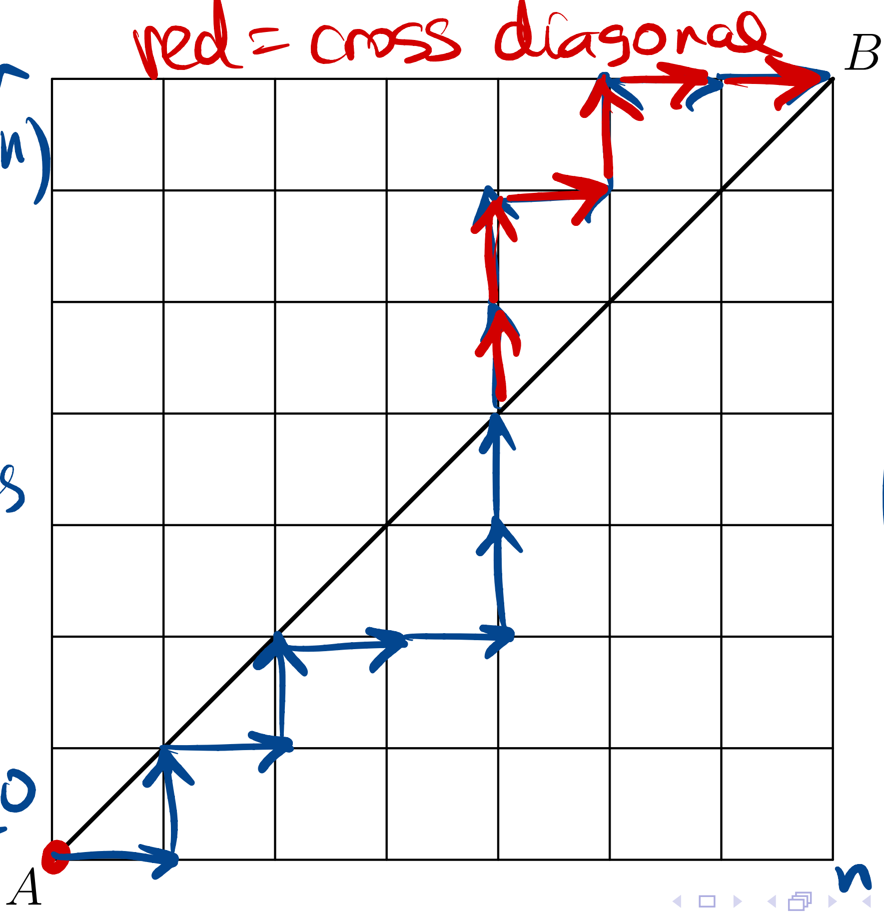
How many shortest paths from A to B do *not* pass above the diagonal?

any path
A(0,0) → B(n,n)

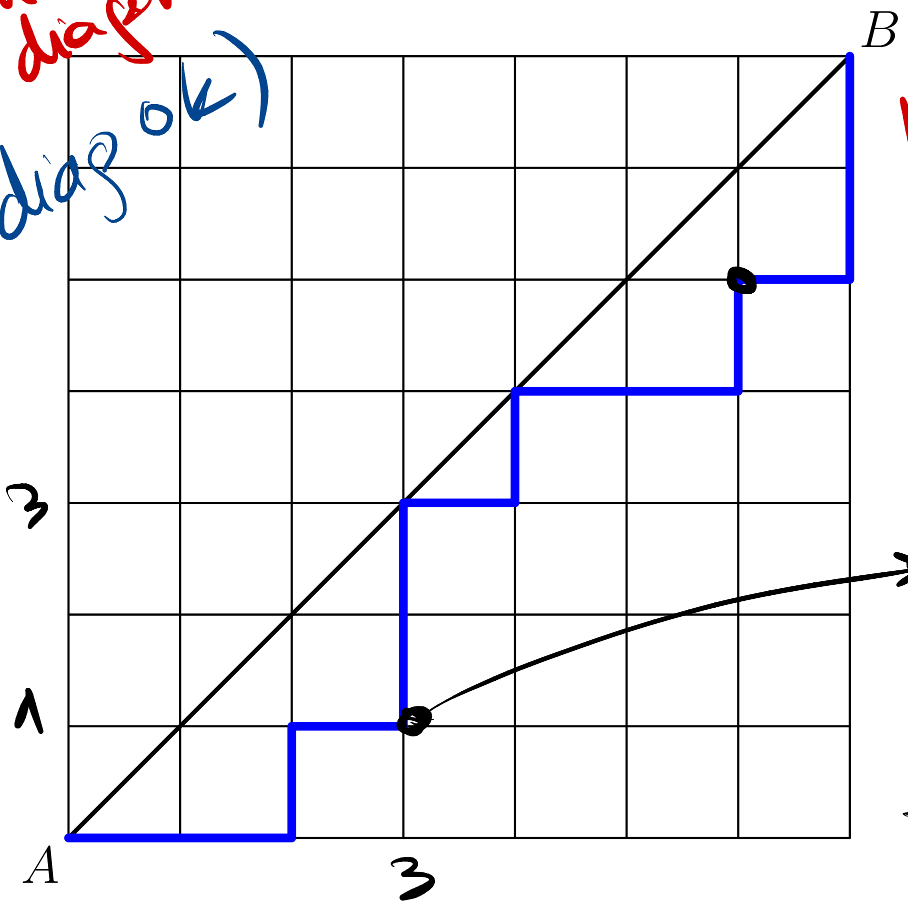


walk moves
"→" or "↑"

n times "→"
n times "↑"

$\binom{2n}{n}$ total moves

restriction: don't cross diagonal (touch diag ok)



Catalan number C_n

anywhere # moves → ≥ # move ↑ (so far)

path →, →, ↑, →, ↑, ↑, →, ↑, →, →, ↑, →, ↑, ↑

n=3 sets of ( ) nested: ((())) ; (()()) ; (())() ; ()(()) ; ()()()

