assuming UG knowledge of Graphs

Lecture: Graphs part 1 (part 2: SP, 3: Net Flow)

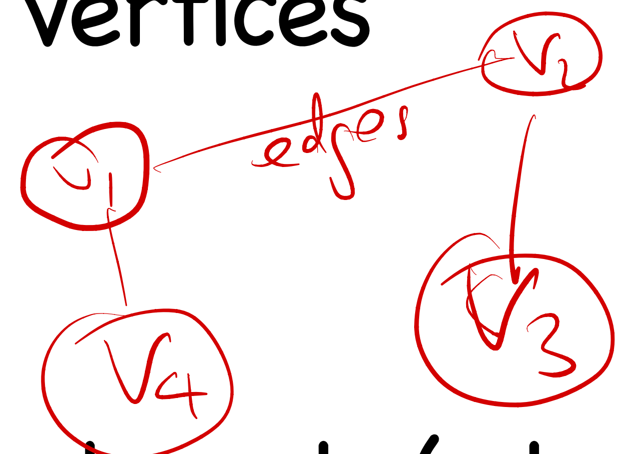
Intro to graphs

Minimum Spanning Trees

- Intro to graphs: cycles, paths, edges, representation (prereq)
- BFS + DFS (prereq)
- DFS (MS-level) details
- DFS ⟹ Strongly connected components
- Min Span Tree (Kruskal, Prim, Cut Theorem)

Graphs

- nodes/vertices and edges between vertices
  - set V for vertices, set E for edges
  - we write graph G = (V,E)

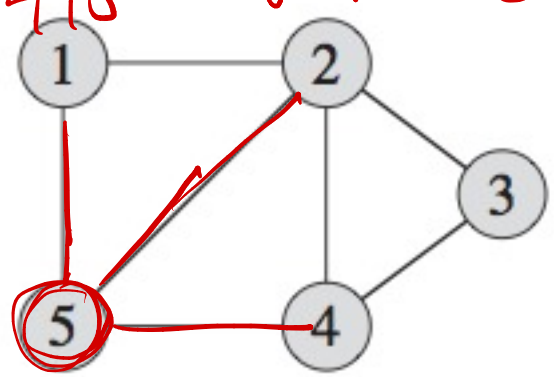


- example : cities on a map (nodes) and roads (edges)

Adjacency matrix

- a_{ij} =1 if there is an edge from vertex i to vertex j
- if graph is undirected, edges go both ways, and the adj. matrix is symmetric

1,2,3,4,5 = vertices ids

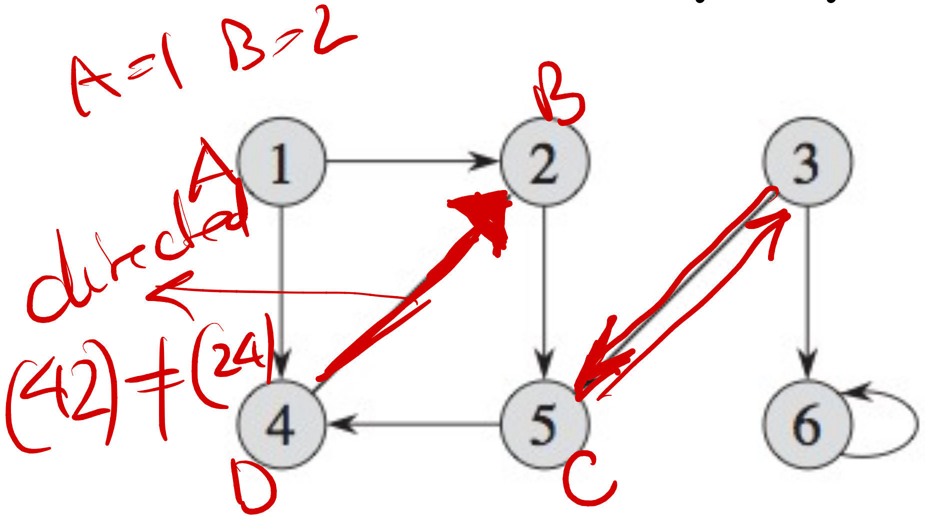


	1	2	3	4	5
1	0	1	0	0	1
2	1	0	1	1	1
3	0	1	0	1	0
4	0	1	1	0	1
5	1	1	0	1	0

→ edge (2,3)
= edge (3,2)

- if the graph is directed, the adj. matrix is not necessarily symmetric

A=1 B=2



	1	2	3	4	5	6
1	0	1	0	1	0	0
2	0	0	0	0	1	0
3	0	0	0	0	1	1
4	0	1	0	0	0	0
5	0	0	0	1	0	0
6	0	0	0	0	0	1

→ missing edge
edge (3,5)
edge (5,3) added.

