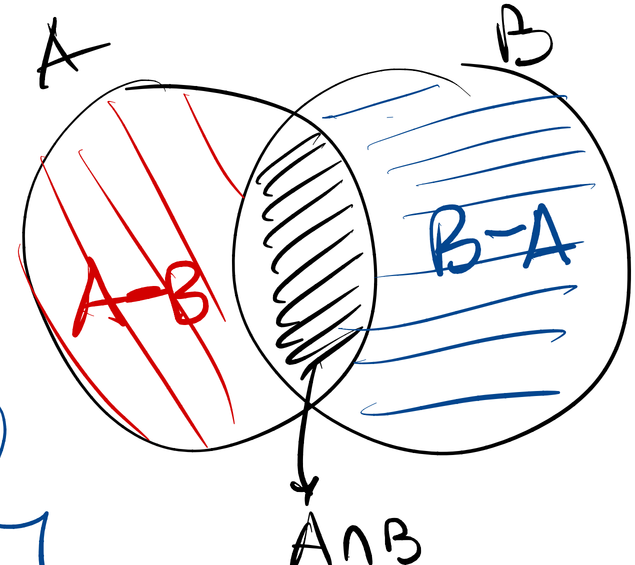
$A \Delta B = (A \setminus B) \cup (B \setminus A)$ } proof?

$= (A \cup B) - (A \cap B)$

$= \{1, 3, 5, 6, 8, 10\}$

not B = complement of $B = \overline{B}$

$=$ universe $\setminus B = \{6, 7, 8, 9, 10\}$



power set = set of subsets

$\mathcal{P}(B) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1,2\}, \{1,3\}, \{1,4\}, \ldots, \{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \ldots\}$

$|\mathcal{P}(B)| = 2^5 = 2^{|B|}$

Bit-vector $\quad U = \{A, B, \ldots Z\}$

$$[\,0/1 \;\_\; \_ \quad \_\,]$$

$$\longleftarrow 26 \longrightarrow$$

$$S_1 = \{A, C, Z\} = [1\,0\,1\,0 \ldots 0\,1]$$

$$S_2 = \{B, C\} = [0\,1\,1\,0 \ldots 0]$$

$$S_1 \cup S_2 = S_1 \vee S_2 = [1\,1\,1\,0\,0 \ldots 0\,1]$$

OR bitwise

$\vee$ $\quad$ //

$S_1 \cap S_2 = S_1 \wedge S_2$

AND bitwise.

$\overline{S}$ complement $\neg S$

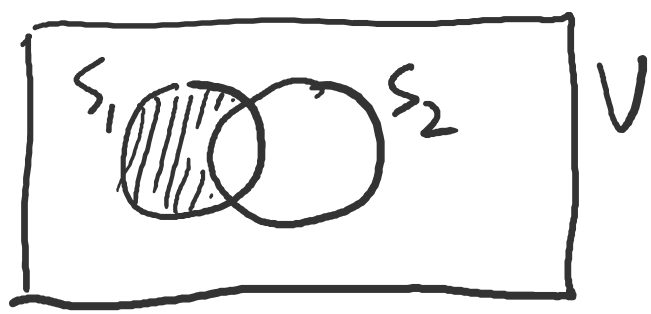
NOT bitwise

flip bits.

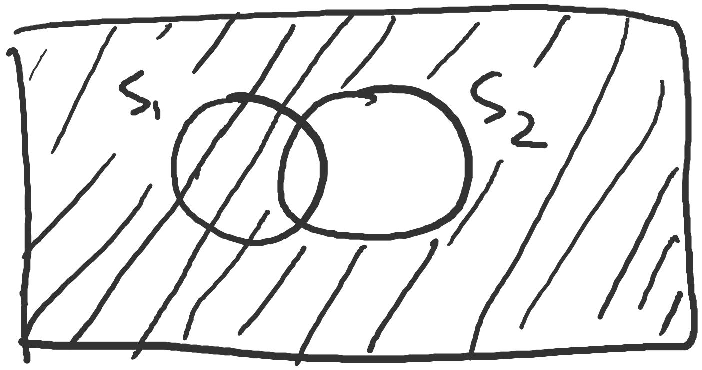
$S_1 - S_2 = S_1 \cap \overline{S_2} = S_1 \wedge (\neg S_2)$