#"(" ≥ #")" at any point in sequence

Stacks push → at the top (LIFO)
pop → from the top

valid sequ. of stack ops: Push, Pop, push, push, pop ...
( ) (( ) ...

same property

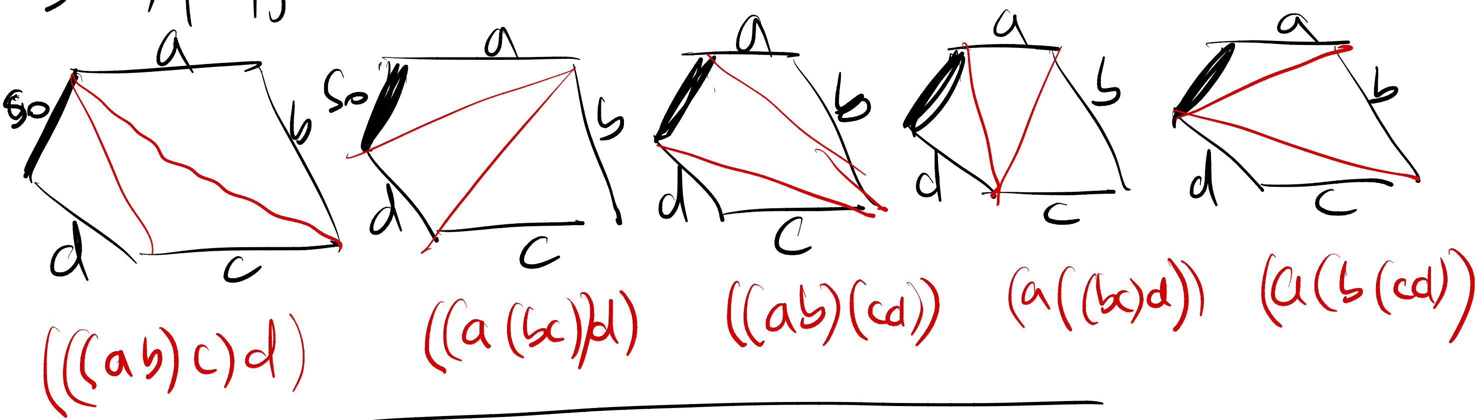
histories = # valid paths under diagonal

multiply a · b · c · d — decide the order → n+1 items

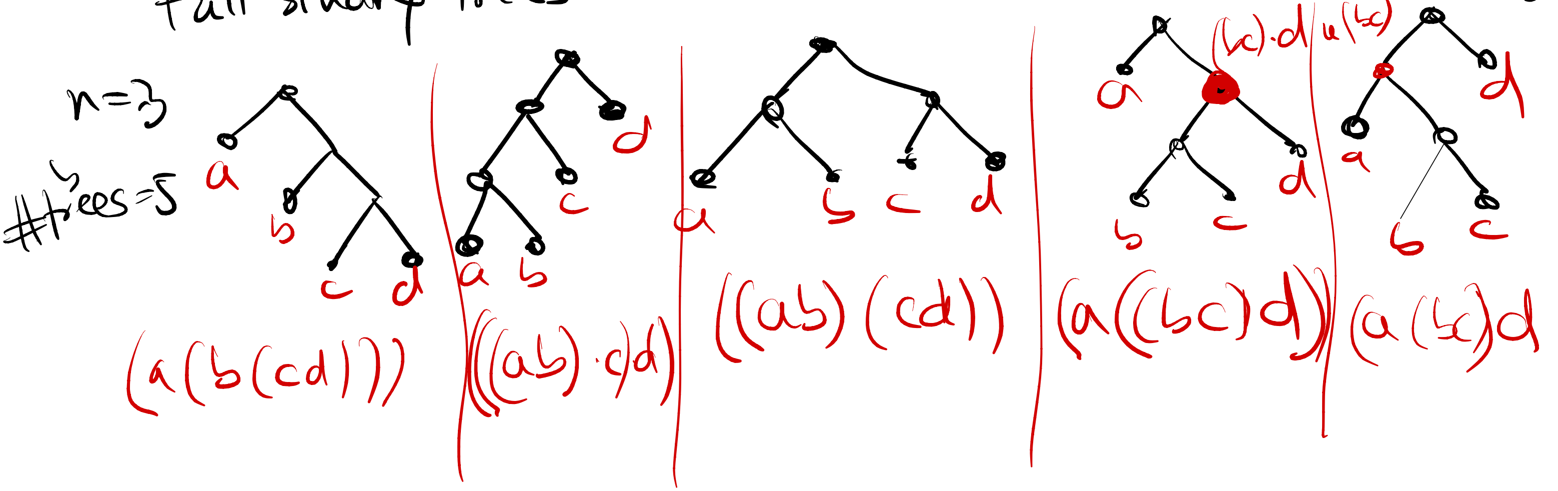
n=3

((a·b)·(c·d))	(((a·b)·c)·d)	(a·(b·c))·d	a·((b·c)·d)	(a·(b·(c·d)))
Keep "·" and ")" ·)··))	·)·)·)	··))·)	··)·))	···)))
replace "·" with "(" ()(())	()()()	(())()	(()())	((()))

