CS1800

Discrete Structures

Fall 2019

Lecture 15

10/29/19

Sequences & Series

- Sequence: a_n : a_1, a_2, a_3 ---

- Examples

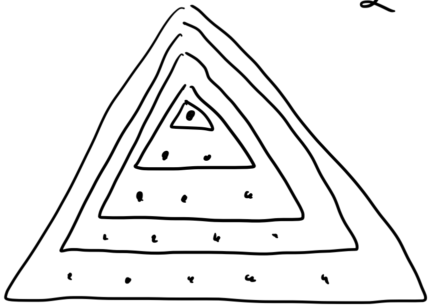
  - $1, 1/2, 1/3, 1/4, 1/5, \ldots$ $a_n = 1/n$

  - $2, 4, 6, 8, 10, \ldots$ $a_n = 2 \cdot n$

  - $7, 8, 9, 10, 11, \ldots$ $a_n = n+6$

  - $1, 3, 6, 10, 15, 21, \ldots$ $a_n = \dfrac{n(n+1)}{2}$

    triangular numbers



Common types of sequences: arithmetic, geometric, quadratic (harmonic)

① Arithmetic Sequences ← difference between consecutive elements is constant.

1 2 3 4 5 ...

e.g. 4, 7, 10, 13, 16, ...

3 3 3 3

a

d

$a = 4$ (starting value)

$d = 3$ (diffence)

$$a_n = a + (n-1)\cdot d$$

starting value

constant difference

$$\Longrightarrow \quad a_n = 4 + (n-1)\cdot 3$$

$$= 3n + 1$$

$$a_n = a + (n-1)\cdot d = d\cdot n + (a-d)$$

informative

simplified

E.g.

$$-7,\ -1,\ 5,\ 11,\ 17,\ 23,\ 29,\ \ldots$$

$$6 \quad 6 \quad 6 \quad 6 \quad 6 \quad \ldots$$

$$a_n = -7 + (n-1)\cdot 6$$

$$= 6n - 13$$

---

② Geometric Sequences ← constant factor between consecutive elements.

$$a_n = a \cdot r^{n-1}$$

a: starting value

r: constant ratio

e.g.

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 1, & 1/2, & 1/4, & 1/8, & 1/16, \ldots \end{array}$$

$$3,\ 6,\ 12,\ 24,\ 48,\ \ldots$$

$a = 3$

$r = 2$

$$a_n = 3\cdot 2^{n-1}$$

$$= \frac{3}{2}\cdot 2^n$$

$a = 1$

$r = 1/2$

$$a_n = 1\cdot (1/2)^{n-1}$$

$$= (1/2)^{n-1}$$

Quadratic sequences ← second differences are constant

e.g.

$$\begin{array}{ccccccccccc} 1 & & 3 & & 6 & & 10 & & 15 & & 21 \quad \ldots \\ & 2 & & 3 & & 4 & & 5 & & 6 & \\ & & 1 & & 1 & & 1 & & 1 & & \end{array}$$

← first differences

← second differences

$a_n = a \cdot n^2 + b \cdot n + c$ for some a, b, c

$n=1 \qquad 1 = a \cdot 1^2 + b \cdot 1 + c \Rightarrow a + b + c = 1$

$n=2 \qquad 3 = a \cdot 2^2 + b \cdot 2 + c \Rightarrow 4a + 2b + c = 3$

$n=3 \qquad 6 = a \cdot 3^2 + b \cdot 3 + c \Rightarrow 9a + 3b + c = 6$

$3a + b = 2$

$5a + b = 3$

$2a = 1$

$\boxed{a = 1/2}$

$3 \cdot 1/2 + b = 2$

$\boxed{b = 1/2}$

$1/2 + 1/2 + c = 1$

$\boxed{c = 0}$

$$\Rightarrow a_n = \frac{1}{2} \cdot n^2 + \frac{1}{2} \cdot n + 0$$

$$= \frac{1}{2} n^2 + \frac{1}{2} n = \frac{1}{2}(n^2 + n)$$

$$= \frac{n(n+1)}{2}$$

Last time

- Finished motivation
  - sorting
- Sequences
  - arithmetic
  - geometric
  - quadratic