Venn diagram

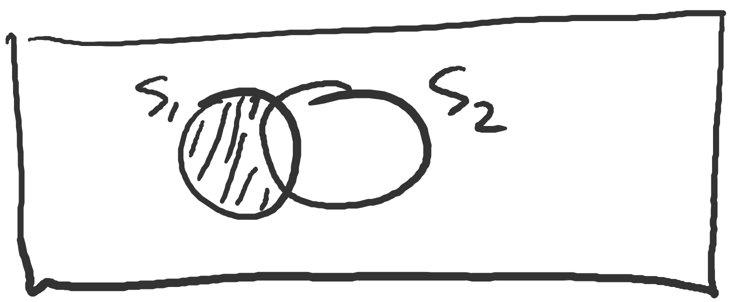
$S_1 - S_2$



$\overline{S_2} = \neg S_2$

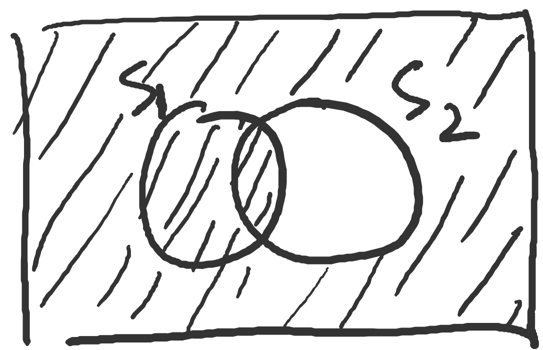


$S_1 \cap \overline{S_2}$



$= S_1 - S_2$

$S_1 \cup \overline{S_2}$



$$S_1 \Delta S_2 = S_1 \oplus S_2 = [1\ 1\ 0\ \_\ \_\ .\ 0\ 1]$$

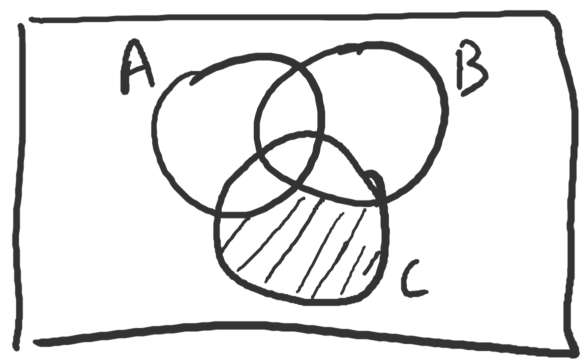
XOR, parity

$$= \{A, B, Z\}$$

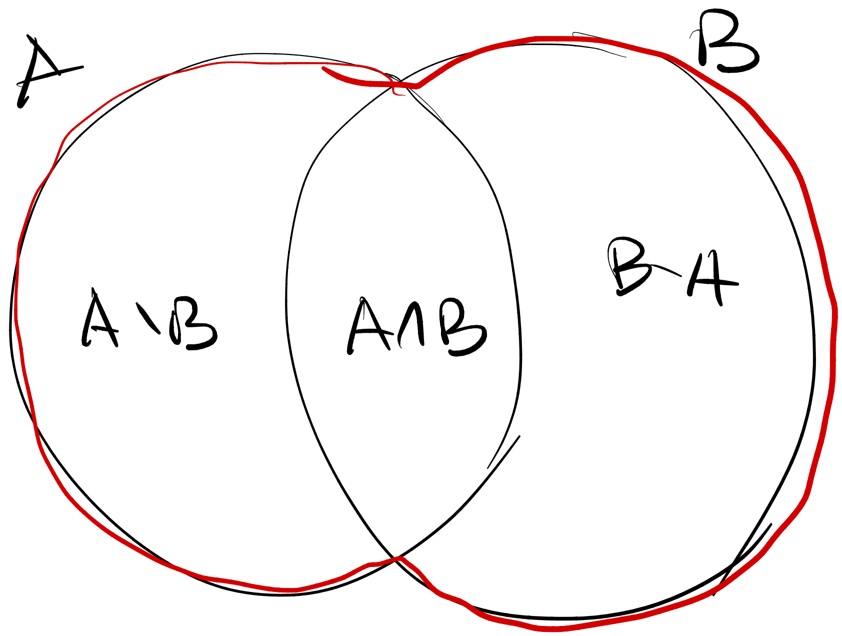
not

(C-A)-B

$$\overline{(A \cup B)} \cap C = (\neg(A \vee B)) \wedge C$$



Sum Rule: size of union.



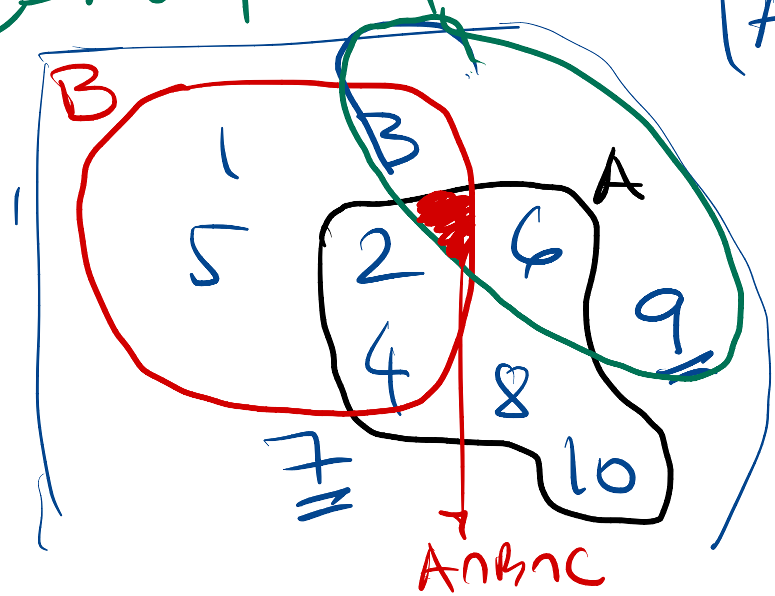
$|A\cup B| = |A\setminus B| + |A\cap B| + |B\setminus A|$

• $\boxed{\text{partition}}$ into 3 sets
→ disjoint sets
→ union = total

was double counted.