n=3 ⟹ polygon n+2=5 sides



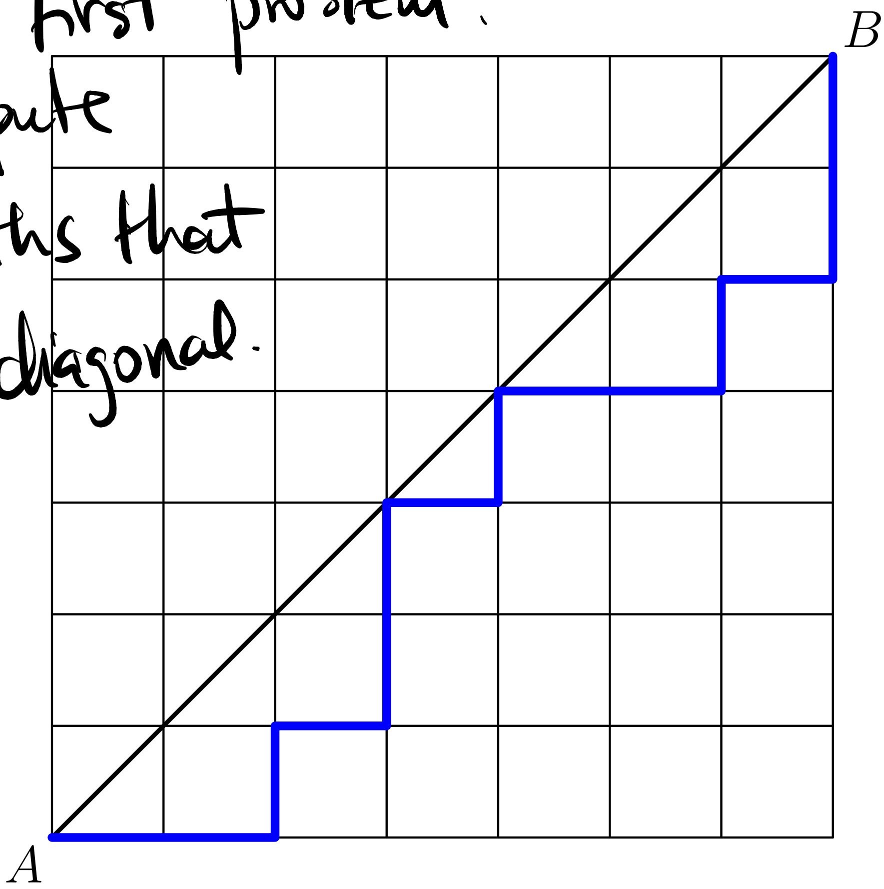
Full binary trees — every node has 2 children (or leaf)



Back to first problem:

Lets compute

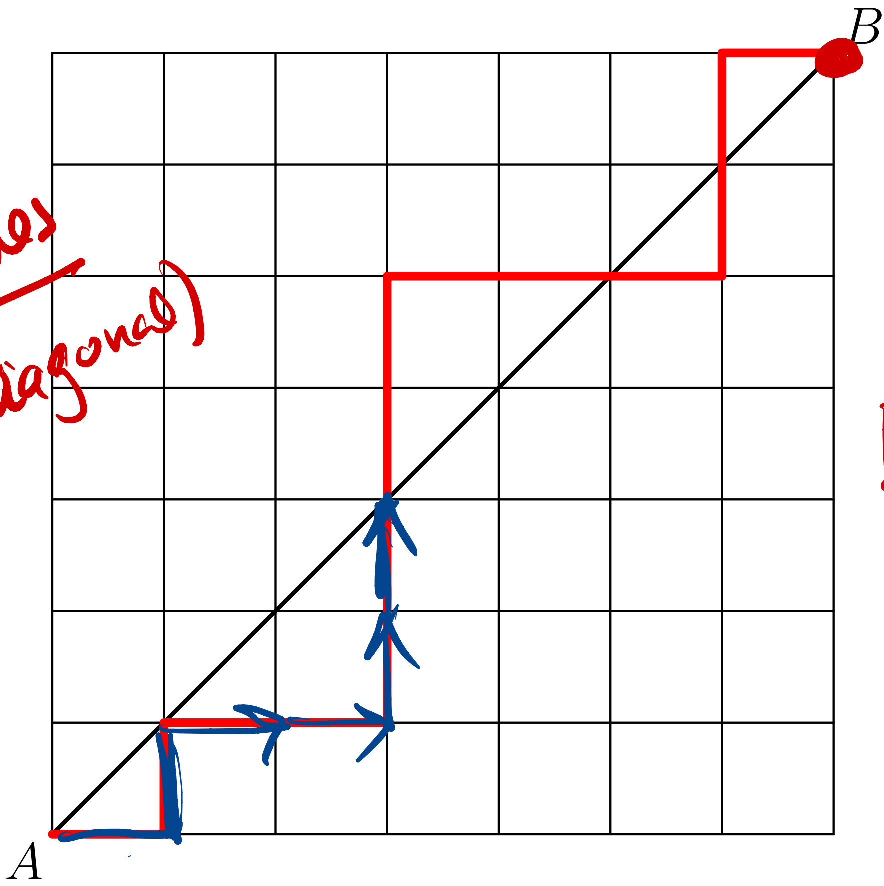
C_n = # paths that dont cross diagonal.



