Recitation 5: Binary, Bases, Two's Complement, Square Game

Problem 1 Base conversions, operations

i. Give the integer (in standard base-10 notation) which is represented by each of the following 8-bit two's complement numbers: 01001110, 11101101, 11111111.

ii. Give the 8-bit two's complement representations of the following integers: 52, -101, -77.

iii. Compute the following using 8-bit two's complement representations, as shown in class and described in the text: $52 - 101$, $-77 - 101$, and $52 - 77$. In each case, indicate whether the calculation results in an overflow. If it does not result in an overflow, then verify that your answer is correct by converting the result back to standard base-10 notation.

iv. What range of numbers can be represented in (a) 11-bit two's complement? What are the *minimum* number of bits necessary to represent (b) 2048 and (c) -195 in two's complement?

v. Convert the hexadecimal number $BEEF_{16}$ to sixteen digits of binary. Show your work.

Problem 2 Base conversions and IP address formats

IP addresses are 32-bit binary numbers, most commonly expressed in the dotted-decimal format in which the 32 bits are grouped into four bytes of 8 bits each, separated by the dot symbol, and each byte is written out in decimal form.
Thus this IP address in binary 10000001000010100111010011001000 corresponds to the dotted decimal notation: 129.10.116.200
The first eight bits 10000001 form the number 129 in decimal notation, the next eight bits 00001010 form the number 10, the next eight bits 01110100 form the number 116, and finally the last eight bits 11001000 form the number 200.

IP addresses can also be represented in other formats, including hexadecimal and decimal. In fact, most browsers will accept IP addresses in these representations as well. These formats are used sometimes for purposes of obfuscation and identity-hiding.

Notation	**Value**	**Conversion from dot-decimal**
Dotted decimal	192.0.2.235	N/A
Hexadecimal	0xC00002EB	The 32-bit number is expressed as the concatenation of the octets from the dotted hexadecimal. (We are omitting the '0x" here.)
Decimal	3221226219	The 32-bit number is expressed in decimal.
Octal	030000001353	The 32-bit number is expressed in octal.

This table is from Wikipedia on IPv4

iv. Convert the IP address 172.16.254.1 from dotted decimal format to the hexadecimal and the decimal formats.

v. Convert the IP address ABC58E3D from hexadecimal format to the dotted decimal and decimal formats.

vi. Convert the IP address 2153042310 from decimal format to the hexadecimal and dotted decimal formats.

Problem 3 Multiplication

Perform the following multiplications in binary. For each problem part, you must (1) convert each decimal number to binary, (2) perform the multiplication in binary, and (3) convert the binary result back to decimal. You must show your work.

Note: For consistency, place the binary representation of the left multiplicand in the top row of your multiplication and place the binary representation of the right multiplicand on the bottom row of your multiplication. Thus, "4×7" would be

$$\begin{array}{cccc} & 1 & 0 & 0 \\ \times & 1 & 1 & 1 \\ \hline \end{array}$$

i. 57×15

ii. 19×71

iii. Just as in addition, multiplication of two binary numbers may result in overflow if the size of the representation is not large enough.
What is the maximum number of bits that may be needed to represent the multiplication of a 17-bit number with a 9-bit number? Justify your answer.

Problem 4 Square Game Play these boards with other students in the classroom. At your turn choose a row and eat at least one tile from that row (can eat entire row). The winner is the player who eats the last tile.

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Problem 5 Bitwise Logical Operations

Decide whether those statements are true (briefly explain) or false (counterexample). Variables a, b, c are boolean (bits) and variables A, B, C are boolean sequences (or sequences of bits), for example $A = a_1a_2...a_k = a_1, a_2, ..., a_k$ is a sequence of k bits (k is fixed for example $k = 32$).
A, B, C allow boolean operations "bitwise": $A = a_1a_2...a_k$ and $B = b_1b_2...b_k$ then
$A \wedge B = a_1 \wedge b_1, a_2 \wedge b_2, ..., a_k \wedge b_k$
Similarly $A \vee B = a_1 \vee b_1, a_2 \vee b_2, ..., a_k \vee b_k$, etc

1. $\forall a, b, c \in \{T, F\} : a \wedge b = a \wedge c \Rightarrow b = c$

2. $\forall A, B, C \in \{T, F\}^k : A \wedge B = A \wedge C \Rightarrow B = C$

3. $\exists A, \forall B, C \in \{T, F\}^k : A \wedge B = A \wedge C \Rightarrow B = C$

4. $\forall a, b, c \in \{T, F\} : a \vee b = a \vee c \Rightarrow b = c$

5. $\forall A, B, C \in \{T, F\}^k : A \vee B = A \vee C \Rightarrow B = C$

6. $\exists A, \forall B, C \in \{T, F\}^k : A \vee B = A \vee C \Rightarrow B = C$

7. $\forall a, b, c \in \{T, F\} : a \text{ XOR } b = a \text{ XOR } c \Rightarrow b = c$

8. ★ $\forall A, B, C \in \{T, F\}^k : A \text{ XOR } B = A \text{ XOR } C \Rightarrow B = C$

Problem 6 Poisoned bottle Prince Humperdinck has been presented with 127 bottles of wine, but one of them has been poisoned(!) with iocane powder. 7 members of the Brute Squad volunteer to sacrifice themselves to find the poisoned bottle. The poison has absolutely no effect on the drinker until exactly 24 hours later, when you instantly drop dead.

In other words, the Prince has 24 hours hours to discover a poison that takes 24 hours to take effect. He doesn't mind if Brute Squad members die. How can he use the 7 to find the poisoned bottle?

Part B ★★ (optional, no credit) Same problem, but up to 3 bottles might be poisoned - that is, the number of poisoned bottles can be 0, 1, 2, or 3. How many volunteers we need, and how do we assign them to drink from bottles, in order to identify exactly which bottles are poisoned?

Part C ★★★ (optional, no credit) Same problem as in part B: up to 3 bottles might be poisoned. But now the volunteers die quickly, so the Prince can design an adaptive strategy:
Use first voluneter to drink from a certain set of bottles and wait 5 min to see if he dies;
Depending on result dead/alive for that, same/another volunteer is assigned to a certain set of bottles with result dead/alive in 5 min
and so on.
What is the strategy now, and how many voluneteers are needed to be sure we find the poisoned bottles?