$$|A\cup B| = |A| + |B| - |A\cap B|$$

C = multiples of 3 = $\{x\in U \mid x=3k,\ k\in\mathbb{Z}\}$



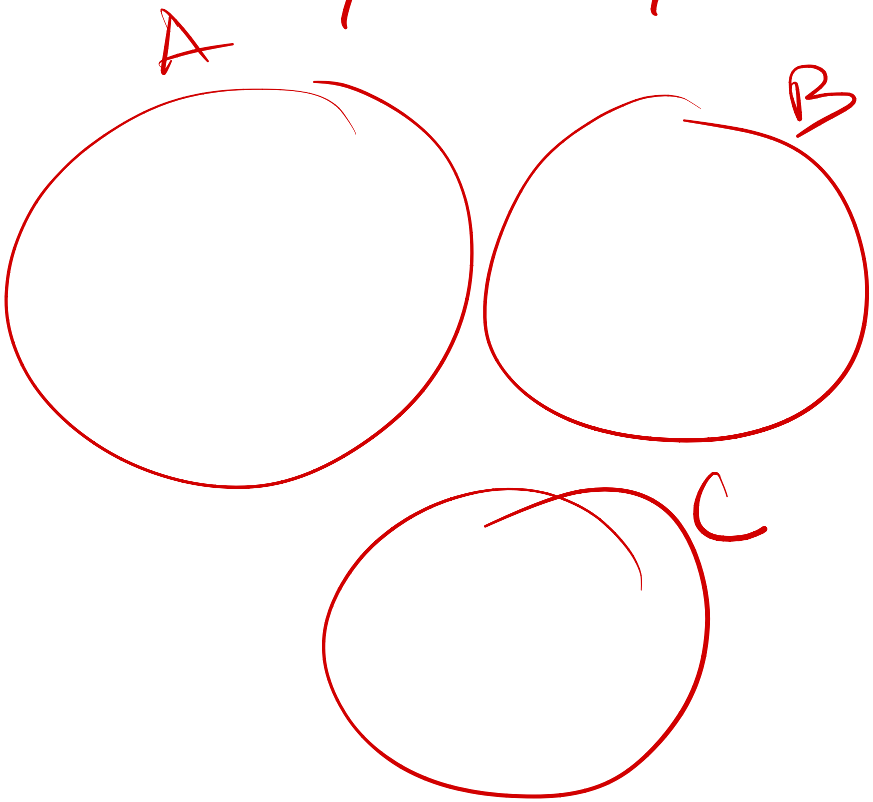
$$|A\cup B| = |A| + |B| - |A\cap B|$$
$$= 5 + 5 - 2 = 8$$

$$|A\cup B\cup C| = |A| + |B| + |C| - |A\cap B| - |B\cap C| - |C\cap A| + \boxed{|A\cap B\cap C|}$$

Principle of Inclusion-Exclusion

Sum Rule for Partition (Partition Rule) (no intersection)

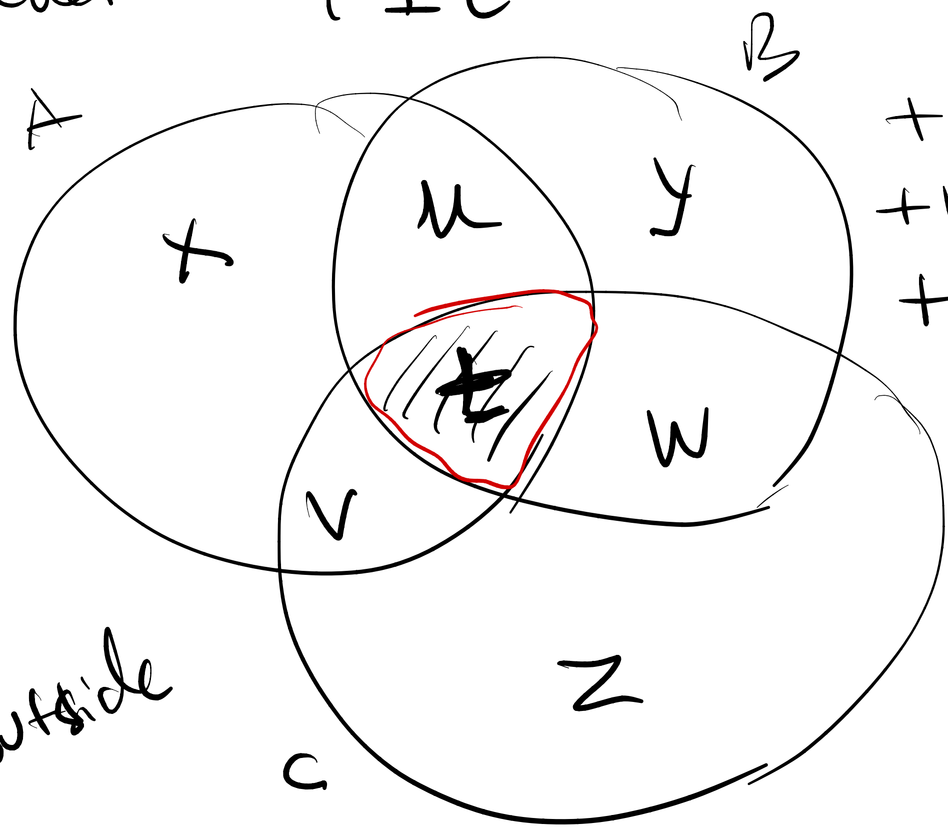
$A \cap B = \emptyset$, $B \cap C = \emptyset$, $C \cap A = \emptyset$