Blue path is good

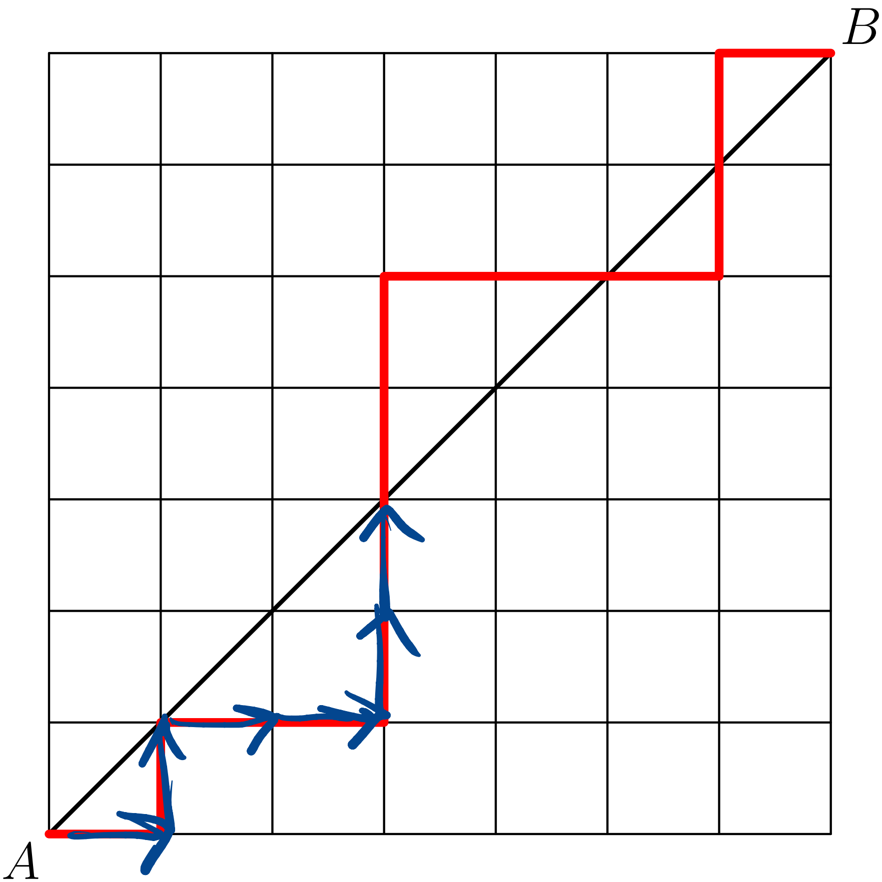
paths = all paths − invalid ones

(paths cross diagonal)



all paths $\binom{2n}{n}$

bad ones?



Red path is bad

$$\begin{aligned}\text{Number of good paths} &= \text{Total Number of paths} - \text{Number of bad paths}\\ &= \frac{(2n)!}{n!n!} - \text{Number of bad paths}\end{aligned}$$