G sparse ⇒ lots of "0" in Adj Mat

edge
link,
connect } ⇒ undirected G

Adjacency lists

→ path, direction

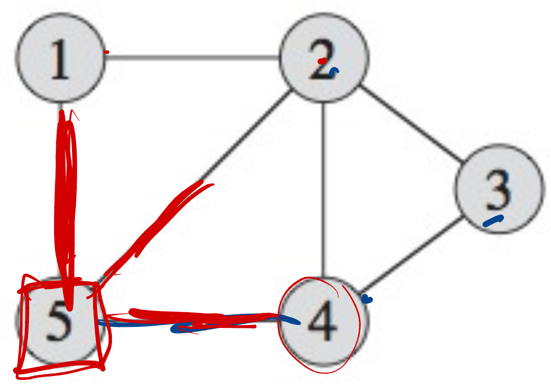
→ edge

vertices

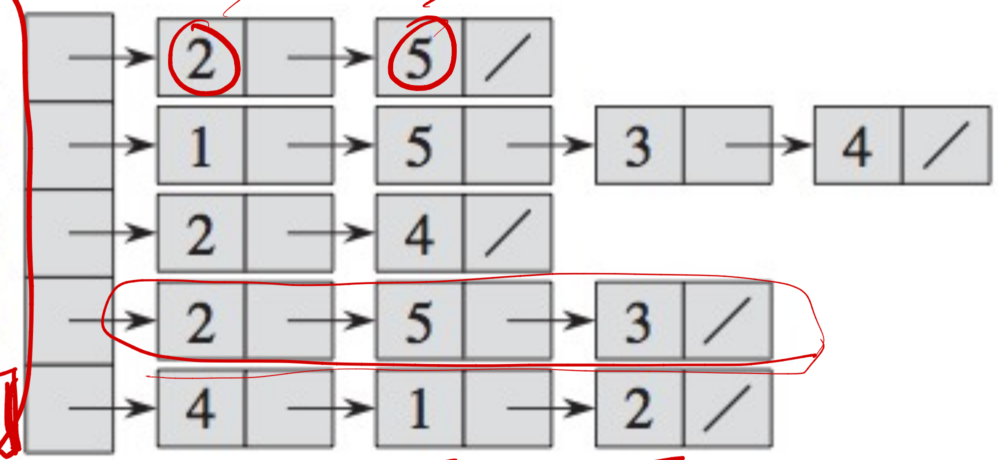
"linked" to 1

"reachable" from 1

Adj Matrix



	1	2	3	4	5
1	0	1	0	0	1
2	1	0	1	1	1
3	0	1	0	1	0
4	0	1	1	0	1
5	1	1	0	1	0



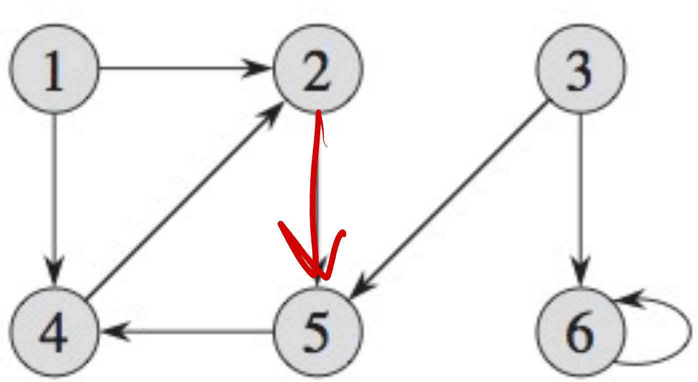
Adj Lists

! Find edge(5→2): Traverse AdjList(5) look for "2"

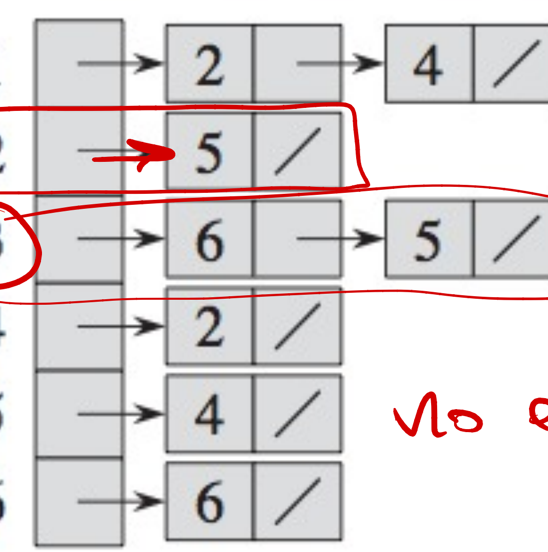
Good: Enumerate all vertices w/ income edges from 5

- linked list marks all edges starting off a given vertex

2→5



	1	2	3	4	5	6
1	0	1	0	1	0	0
2	0	0	0	0	1	0
3	0	0	0	0	1	1
4	0	1	0	0	0	0
5	0	0	0	1	0	0
6	0	0	0	0	0	1



3→6
3→5

no edge 5→2

reachable = path
1→2→5; 1 ⇝ 5

edge(5,2) missing

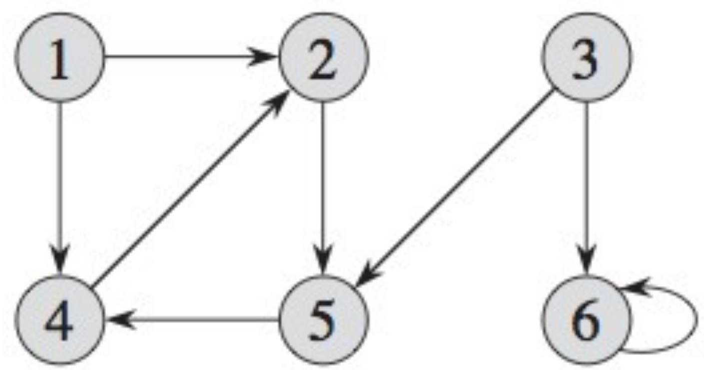
where can I go 1 step

backwards adj list : where can I come from 1 step.

1	null
2	1 4 null
3	null
4	5 1 null
5	2 3 null
6	3 6 null

paths and cycles

- path: a sequence of vertices $(v_1,v_2,v_3,\ldots,v_k)$ such that all (v_i,v_{i+1}) are edges in the graph



- edges can form a cycle = a path that ends in the same vertex it started

- paths and cycles are defined for both directed and undirected graphs

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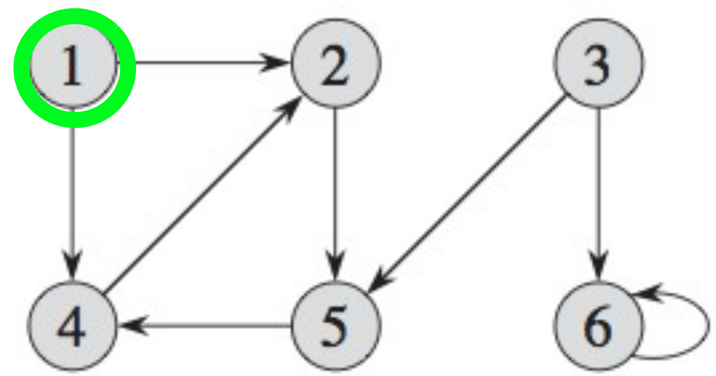

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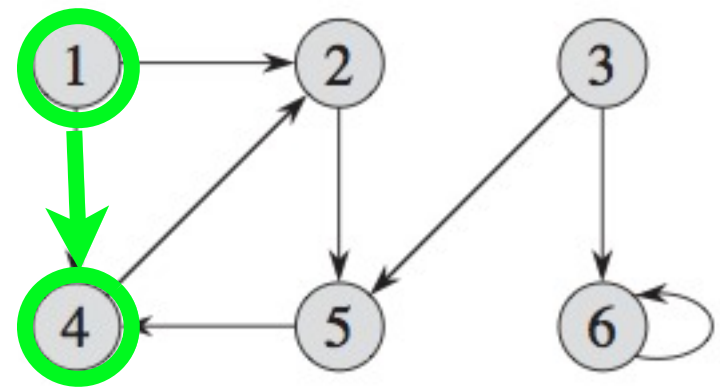

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paths and cycles

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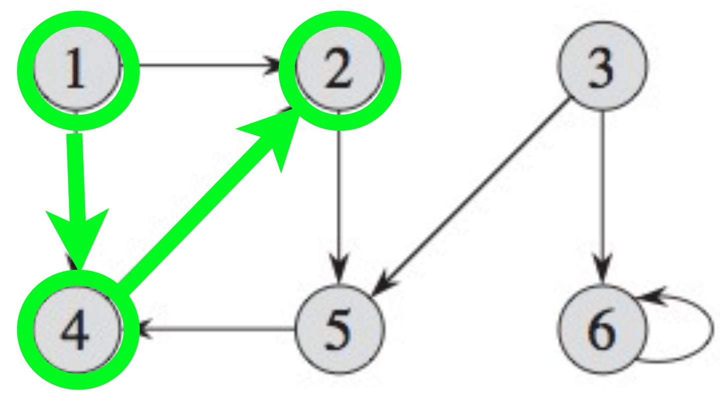
1 ⇝ 2 two options