$|A \cup B \cup C| =$

$|A| + |B| + |C|$

General PIE



m = outside

$$|A\cup B\cup C| = x+u+y+v+t+w+z$$

$$+|A| \rightarrow x+\cancel{u}+\cancel{t}+v$$

$$+|B| \rightarrow u+y+\cancel{t}+\cancel{w}$$

$$+|C| \rightarrow z+\cancel{t}+\cancel{v}+w$$

$$-|A\cap B| \rightarrow -\cancel{u}-\cancel{t}$$

$$-|B\cap C| \rightarrow -\cancel{t}-\cancel{w}$$

$$-|C\cap A| \rightarrow -\cancel{t}-\cancel{v}$$

$$+|A\cap B\cap C| \rightarrow +t$$

$$x+v+u+y+z+w+t$$

Partition: Split everything into <u>disjoint</u> parts

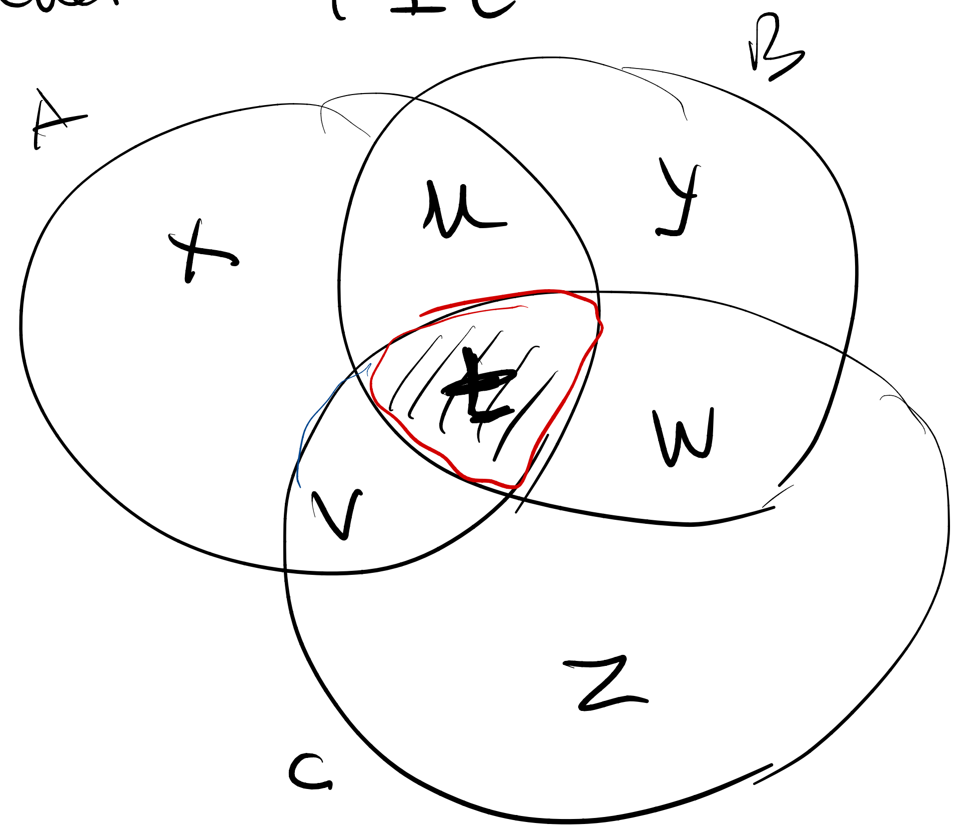
$$x = A\setminus B\setminus C$$

$$u = A\cap B\setminus C$$

$$m = \overline{A\cup B\cup C}$$

⋮

General : PIE



PB1

$(A\cap B)\cup C \stackrel{?}{=} A\cap(B\cup C)$

$u + \cancel{t} + \cancel{t}\; t + v + w + z = u + v + t$

$u+t+v+w+z$

NO

PB2

$(A\setminus C)\cap B \stackrel{?}{=} A-(C-B)$

$x \stackrel{?}{=} x+u+t$

NO

PB3

$(A\cup B)-C \stackrel{?}{=} (A\cup B\cup C)-C$

$x+u+y \stackrel{\text{YES}}{=} x+u+y$

PB4

$$|A\cup B\cup C| = |A|+|B|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C|$$

$$x+u+y = x+\cancel{u}+\cancel{t}+\cancel{v} - \cancel{u}-\cancel{t} - \cancel{v}-t - \cancel{t}-\cancel{w} + \cancel{t}$$

$$u+y+\cancel{t}+\cancel{w}$$

YES

$$|(A\cup B)\setminus C| \stackrel{?}{=} |A| + |B| - |A\cap B| - |A\cap C| - |B\cap C| + |A\cap B\cap C|$$

Set algebra:

$$|(A\cup B)\setminus C| = |A\cup B\cup C \setminus C| = |A\cup B\cup C| - |C| =$$