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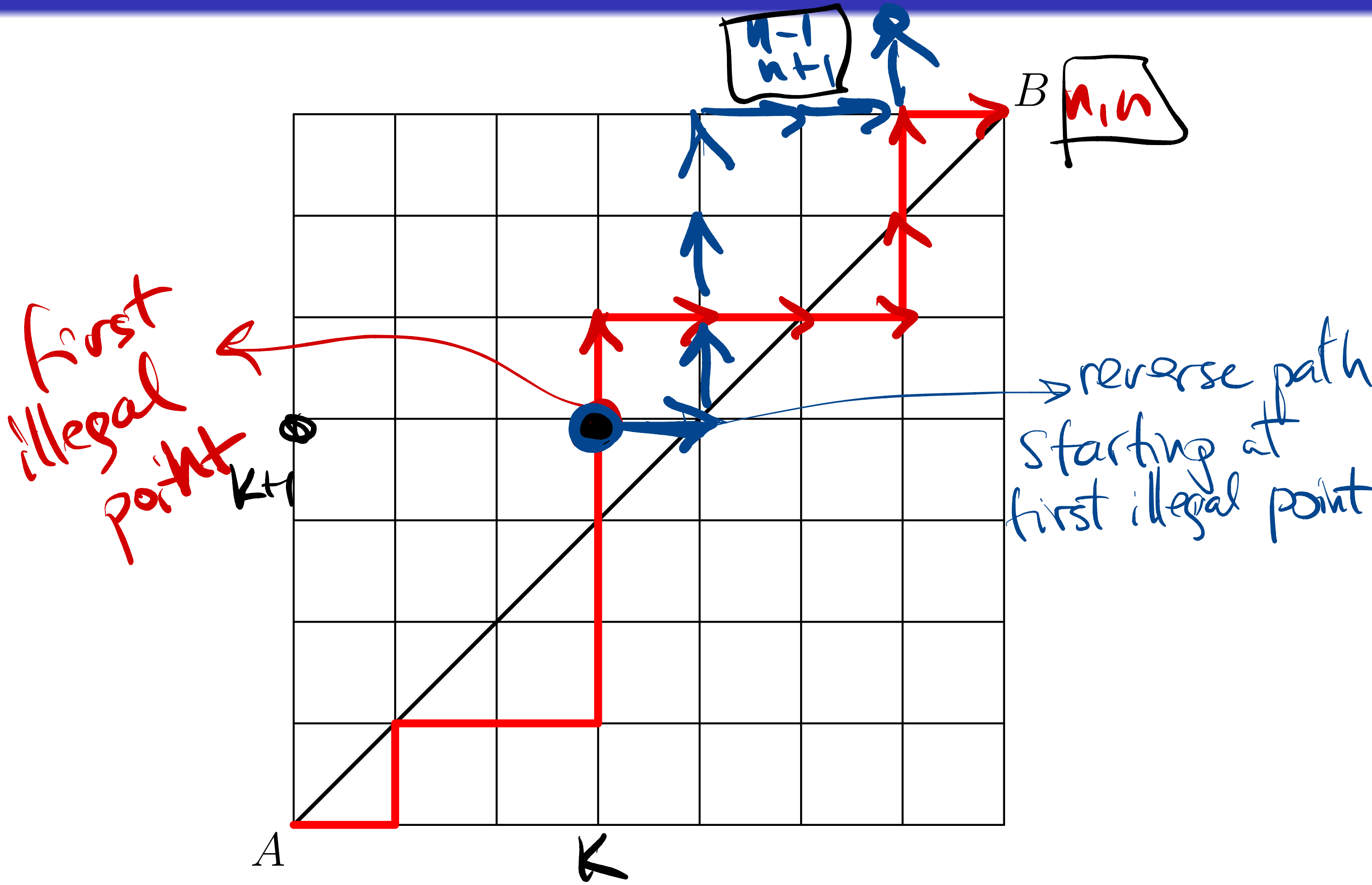
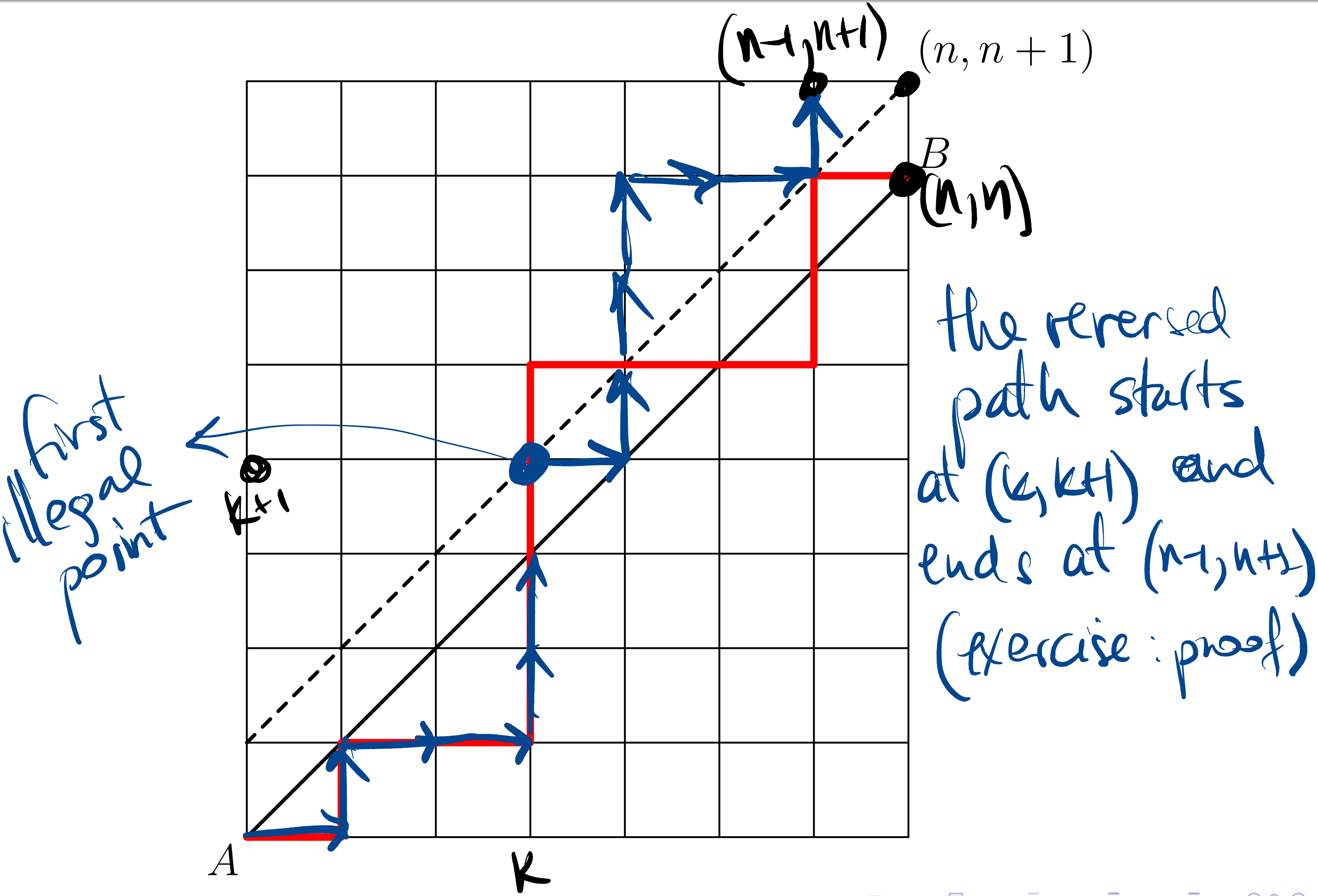
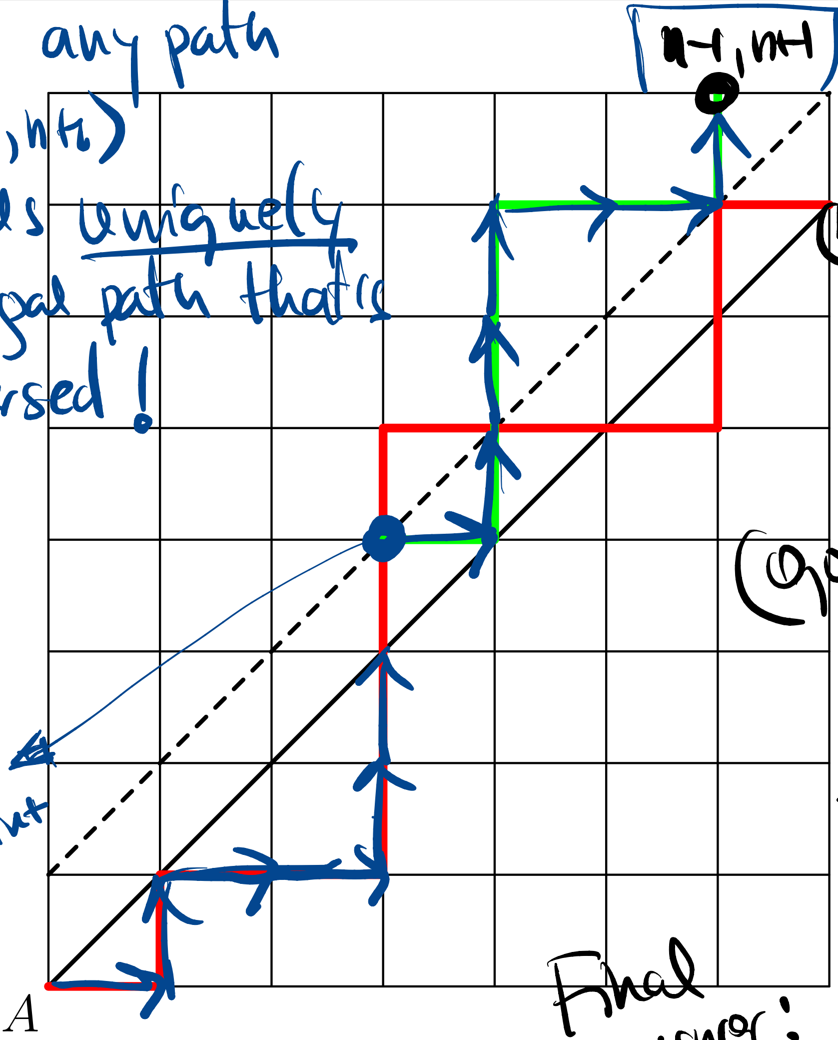
So it is sufficient to count the number of bad paths.

How to count the number of bad paths?




