virus

LECTURE 13

test for virus not perfect "false positives"

$$P(V) = \overset{V,\,T}{P}(\text{virus}) = P(\text{infected with virus}) = \frac{1}{10^4}$$

$$P(\text{pos test} \mid \text{virus infection}) = P(T \mid V) = \frac{99}{100}$$

$$P(\text{pos test} \mid \text{no virus}) = P(T \mid \bar{V}) = 1/1000$$

false positive

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if test = pos, chance of being infected?

$$P(V \mid T) = ? = \frac{P(T \mid V) \cdot P(V)}{P(T)} = \frac{99/100 \cdot 1/10{,}000}{P(T)}$$

$$\frac{99/10^6}{P(T \mid V) \cdot P(V) + P(T \mid \bar{V}) \cdot P(\bar{V})} = \frac{99/10^6}{99/10^6 + 1/1000 \cdot (1 - 1/10^4)}$$

cabs 60% white 40% yellow

accident; witness says "cab was yellow"

witness tells truth 80%; lies 20% indep of cab color

Q: What prob(cab = Yellow) = ?

R.V cab color: Y vs $\bar{Y}$

Second RV: A witness says cab = yellow GOOD

$\bar{A}$: says cab = white"

match

match

Second RV - NO GOOD

witness tells Truth W

lies $\bar{W}$

tells yellow

no best

cab is yellow

$$\boxed{P(Y \mid A)} = \frac{P(A \mid Y) \cdot P(Y)}{P(A)} = \frac{0.8 \times 0.4}{P(A \mid Y) \cdot P(Y) + P(A \mid \bar{Y}) \cdot P(\bar{Y})}$$

Lies

$$= \frac{80\% \cdot 40\%}{P(A, Y) + P(A, \bar{Y})}$$

marginal

$$= \frac{0.8 \times 0.4}{0.8 \times 0.4 + 0.2 \times 0.6}$$ → cab is white

$$\boxed{P(A|Y) = \frac{P(A,Y)}{P(Y)}}$$

out of "Y possib" (restricted)

P(A,Y) = the ones with A

$$\text{cond} = \frac{\text{joint}}{\text{marginal}}$$

Alien Planet 3 Parties Red Blue Purple

4 states E W N S

Exercise

each region elects 2 senators at random (from that region)

	Red	Blue	Purple
E	12	20	8
W	16	14	18
N	9	18	14
S	22	10	12

Task: if we meet a purple elected senator, what is chance its from S region?

Hints $X =$ Region $\{N, S, W, E\}$

$Y =$ Party $\{R, B, P\}$

$G =$ elected (senator) $\{T, F\}$

$P(X=S \mid Y=P, G=T)$

$$Pr(S \mid P, G) = \frac{Prob(P, G \mid S) \cdot P(S)}{P(P, G)}$$

Expectation $E[X] = \sum_{x \in \Omega(X)} x \cdot Pr(X=x) = \sum x \cdot P(x)$

R.V. X ~~requirement~~ values(X) = numbers

intuition

$E[X]$ = weighted average of values (x) weighted

By probabilities of seeing that value

$P(x)$

$P(X=x)$

uniform: $p(x)$ = constant in x $= \frac{1}{|\Omega(X)|} = \frac{1}{n}$

n = # of values

$$E[X] = \sum_x x \cdot P(x) = \frac{1}{n} \sum_x x = $$

$$= \frac{x_1 + x_2 + \dots + x_n}{n}$$

arithmetic average

E[ ] rules   c = constant

$$E[c \cdot X] = c\,E[X]$$ exercise

$$E[c+X] = c + E[X]$$

X, Y 2 R.V with values = numbers (even DEPENDENT)

Linearity of Expectation

Th $E[X+Y] = E[X] + E[Y]$ $X+Y = $ R.V

proof $$E[X+Y] = \sum_{w \in \Omega(X+Y)} w \cdot P(X+Y=w) =$$

$$\sum_{\substack{w=x+y \\ \text{values}}} (x+y)\, P(X+Y=x+y) =$$

$$= \sum_{x \in \Omega(X)} \sum_{y \in \Omega(Y)} (x+y)\, P(X=x, Y=y).$$

$$= \sum_x \sum_y x \cdot P(X,Y) + \sum_x \sum_y y \cdot P(X,Y)$$

$$= \sum_x x \sum_y P(X) \cdot P(Y|X) + \sum_y y \sum_x P(Y) \cdot P(X|Y)$$

$$= \sum_x \left[x\, P(x) \cdot \underbrace{\left(\sum_y P(Y|X=x) \right)}_{1} \right] + \sum_y \left[y \cdot P(y) \underbrace{\left(\sum_x P(X|Y=y) \right)}_{1} \right]$$

$$= \sum_x x \cdot P(x) + \sum_y y \cdot P(y)$$

$$= E[X] + E[Y]$$

- $E[X+Y+Z] = E[X+Y] + E[Z] = E[X] + E[Y] + E[Z]$
- $X_1, X_2, \ldots X_n$ indicator R.V $\begin{cases} 1 & \text{"present" / T,} \\ 0 & \text{"not"} \end{cases}$

$X = X_1 + X_2 + \cdots + X_n$ = count of items present

$E[X]$ = expected # items present

$= E[X_1 + X_2 + \cdots + X_n] = \sum_i E[X_i]$

"expected number of ..."