Today

- Series
  - arithmetic
  - geometric
    + infinite

optional

  - harmonic
  - telescoping
  - math trick
    - Fibonacci Representations

Next time

- Skip lists
- Induction

Series: Sum of elements from a sequence

E.g. Arithmetic sequence: $1, 2, 3, 4, \_\_ n$

Arithmetic Series: $1+2+3+4+\cdots+n = \frac{n(n+1)}{2}$

E.g. Geometric series: $3 + 3/2 + 3/4 + 3/8 + 3/16 + \cdots = ?$

Arithmetic Series: Gauss's Trick

$$
\begin{aligned}
S &= 1 + 2 + 3 + \cdots + 100 \\
+ \quad S &= 100 + 99 + 98 + \cdots + 1 \\
\hline
2 \cdot S &= \underbrace{101 + 101 + 101 + \cdots 101}_{100}
\end{aligned}
$$

$$= 100 \cdot 101 = 10100$$

$$S = 10100 \div 2 = 5050$$

Examples

①

$$
\begin{aligned}
S &= 1 + 2 + 3 + 4 + \cdots + n \\
+\quad S &= n + (n-1) + (n-2) \cdots + 1 \\
\hline
2S &= \underbrace{(n+1) + (n+1) + \cdots}_{n} (n+1) \\
&= n\cdot(n+1)
\end{aligned}
$$

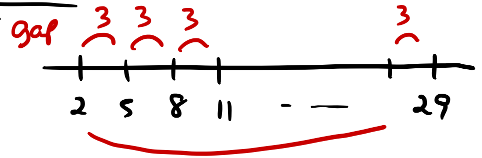
$$\Rightarrow \quad 2S = n(n+1) \quad \Rightarrow \quad S = \frac{n\cdot(n+1)}{2}$$

$$\rightarrow \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

②

$$
\begin{aligned}
S &= 2 + 5 + 8 + 11 + 14 + \cdots + 29 \\
S &= 29 + 26 + 23 + \cdots + 2 \\
\hline
2S &= \underbrace{31 + 31 + 31 + \cdots}_{10} + 31 \\
&= 31\cdot 10
\end{aligned}
$$

$$\Rightarrow \quad S = \frac{31\cdot 10}{2}$$



span $= 29 - 2 = 27$

$$\#\text{gaps} = \frac{\text{span}}{\text{gap size}} = \frac{27}{3} = 9$$

$$\#\text{numbers} = \#\text{gaps} + 1 = 10$$

③

$$S = -11 - 7 - 3 + 1 + 5 + 9 + \cdots + 17$$

$$S = 17 + 13 + 9 + \cdots \quad -11$$

$$2S = \underbrace{6 + 6 + 6 + \cdots + 6}$$

$$= \frac{(17 - (-11))}{4} + 1 = \frac{28}{4} + 1 = 7 + 1 = 8$$

$$2S = 6 \cdot 8$$

$$S = \frac{6 \cdot 8}{2} = 24$$

Geometric Series: $3 + 3/2 + 3/4 + \text{---} + 3/32$

$$S = 3 + \cancel{3/2} + \cancel{3/4} + \cancel{\text{---}} + \cancel{3/32}$$

$$- \quad 1/2 \cdot S = \cancel{3/2} + \cancel{3/4} + \cancel{3/8} + \cancel{\text{---}} + 3/64$$

$$S - 1/2 \cdot S = 3 - 3/64$$

$$\downarrow$$

$$1/2 \, S = 3 - 3/64$$

$$S = 6 - 3/32$$

Constant ratio r ; Start at 1

$$S = 1 + \cancel{r} + \cancel{r^2} + \cancel{r^3} + \cdots \cancel{r^n}$$

$$- r \cdot S = \cancel{r} + \cancel{r^2} + \cancel{r^3} + \cdots \cancel{r^n} + r^{n+1}$$

