Homework 5: Series, Recurrences, Induction

Problem 1 Sequences, Series, Summations

i. Give a general formula for the sequence $a_0 = -2, a_1 = 6, a_2 = -18, a_3 = 54, a_4 = -162, a_5 = 486, \ldots$

ii. Compute a close form as function of n

$$S(n) = \sum_{k=4}^{n}(3k - 1)$$

iii. Compute a close form as function of n

$$S(n) = \sum_{k=3}^{n} 4 \cdot 5^{k+2}$$

iv. ★ Compute a close form as function of n. Dont use induction; instead subtract $2S(n) - S(n)$ term by term to get a familiar series

$$S(n) = \sum_{k=1}^{n} k \cdot 2^k$$

v. Prove the formula obtained from previous exercise by induction over n

Problem 2 Functions Order of growth

Rank the following functions in terms of asymptotic growth. In other words, find an arrangement of the functions $f_1, f_2, f_i, f_{i+1} \ldots$ such that $\forall i, \exists n_0, \forall n \geq n_0 : f_i(n) \leq f_{i+1}(n)$.

$\sqrt{n}\ln n \quad \ln\log n^2 \quad 2^{\log^2 n} \quad n^{1.5} \quad n! \quad n^{0.001} \quad n\log n \quad 2^{2\log n} \quad (\log n)! \quad n^3 \quad n^{logn}$

If youre not sure, you can easily plot the functions up to a reasonable large n to get a visual clue. Once you decide the ranking you have to prove each inequality between consecutive functions. Some of these inequalities might be esier by induction, others by change of variable, but you can use any method. For logs you can choose a convenient base. You can take for granted that $log(n) < n^c < d^n < n!$ for large-enough n, and any constants $c > 0$, $d > 1$.

Problem 3 Fibonacci base of numeration

The Fibonacci numbers $1, 1, 2, 3, 5, 8, \ldots$ is a sequence defined by the equation

$$F_n = F_{n-1} + F_{n-2}$$

where $F_1 = 1$ and $F_2 = 1$. Consider a currency system with Fibonacci denominations. In other words, unlike the US currency system which has \$1, \$5, \$10, \$20, \$50, … denominations, we would have \$1, \$2, \$3, \$5, \$8, \$13, … denominations. In what follows, we will show that the Fibonacci currency system is efficient in that very few bills are needed to make change for any value \$$d$.

Consider the standard greedy algorithm for making change: To make change for \$$d$, you would choose the largest denomination less than or equal to d, subtract that denomination value from d, and then repeat for the remainder. For example, to make change for \$19 in the US currency system, we would choose a \$10 bill, yielding a remainder of \$9. We would then choose a \$5 bill, yielding a remainder of \$4. We would then choose a \$1 bill, yielding a remainder of \$3, and so on. Our change would then be

\$10, \$5, \$1, \$1, \$1, \$1.

Applying the same greedy strategy in the Fibonacci system, our change would be

\$13, \$5, \$1.

i. In making change for \$$d$ using the greedy strategy, let F_n be the largest Fibonacci denomination value less than or equal to d. Prove that after choosing an F_n bill, the remainder $d - F_n$ must satisfy $d - F_n < F_{n-1}$.

ii. Prove by induction that for any dollar value \$$d$, you can make change for \$$d$ without using any denomination more than once and without the use of adjacent denominations.

For example, while you could make change for \$14 using two \$5 bills, one \$3 bill, and one \$1 bill, you would be using a denomination twice (the \$5 bill) and you would be using adjacent denominations (the \$5 and \$3 bills). One could instead use one \$13 and one \$1 bill, which are non-adjacent and without repetition.

iii. Prove that if change is being made with n denominations $d_1, d_2, \ldots, d_n$ and adjacent denominations are not allowed, then at most $n/2$ unique denominations can be used. For simplicity, you may assume that n is even; the more general result for n odd or even is that at most $\lceil n/2 \rceil$ unique denominations can be used.

iv. Prove by induction that $\forall n \geq 6, F_n \geq 2^{n/2}$. Note that $2^{n/2} = (\sqrt{2})^n \approx 1.414^n$.

v. Prove (not necessarily by induction) that $\forall d \geq 8$ it is always possible to make change for \$$d$ in the Fibonacci currency system using at most $\log_2 d$ bills.