subset of first term

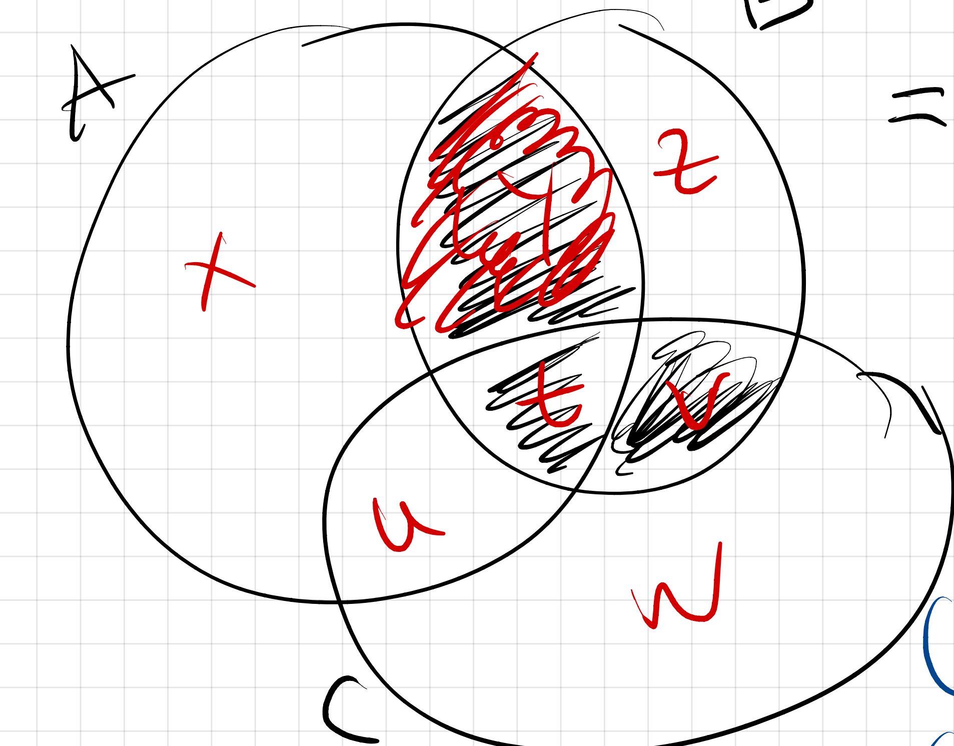
$$|A| + |B| + \cancel{|C|} - |A\cap B| - |A\cap C| - |B\cap C| + |A\cap B\cap C| - \cancel{|C|}$$

Sum Rule $A, B, C \ldots$ disjoint ($\cap = \emptyset$)
(partition)

$$|A\cup B\cup C \ldots| = |A| + |B| + |C| + \ldots$$

- counting objects in set S:
  - partition $S = S_1 \cup S_2 \ldots S_n$ ($S_i \cap S_j = \emptyset$)
  - count each part $|S_i|$
  - sum up $|S_1| + |S_2| + \ldots + |S_n|$

PB3 Sets Rule Algebra



$y \cup t \cup v$

$\boxed{(A\cup C)\cap B} =$

$= (A\cap B\setminus C)\cup(A\cap B\cap C)\cup(B\cap C\setminus A)$

$y \qquad t \qquad t\cup v\setminus A$

$(A\cap B\cap\bar{C})\cup(A\cap B\cap C)\cup(B\cap C\cap\bar{A})$

$(A\cap B\cap\bar{C})\cup\left((B\cap C)\cap(\underbrace{A\cup\bar{A}}_{all})\right)$

$(A\cap B\cap\bar{C})\cup(B\cap C)$

$B\cap\left((A\cap\bar{C})\cup C\right)$

$B\cap\left((A\cup C)\cap(\underbrace{C\cup\bar{C}}_{all})\right)$

$B\cap(A\cup C) = (A\cup C)\cap B$

Counting $|A \dot{\cup} B| = |A| + |B|$ - sum rule

disjoint

$$|A \times B| = |A| * |B|$$ - product rule

---

password length 6, digits + upper-case + lower-case + 12 special

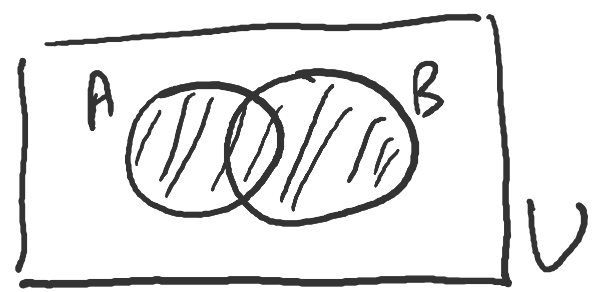
$$|\text{passwords}| = \underbrace{\binom{10+}{\substack{26+\\26+\\12}}}_{=74 \times 74}\ \_\ \_\ \_\ \_\ \underset{\times 74}{\_} = 74^6.$$

Same as before : password length at least 4 and at most 6.

$$|\text{password}| = 74^4 + 74^5 + 74^6$$

Inclusion - Exclusion

2 sets A & B $\quad |A \cup B| = |A| + |B| - |A \cap B|$



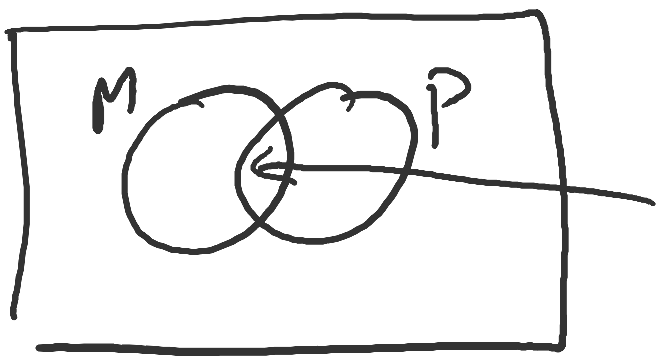
<u>Example</u> 20 courses contain some math

30 " " " programming

15 " " both

How many courses containing math or programming

<u>Answer</u> 35



15

$20+30-15$
$=35.$

Example initials - 2 upper case RS

How many initials with a C ?