1 → 2

1 → 4 → 2



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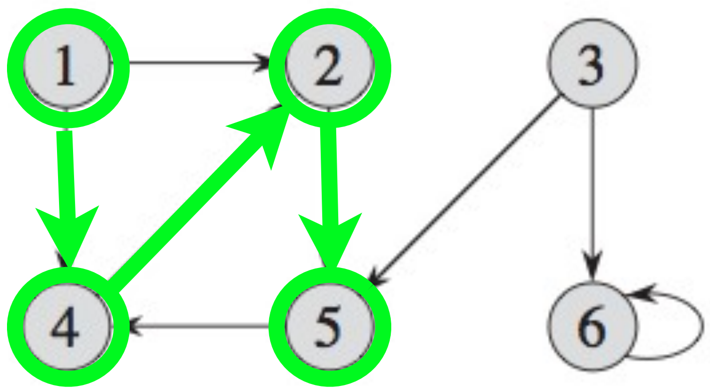

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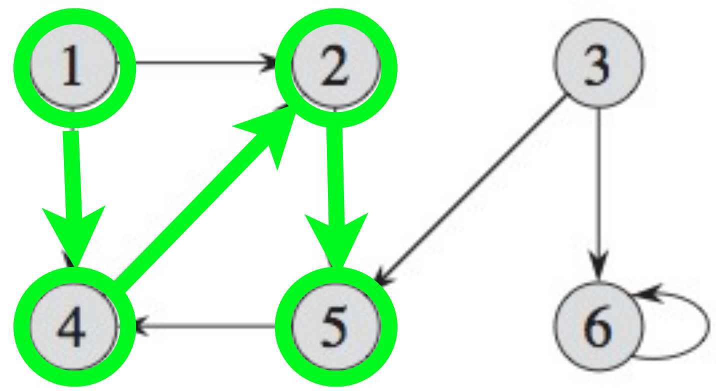

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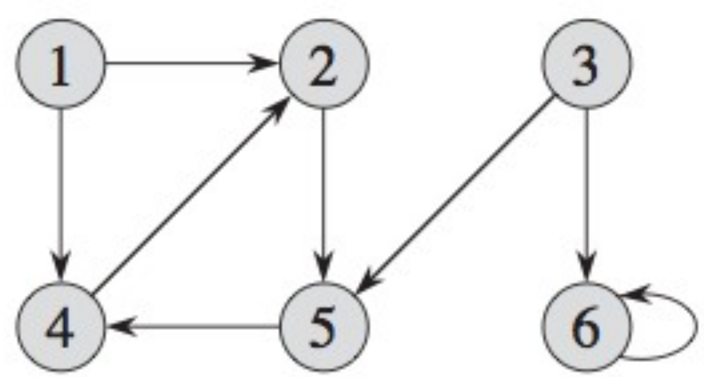

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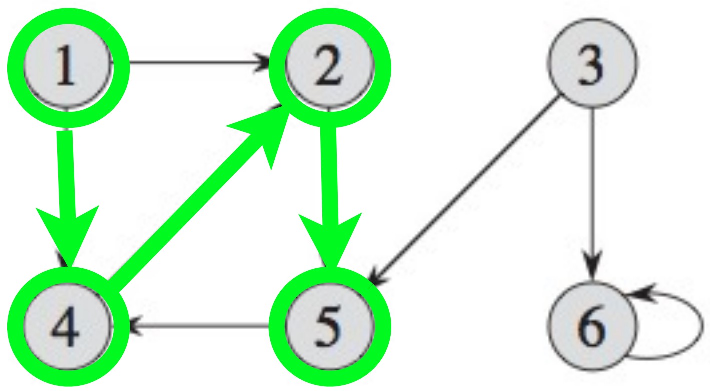

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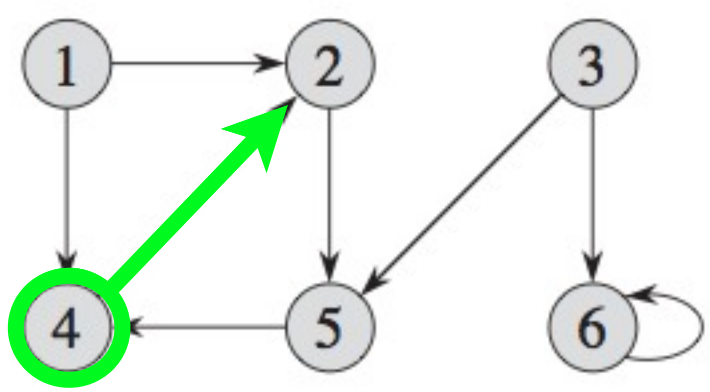

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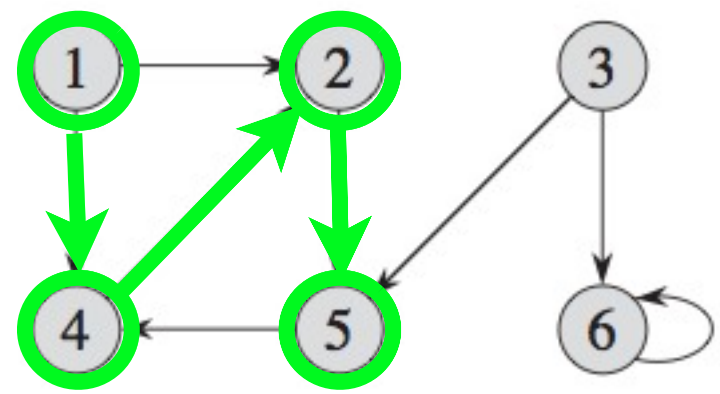

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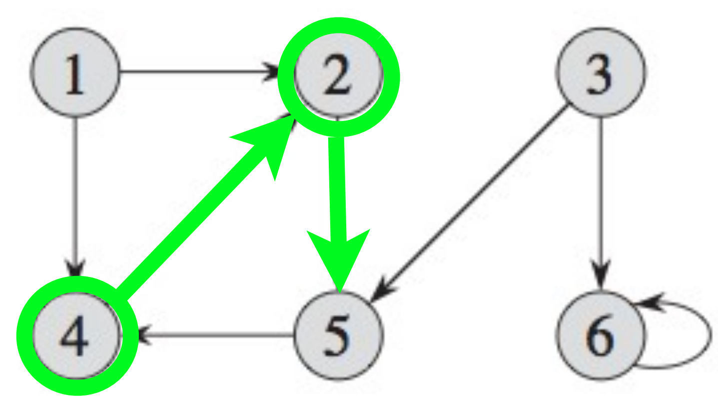