$$S - r \cdot S = 1 - r^{n+1}$$

$$\downarrow$$

$$(1-r) \cdot S = 1 - r^{n+1}$$

$$S = \frac{1 - r^{n+1}}{1 - r}$$

$r = 1/2$

$$1 + 1/2 + 1/4 + 1/8 + \cdots + 1/2^n$$

$$S = \sum_{i=0}^{n} r^n$$

$$= \frac{1 - r^{n+1}}{1 - r}$$

$r = 2$

$$S = 1 + 2 + 4 + 8 + \cdots 2^n$$

$$S = \frac{1 - 2^{n+1}}{1 - 2}$$

$$= \frac{1 - 2^{n+1}}{-1}$$

$$= 2^{n+1} - 1$$

Infinite geometric series, e.g.

$$S = 1 + r + r^2 + r^3 + \cdots = \sum_{k=0}^{\infty} r^k \qquad |r| < 1$$

e.g. $1 + 1/2 + 1/4 + 1/8 + \cdots$

not precise

$$S = 1 + \cancel{r} + \cancel{r^2} + \cancel{r^3} + \cancel{\cdots}$$

$$- r \cdot S = \cancel{r} + \cancel{r^2} + \cancel{r^3} + \cancel{\cdots}$$

$$S - r \cdot S = 1$$

$$(1-r) \cdot S = 1$$

$$S = \frac{1}{1-r}$$

$$\frac{1}{1 - 1/2} = \frac{1}{1/2} = 2$$

precisely...

$$\sum_{k=0}^{\infty} r^k = \lim_{n \to \infty} \sum_{k=0}^{n} r^k$$

$$= \lim_{n \to \infty} \frac{1 - r^{n+1}}{1-r}$$

$$= \frac{1}{1-r}$$

because $\lim_{n \to \infty} r^{n+1} \to 0 \quad |r| < 1$

③ Harmonic Series

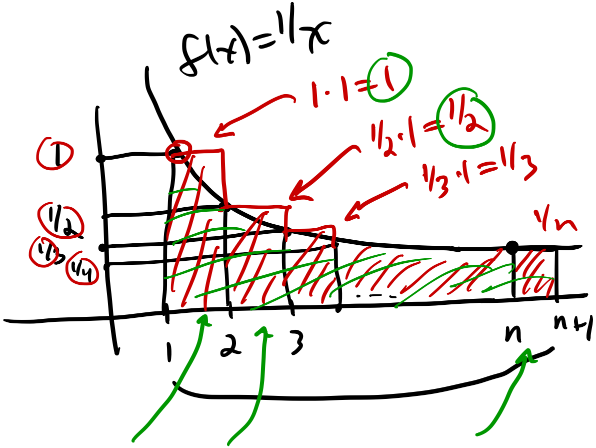
$$\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n}$$

$$= \ln(n) + \sim \text{constant}$$

$\hookrightarrow \gamma \sim 0.577$

e.g. $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{1000} \sim \ln(1000)$

why



[red] – area of rectangles

[green] – area under curve

→ about the same

$$[\text{red}] \approx \int_1^{n+1} \frac{1}{x}\,dx$$

$$= \ln x \Big|_1^{n+1} = \ln(n+1) - \ln 1$$

$$= \ln(n+1) \checkmark$$

Fibonacci Numbers

Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...

2 1.5 1.66 1.6 1.625 1.615 1.619 1.617

Ratio converges to Golden Ratio

$$\phi = \frac{1+\sqrt{5}}{2} \approx \underline{1.618}$$

$$F_n = \begin{cases} 1 & n=1 \text{ or } 2 \\ F_{n-1}+F_{n-2} & n>2 \end{cases}$$

Representing numbers in Fibonacci

	144	89	55	34	21	13	8	5	3	2	1
	1		1	1		1	1		1 1		1
47 =				1	0	1	0	0	0	0	0
26 =					1	0	0	1	0	0	0
40 =				1	0	0	0	1	0	0	1

Coincidence: # km/mi ≈ 1.609

47 miles → km

55 21 = 76

$47 = 34+13 \rightarrow 1.6\times(47) = 1.6\cdot 34 + 1.6\times 13$