Ans C_1 – initials starting with C

C_2 – " ending " "

$$|C_1 \cup C_2| = 26 + 26 - 1 = 51.$$

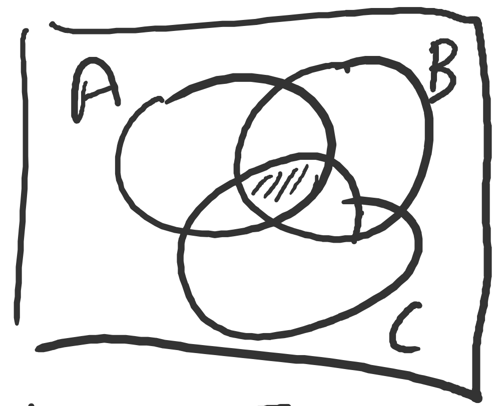
Inclusion - Exclusion (3 sets).

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

Example

Ans 62

video game club - 30
anime society - 20
fencing club - 30



10 VG are fencers, 2 fencers are anime, 7 anime are VG, 1 is in all, How many in Total?

<u>Example</u> initials - 3 upper case.

How many with a C?

C_i - initials with C in i'th place $1 \le i \le 3$

$|C_i| = 26^2$.

$|C_i \cap C_j|$, $1 \le i < j \le 3$ $= 26$

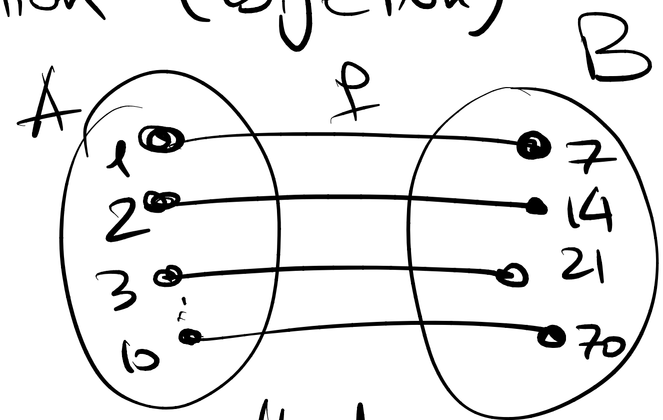
$|C_1 \cap C_2 \cap C_3| = 1$

So $3 * 26^2 - 3 * 26 + 1 = 1951$

Counting technique: indexing / mapping

$f = \text{map}$

map = <u>one-to-one</u> function (bijection)



<u>example</u>

$A = \{1, 2, 3, \dots 10\}$

$B = \{x \in \mathbb{N};\ 2 \leq x \leq 72,\ x = 7k\}$ multiple of 7

$= \{7, 14, 21 \dots 70\}$

$f : A \to B$ bijection (one-to-one)

$f(x) = 7 \cdot x$

x in A $\qquad$ $7x$ in B

called <u>indexing</u> if

$A = \{1, 2, 3, \dots, n\}$

$A = \{1 : n\}$

Th $\exists\ f : A \to B$ one-to-one $\Longrightarrow |A| = |B|$

$\mathbb{Z}_n$ = remainders at integer-division with $n = \{0, 1, 2, \ldots, n-1\}$

$\mathbb{Z}_{10} = \{0, 1, 2, \ldots 9\}$

$\mathbb{Z}_2 = \{0, 1\}$

$\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$.

$\left\{ \text{all pairs } (x \in \mathbb{Z}_2,\ y \in \mathbb{Z}_5) \right\}$

$$\mathbb{Z}_{10} \overset{f}{\longleftrightarrow} \mathbb{Z}_2 \times \mathbb{Z}_5$$

$$f(x) \longleftrightarrow (x \bmod 2,\ x \bmod 5)$$

$$x = 3 \longleftrightarrow (1, 3)$$

$$x = 7 \longleftrightarrow (1, 2)$$

$$x = 4 \longleftrightarrow (0, 4)$$

$X =$ set   example $X = \{a, b, c, d\}$ $a \in X$

$A \subset \mathcal{P}(X)$ $A = \{$subsets of X include "a"$\}$

$\{a\}$ $\{a, b\}$ $\{a, b, c\}$ ...

$A \cup B = \mathcal{P}(X)$

$B \subset \mathcal{P}(X)$ $B = \{$subsets of X do not include "a"$\}$

$\emptyset$, $\{b\}$, $\{c\}$ $\{b, c\}$ $\{b, d\}$ $\{d\}$ ...

$|A| = |B|$?

$A \overset{f}{\underset{\text{map}}{\longleftrightarrow}} B$ one-to-one

$T \in A \overset{f}{\longleftrightarrow}$ S in B one-to-one

$\boxed{T = \text{subset of } X \ a \in T}$ $\boxed{S \text{ subset of } X \ a \notin S}$

$f(T) = T \setminus \{a\}$ ✓ $\Rightarrow |A| = |B| = \dfrac{|\mathcal{P}(X)|}{2}$

* X set $a \in X$ $b \in X$ $a \neq b$. $X = \{a, b, c, d, e\}$

A = set of $\left\{ \begin{array}{l} \text{all subsets of } X \\ \text{contain } a \in T \end{array} \right\}$

$\{a\}$ $\{a,b\}$ $\{a,c\}$ $\{a,b,c\}$ - - -

B = $\left\{ \begin{array}{l} \text{all subsets of } X \\ S \text{ that contain "b"} \end{array} \right\}$