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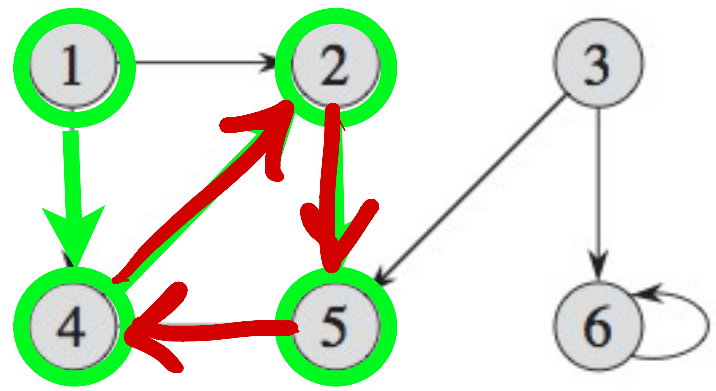
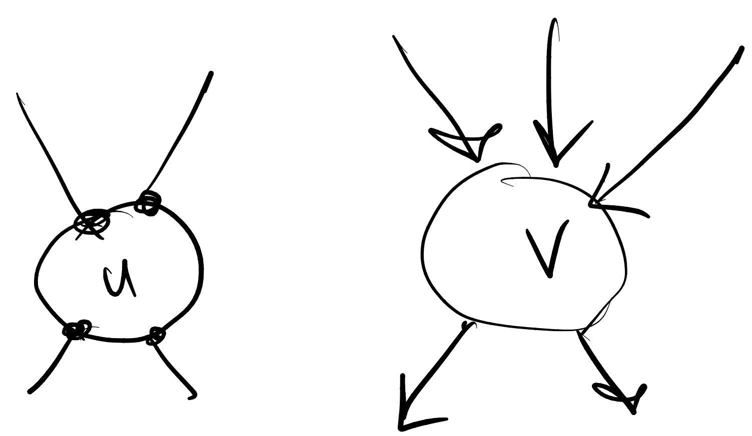


(4,2,5)

cycle path that ends where it started

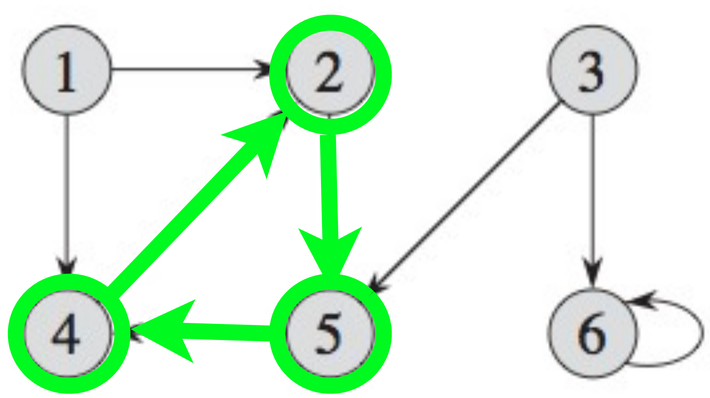
simple cycle: does not repeat vertices

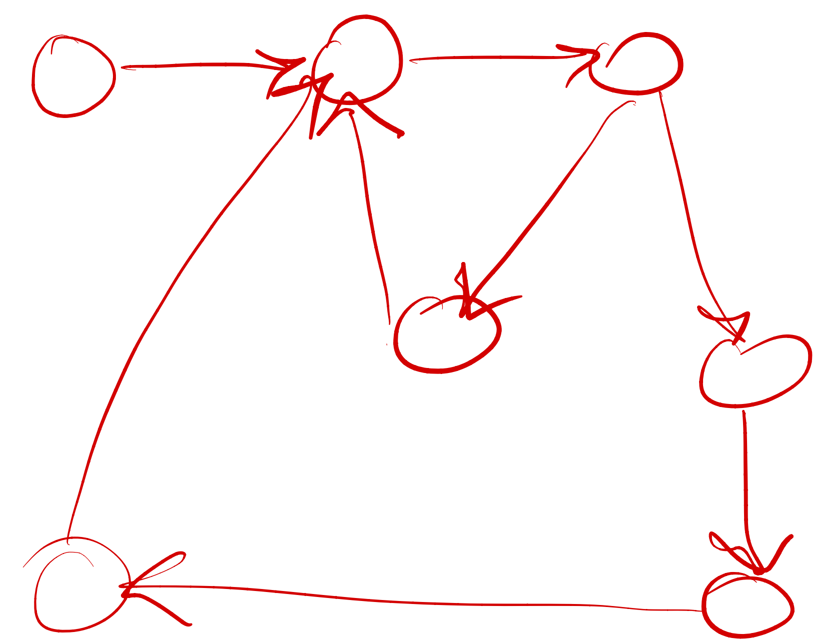
- edges can form a cycle = a path that ends in the same vertex it started



deg(u)=4

indeg(v)=3
outdeg(v)=2





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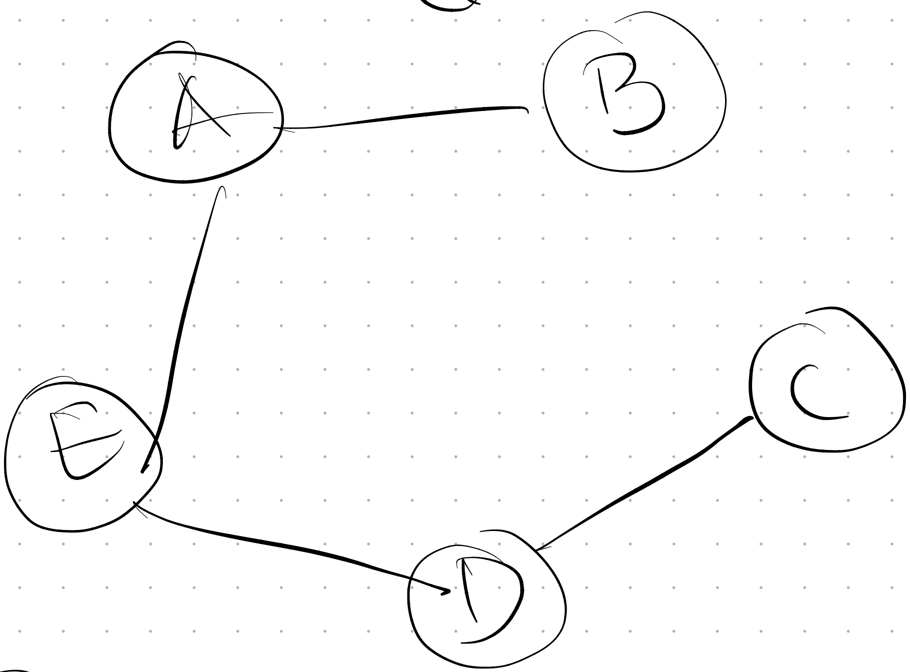
Graph Theory examples