Problem 4 Recurrence bounds by guess and induction

For each one of these recurrences un upper bound or lower bound (or both) guess is given. Your task is to prove it by induction ("substitution" method). You will have to first state the induction step as an implication. C, D, F etc are positive constants that you can set or prove existent (but cannot change).

i. $T(n) = 8T(n/2) + n^3$ Lower Bound : $Cn^3 \log n$; Upper Bound : $Dn^3 \log n$

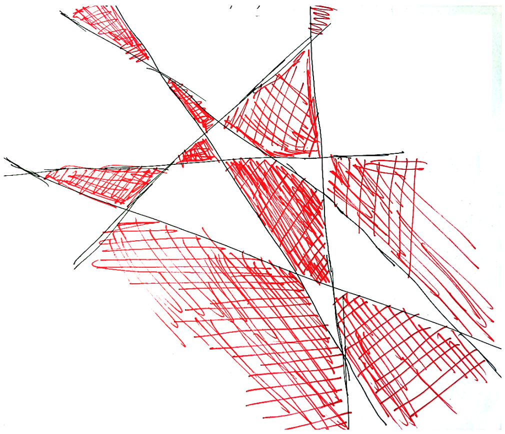
ii. $T(n) = 8T(n/2) + n^2$ Upper Bound : Dn^3

Hint: Prove a harder bound by subtracting a lower degree term : Upper Bound $Dn^3 - Fn^2$

iii. ★ $T(/n) = 3T(n/3 - 2) + n$ Lower Bound : $Cn \log n$; Upper Bound : $Dn \log n$

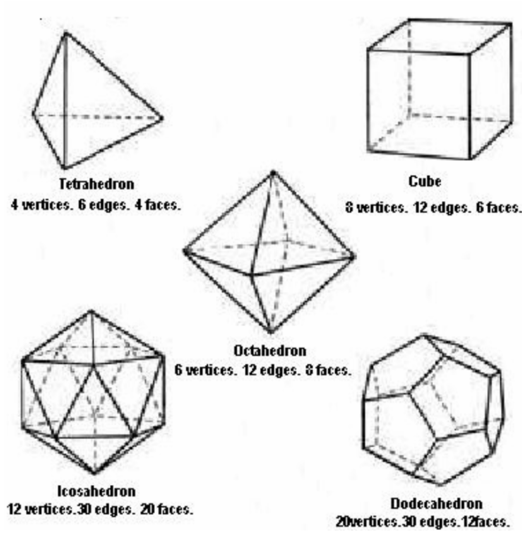
Problem 5 Structural induction (submit with HW6)

i. If n lines are drawn on a plane, and no two lines are parallel, and no 3 lines are concurrent. Show that they dive plane into $(n^2 + n + 2)/2$ regions. Use induction over n the number of lines



ii. If n lines are drawn on a plane, and no two lines are parallel, and no 3 lines are concurrent (previous exercise). Show that it is possible to color the regions formed with only two colors so that no two adjacent regions (a common side) share the same color.Use induction over n the number of lines

iii. Euler's Theorem. Prove that a convex polyhedron with F faces, E edges, V vertices satisfies V+F = E +2. Use induction over number of vertices V.



See `https://en.wikipedia.org/wiki/Euler_characteristic` for a larger article.

Problem 6 ★★★ 3D sphere Coverage (optional, no credit) A sphere is covered with some number of "caps" which are hemispheres. Prove that it is possible to choose four of these hemispheres, and remove all others, such that the sphere is still covered.