(n-1, n+1)
(n, n+1)
B
(n, n)
first illegal point
k+1
A
k
the reversed path starts at (k, k+1) and ends at (n-1, n+1)
(exercise: proof)

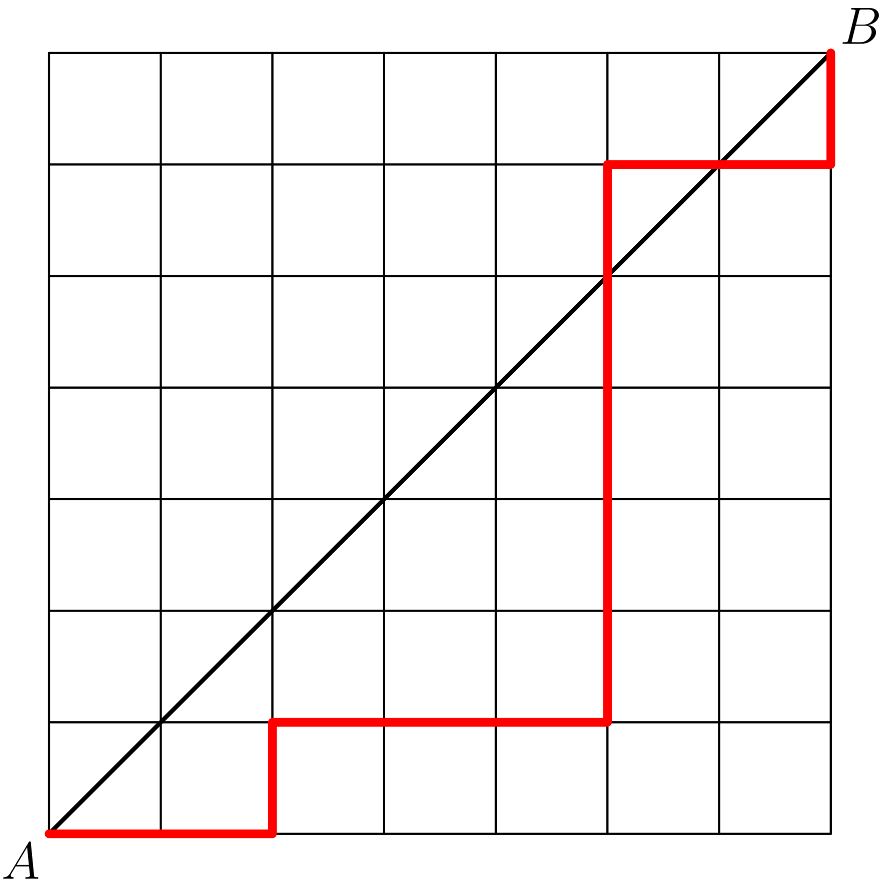
actually any path $(0,0) \to (n-1, n+1)$ corresponds uniquely to an illegal path that is seen reversed!