Birthday paradox $n=244$ people bdays are at random

probab different bdays?

"no collisions"

preview

$1+x \simeq e^{x} \mid x$ close to 0

probab no collisions product rule

$$\underset{1^{st}}{1} \times \underset{2^{nd}}{\frac{364}{365}} \times \underset{3^{rd}}{\frac{363}{365}} \times \cdots \times \underset{n}{\frac{366-n}{366}}$$

$$= \left(1-\frac{0}{365}\right)\left(1-\frac{1}{365}\right)\cdots\left(1-\frac{n-1}{365}\right)$$

$$\simeq e^{-0} \cdot e^{-1/365} \cdot e \cdots e^{-\frac{n-1}{365}}$$

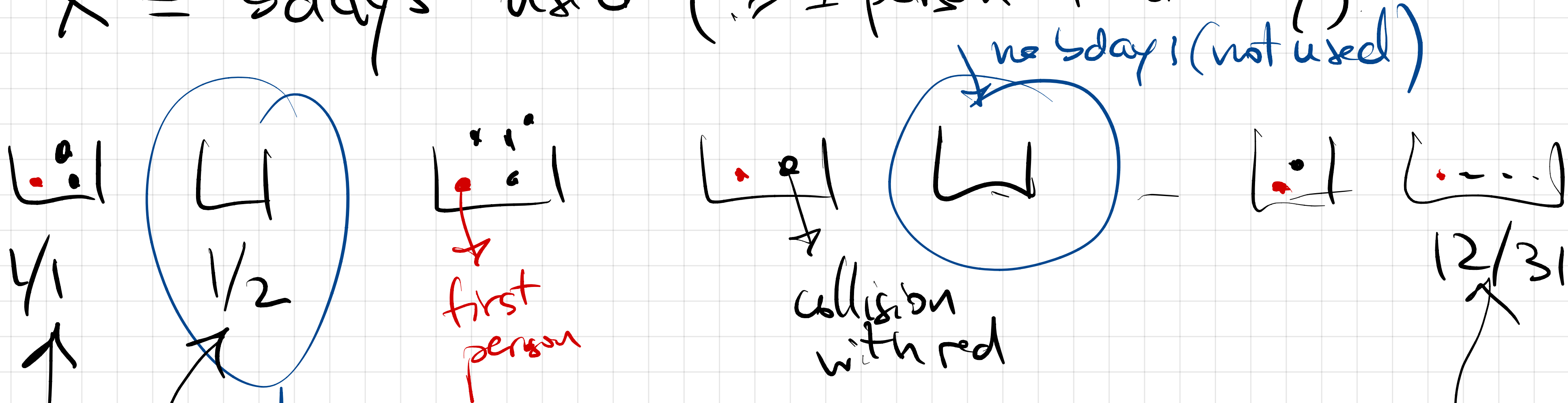
$$= e^{-(0+1+2\cdots+(n-1))/365}$$

$$= e^{-\frac{n(n-1)}{2}/365} \simeq 0.5 \quad (n=244)$$

Expected # of collisions? # 2^{nd} bdays in a day with already a first-bday.

ex. bday with 3 people → 2 collis
10 people → 9 collis.

X = bdays used (≥ 1 person that day)



RV X_1 X_2 X_3 — — X_{364} X_{365}

1/0 used 1/0 1/0 1/0

$$X_i = R_iV \begin{cases} 1 & \text{if bday } i \text{ is used (at least 1 person)} \\ 0 & \text{if nobody has that bday.} \end{cases}$$

$$X = X_1 + X_2 + \dots + X_n$$

$$E[X] = E[X_1 + X_2 \dots + X_n] = E[X_1] + E[X_2] \dots + E[X_n]$$

$E[X_i] = E[X_j]$ (all have the same expectation)

$$E[X_1] = \text{prob}(X_1 = 1) \cdot 1 + \text{prob}(X_1 = 0) \cdot 0 =$$

$$= \text{prob}(X_1 = 1)$$

nobody has X_1 bday

$$P(X_1 = 1) = 1 - \text{prob}(X_1 = 0) =$$

$$1 - \left(1 - \frac{1}{365}\right)^{n}$$

n → n people

probab of 1 person not have that bday X_1

X, Y R.V

$$E[X \cdot Y] = \sum_{w = x \cdot y} x \cdot y \ \text{Prob}(X \cdot Y = w)$$

$$= E[X] \cdot E[Y] \quad \text{if } X, Y \text{ independent}$$

$$\neq E[X] \cdot E[Y] \quad \text{in general } X, Y \text{ might be dependent.}$$