V = vertex set
E = edge set

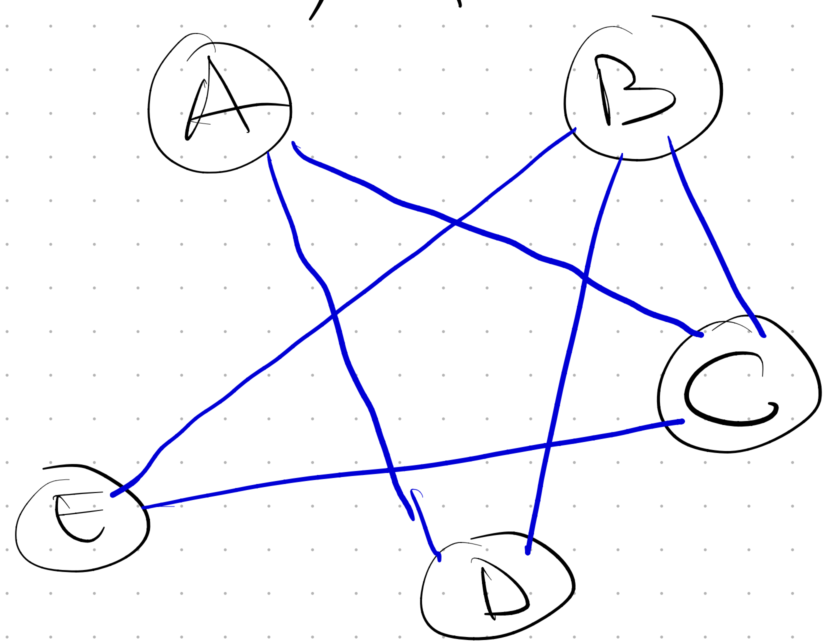
Th $$\sum_{u \in \text{vertices}} \deg(u) = 2 \cdot \#\text{edges}$$