$\{b\}$ $\{b,a\}$ $\{b,c\}$ $\{b,d\}$ $\{b,c,d\}$, $\{b,a,c\}$ - - -

$A \cap B = \cancel{\emptyset}$? $\{a,b\} \in A \cap B$

$\{a,b,c\} \in A \cap B$

$A \xleftrightarrow{f \text{ one to one?}} B$ <u>yes</u> $\Rightarrow |A| = |B|$

<u>exercise</u>

$\mathbb{N} = \{0, 1, 2, 3, 4 \ldots\}$ infinite, countable

$\mathbb{Z} = \{\ldots -3, -2, -1, 0, +1, +2, +3 \ldots\}$

$\mathbb{N} \subset \mathbb{Z}$

$\mathbb{Z} \setminus \mathbb{N} \neq \emptyset$ forexample $-1, -2, \ldots$

$\mathbb{N} \longleftrightarrow \mathbb{Z} \longrightarrow \boxed{|\mathbb{N}| = |\mathbb{Z}|}$ Countable infinite

$\mathbb{N}$	0	1	2	3	4	5	6	7	8	…
$\mathbb{Z}$	0	+1	−1	+2	−2	+3	−3	+4	−4	

$x = 2k \quad f(2k) = -k$

$x = 2k+1 \quad f(2k+1) = k-1$

Technique for counting sets: product rule
(hats × pants × jackets)

$A = \{a, b, c\}$ $B = \{1, 2\}$ $C = \{\alpha, \beta, \gamma\}$

want triplet $(\underline{x \in A}, \underline{y \in B}, \underline{z \in C})$

- any combination works

$(a, 1, \beta) \neq (1, \beta, a)$

triplets : $|A| \cdot |B| \cdot |C|$

$3 \cdot 2 \cdot 3$

is $(1, a, \alpha)$ triplet?
NO

triplet = sequence

$$A \times B \times C = \left\{ \begin{array}{lll} (a, 1, \alpha), & (a, 1, \beta), & (a, 1, \gamma) \\ (a, 2, \alpha), & (a, 2, \beta), & (a, 2, \gamma) \\ \vdots & & \\ (c, 2, \alpha), & (c, 2, \beta), & (c, 2, \gamma) \end{array} \right\}$$

$n = 25$ students
$k = 3$ classrooms $\Rightarrow \exists$ one classroom with $\geq \left\lceil \frac{25}{3} \right\rceil = 9$

---

10 people $x_1, x_2 \ldots x_{10}$ salaries avg $80{,}000$ / sum of salaries is $800{,}000$

$$\frac{x_1 + x_2 + \ldots + x_{10}}{10} = 80{,}000$$

$\Rightarrow$ at least one $\underline{x_i \geq 80{,}000}$
$(\exists i)$

Pigeon Hole principle

non-math version

- n items placed in $n-1$ boxes (spots) $\Rightarrow$ $\exists$ at least one box with 2 items or more
- n items placed on k boxes $\Rightarrow$ $\exists$ at least one box with $\lceil \frac{n}{k} \rceil$ items

math-version

$x_1, x_2, x_3 \cdots x_n \in \mathbb{R}$ $\quad \mu = \dfrac{x_1 + x_2 \cdots + x_n}{n} = E[x]$

- at least one of them $x_i \geq \mu$ $\rightarrow$ prove by contradiction
- at least one of them $x_j \leq \mu$