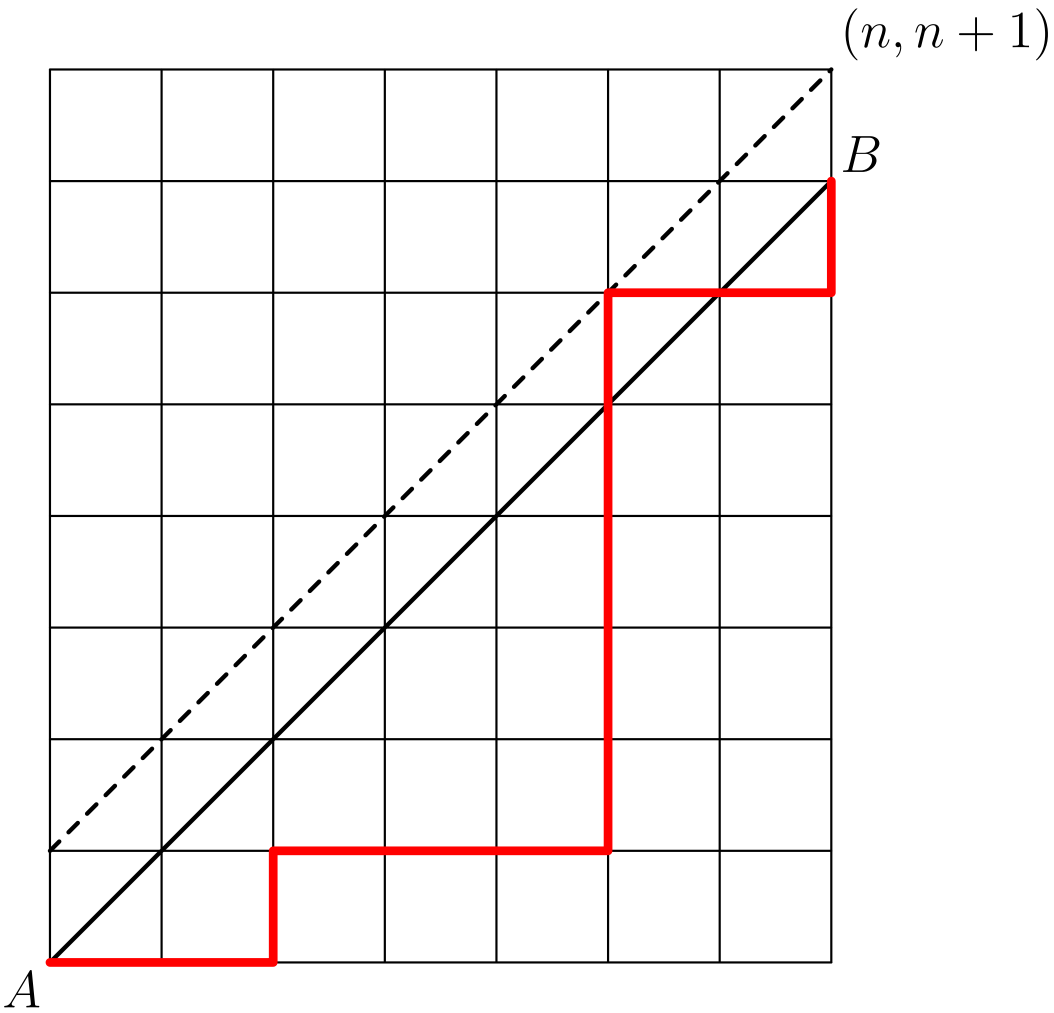


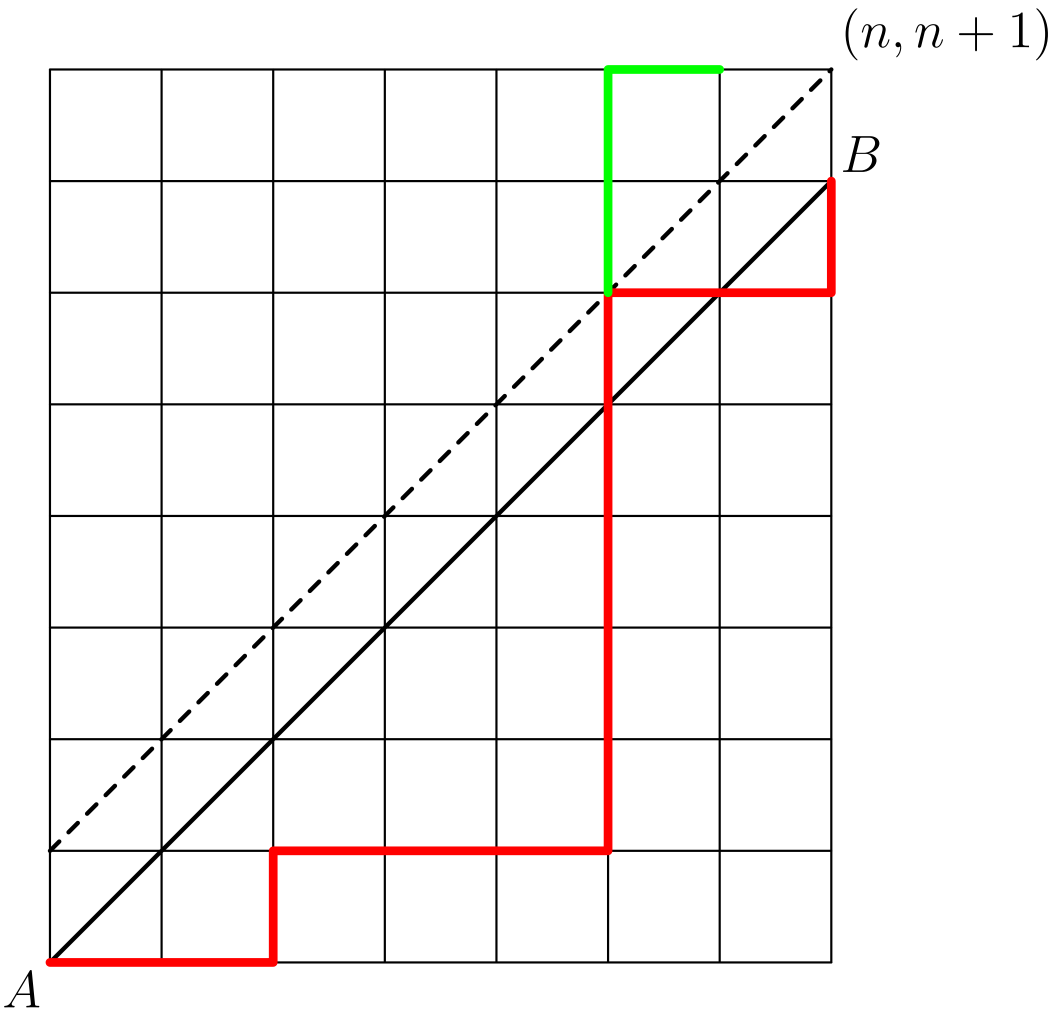
so (n,n) bad paths $\equiv$ all paths $(0,0)$ to $(n-1, n+1)$

$$= \binom{2n}{n-1} = \binom{2n}{n+1}$$

Final answer: $C_n = \binom{2n}{n} - \binom{2n}{n-1}$


B
A


$(n, n+1)$
B
A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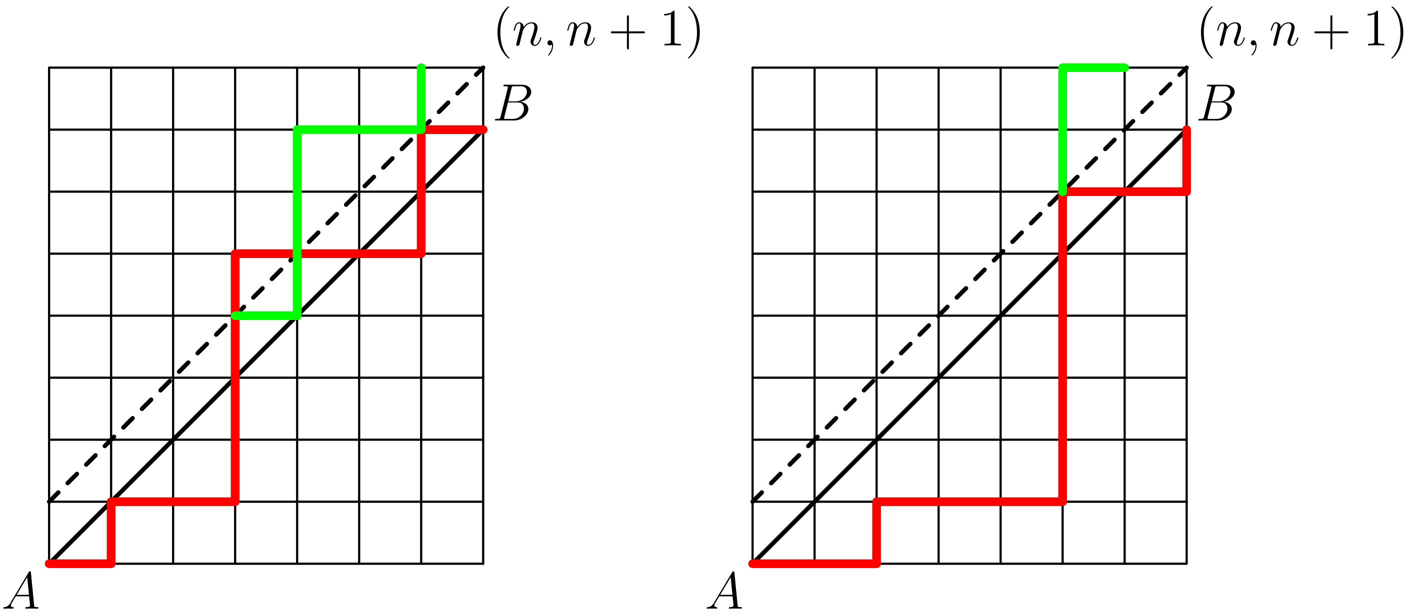