$$\sum_{u \in V} \deg(u) = 2|E|$$

G

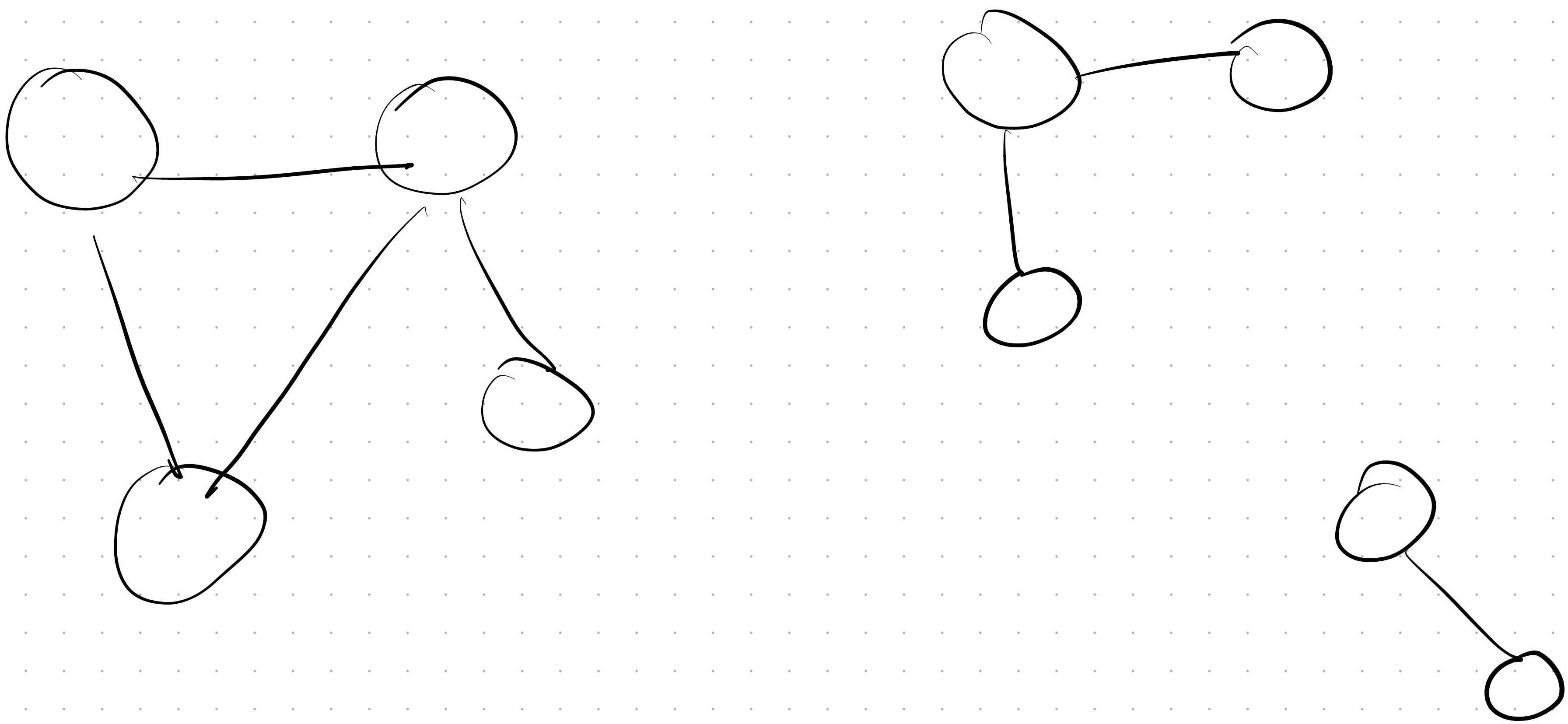


$\overline{G}$ - complement
same V, opposite edges



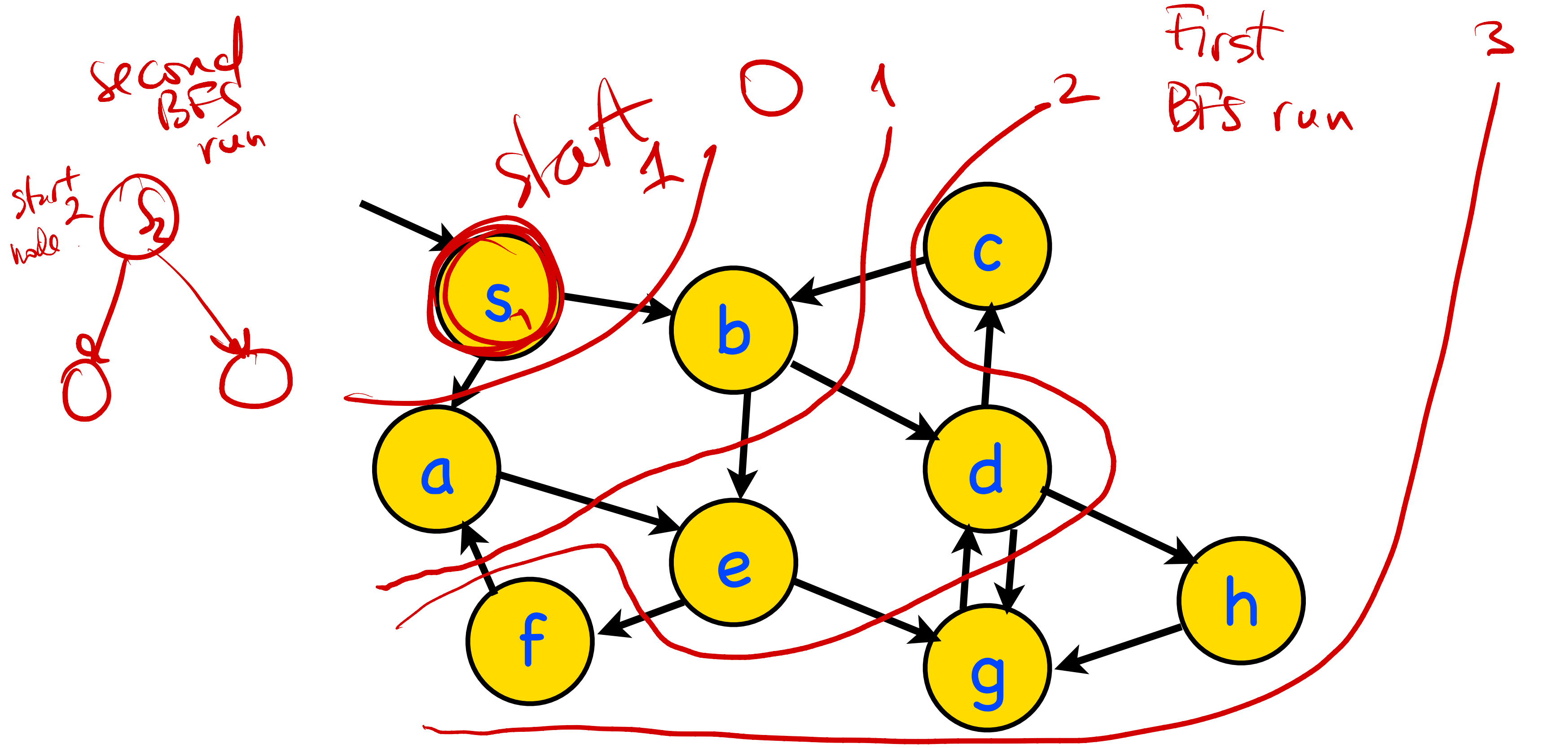
Th G or $\overline{G}$ is connected.

Ex Disconnected Graph : 3 components



Traverse/search graphs : BFS

- BFS = breadth-first search.
- Start in a given vertex s, find all reachable vertices from s
  - proceed in waves partition of (nodes reachable from s) descendants(s)
  - computes d[v] = number of edges from s to v. If v not reachable from s, we have d[v] = ∞.



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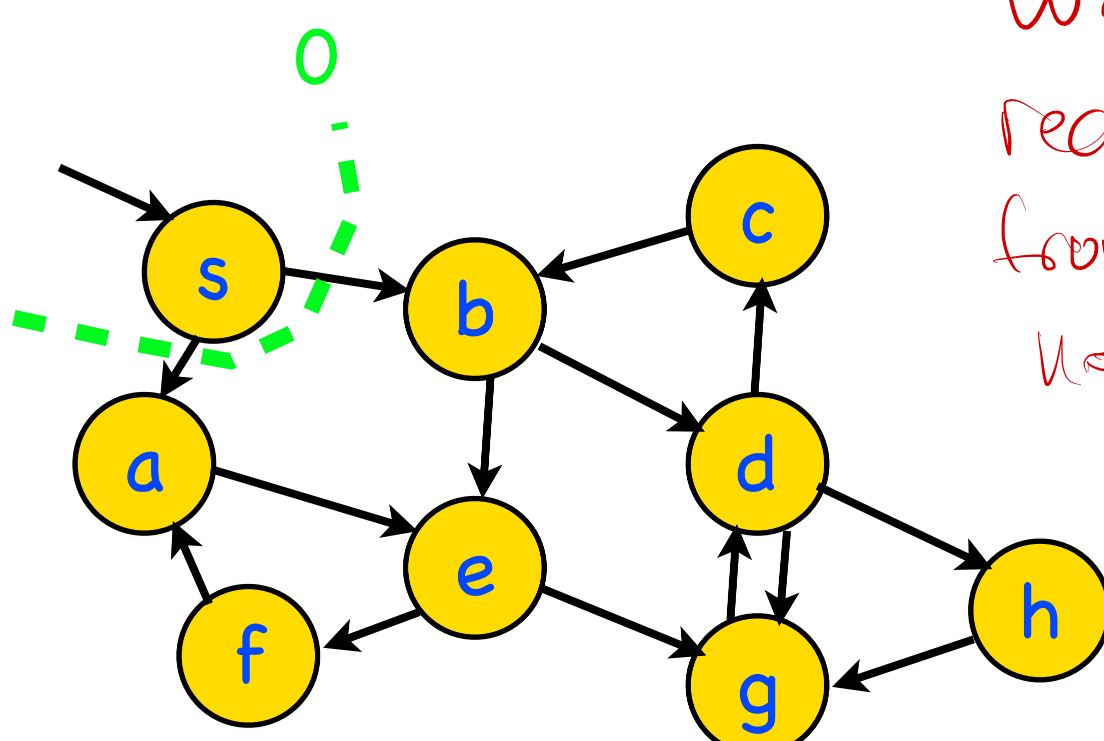
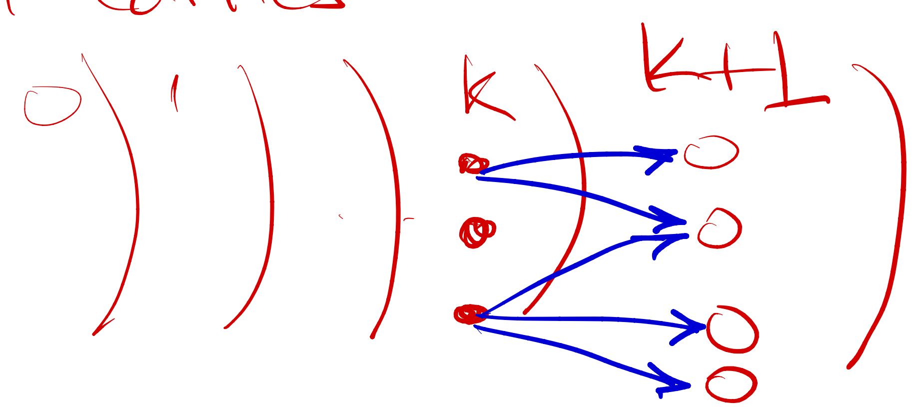
wave k+1 : all nodes reachable w/ direct edge from wave k nodes, but not earlier