assume $x_i < \mu \ \forall i$

$\sum x_i < n \cdot \mu$

$\sum x_i < \sum x_i$! contra

$\Rightarrow \exists i \ x_i \geq \mu$

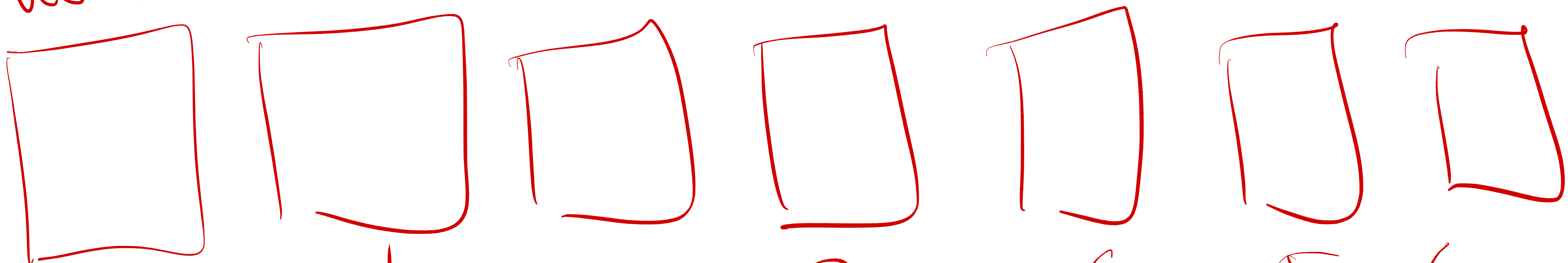
$\boxed{\text{PHP 1}}$

100 integers $\Longrightarrow$ $(\exists)$ select 15 of them

any diff of 2 = multiple of 7

$a-b = 7k \iff 7 \mid a-b \iff \boxed{a \equiv b \mod 7}$

mod 7 $\Longrightarrow$ 7 boxes



rem=0 1 2 3 4 5 6

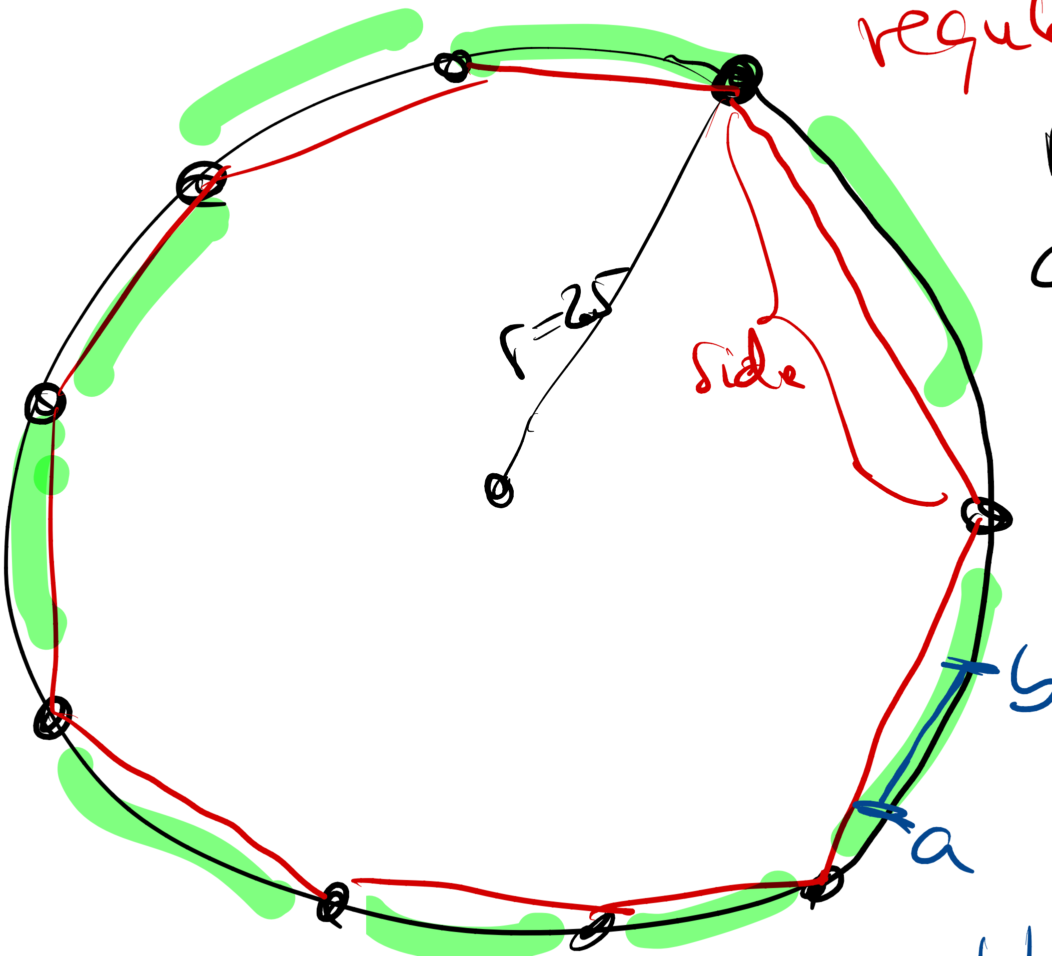
100 integers, 7 boxes $\Big\}$ $\xRightarrow{\text{PHP}}$ $\exists$ box with $\geq$ 1 box has at least $\left\lceil \frac{100}{7} \right\rceil = 15$

same rem at $\Longrightarrow$ $|a-b| =$ multiple of 7

10 point on a circle of diameter $= 2r = 5$

PHP2 $\Rightarrow \exists$ 2 of them at $dist(a,b) < 2$.

a, b



regular 9-gon (equal sides)

$r = 2.5 \Rightarrow$ side $\simeq 1.71$?

9-gon splits circle into 9 regions (green)

10 points on circle

$\Downarrow$ PHP

2 of them same region

geom: $|ab| <$ side 9-gon $\simeq 1.71$

PB6 50 cats + 50 dogs in 9 rooms. What is min

A guaranteed to be in a room?

$$\left\lceil \frac{100}{9} \right\rceil = 12$$

---

per room:
no more than 6 cats ; at least 2 dogs

B what is the maximum # animals in a room

R = room with max animals = 6 cats + max dogs

→ all other 8 rooms (except R) minimize # dogs

$8 \times 2 = 16$ dogs $\Longrightarrow$ R has $50 - 16 = 34$ dogs

$$\begin{array}{r} + 6 \text{ cats} \\ \hline 40. \end{array}$$

General pigeonhole principle. ceiling

p pigeons h holes $\Rightarrow \left\lceil \frac{p}{h} \right\rceil$ pigeons in some hole.

Example 250 students

26 first letter of last name

$10 = \left\lceil \frac{250}{26} \right\rceil$ students with same first letter of last name.

Example 250 students 2letter initials
No guarantee two have same initials.

Example : cabinet with 10 black socks
and 20 white socks. How many
to guarantee matching pair?

Ans : 3

<u>Example</u> In any group of n people there will be two with the exact same number of friends.

(Alt: any graph has two nodes with same degree).

<u>Ans</u> n - pigeons, n - holes, number of friends can be $0 .. n-2$ or $1 ... n-1$ since cannot have extrovert with $n-1$ friends and hermit

with 0 friends simultaneously

So $n \leq n-1$. Thus two people

with exact same number of friends