In fact, reflection turns every bad path into a path reaching $(n-1, n+1)$.

Moreover, every path reaching $(n-1, n+1)$ is obtained from a bad path.

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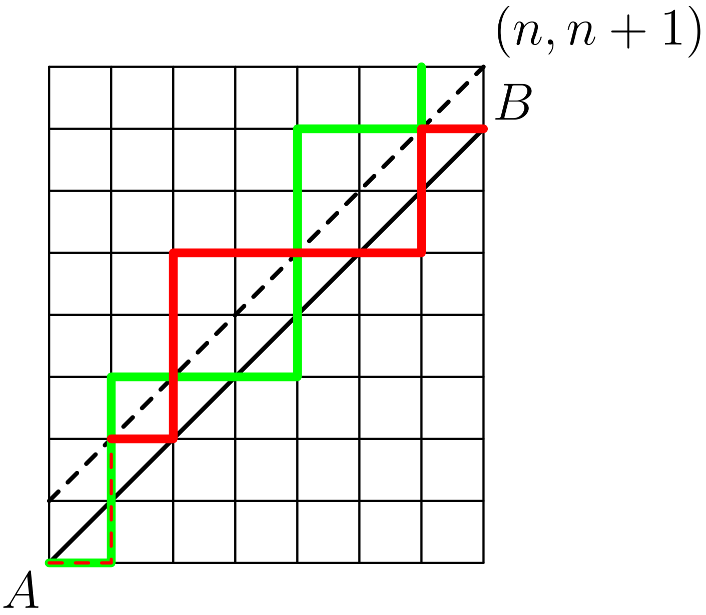

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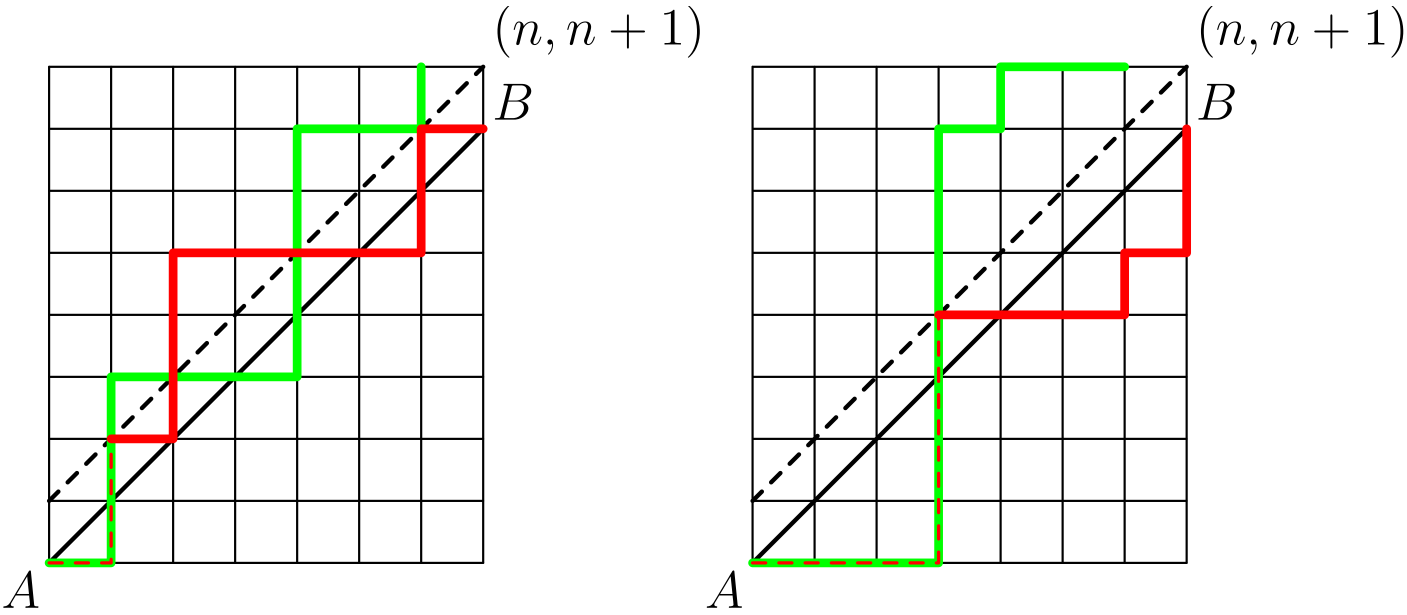

Conclusion: There is a one-to-one correspondence between the set of bad paths and the set of paths reaching $(n-1, n+1)$.

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Finally,

Number of good paths

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Number of good paths = Total Number of paths − Number of bad paths

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Finally,

$$\begin{aligned} \text{Number of good paths} &= \text{Total Number of paths} - \text{Number of bad paths} \\ &= \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n-1)!(n+1)!} \end{aligned}$$

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The number $C_n = \frac{1}{n+1}\binom{2n}{n}$ is called the n^{th} Catalan number and has a lot of applications.