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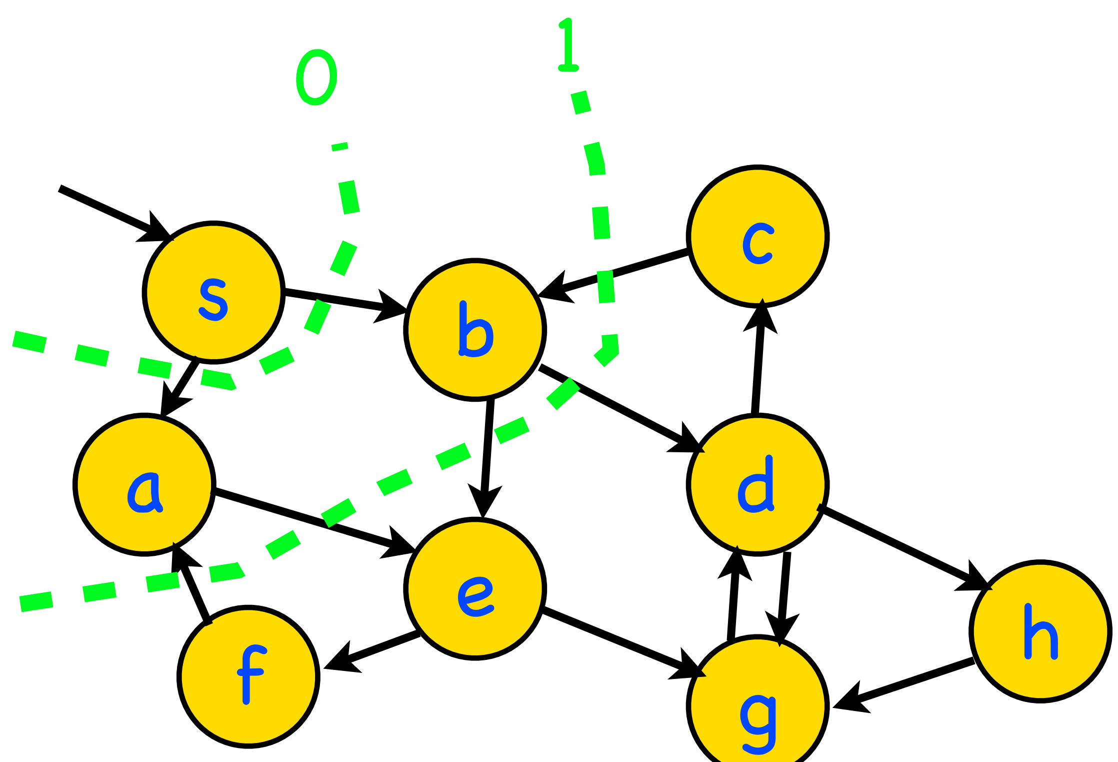

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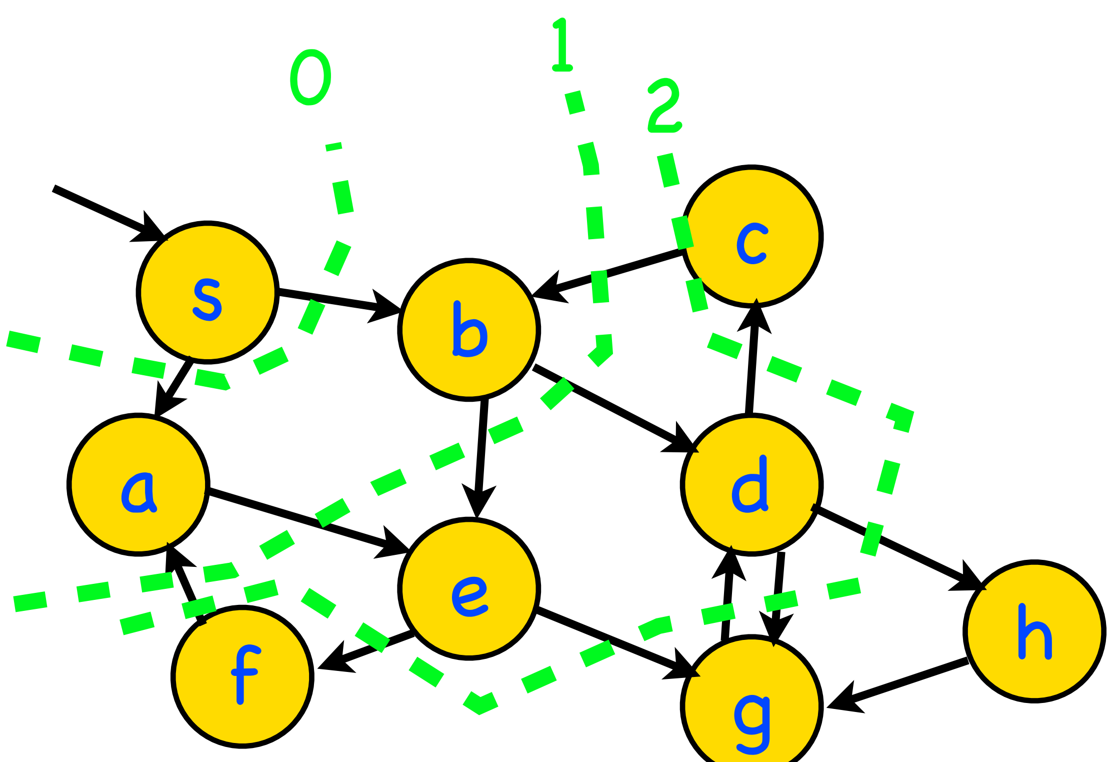

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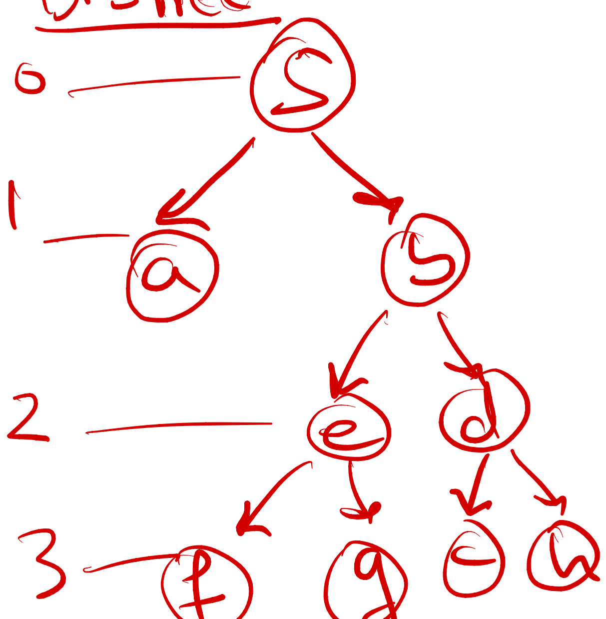
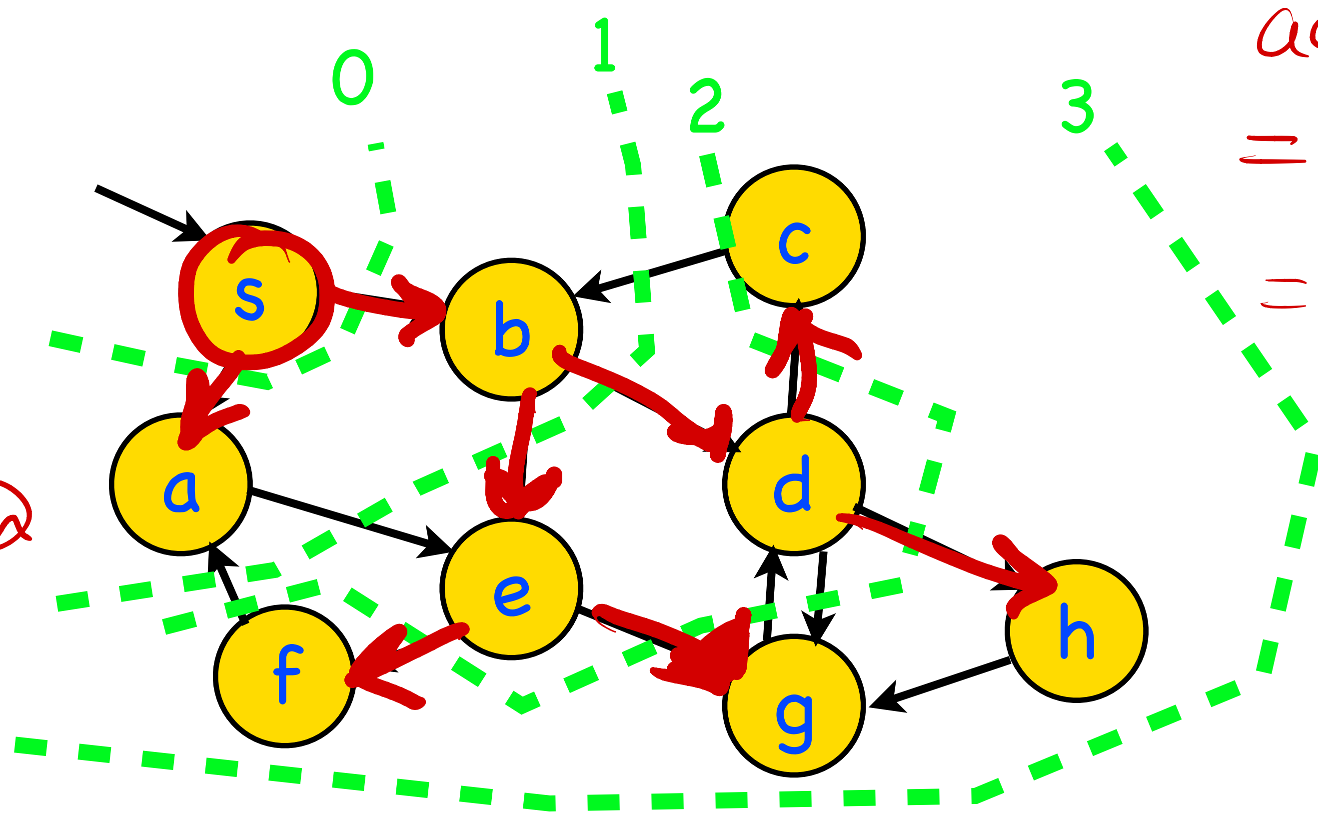
BFS tree



d[e] = 2 // wave
parent[e] = b



adv. edge =
= BFS edge
= edge
(u.parent, u)
k k+1

BFS

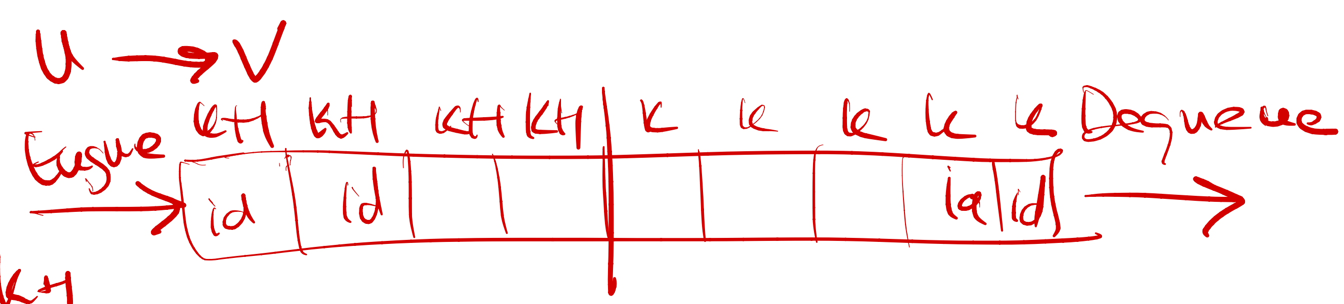
- use a queue to store processed vertices
  - for each vertex in the queue, follow adj matrix to get vertices of the next wave

```
BFS(V,E,s)
for each vertex v≠s, set d[v]=∞
init queue Q; enqueue(Q,s) //puts s in the queue
while Q not empty
    u = dequeue(S) // takes the first elem available from the queue
    for each vertex v ∈ Adj[u]
        if (d[v]==∞) then
            d[v]=d[u]+1
            Enqueue(Q,v)
        end if
    end for
end while
```

k+1 k

BFS

$d[v] = \text{wave}(v)$



- Running time O(V+E), since each edge and vertex is considered once.

Traverse/search graphs : DFS

- DFS = depth-first search : go as far as possible on some direction/branch
  - once a vertex is discovered, proceed to its adj vertices, or "children"(depth) rather than to its "brothers" (breadth)

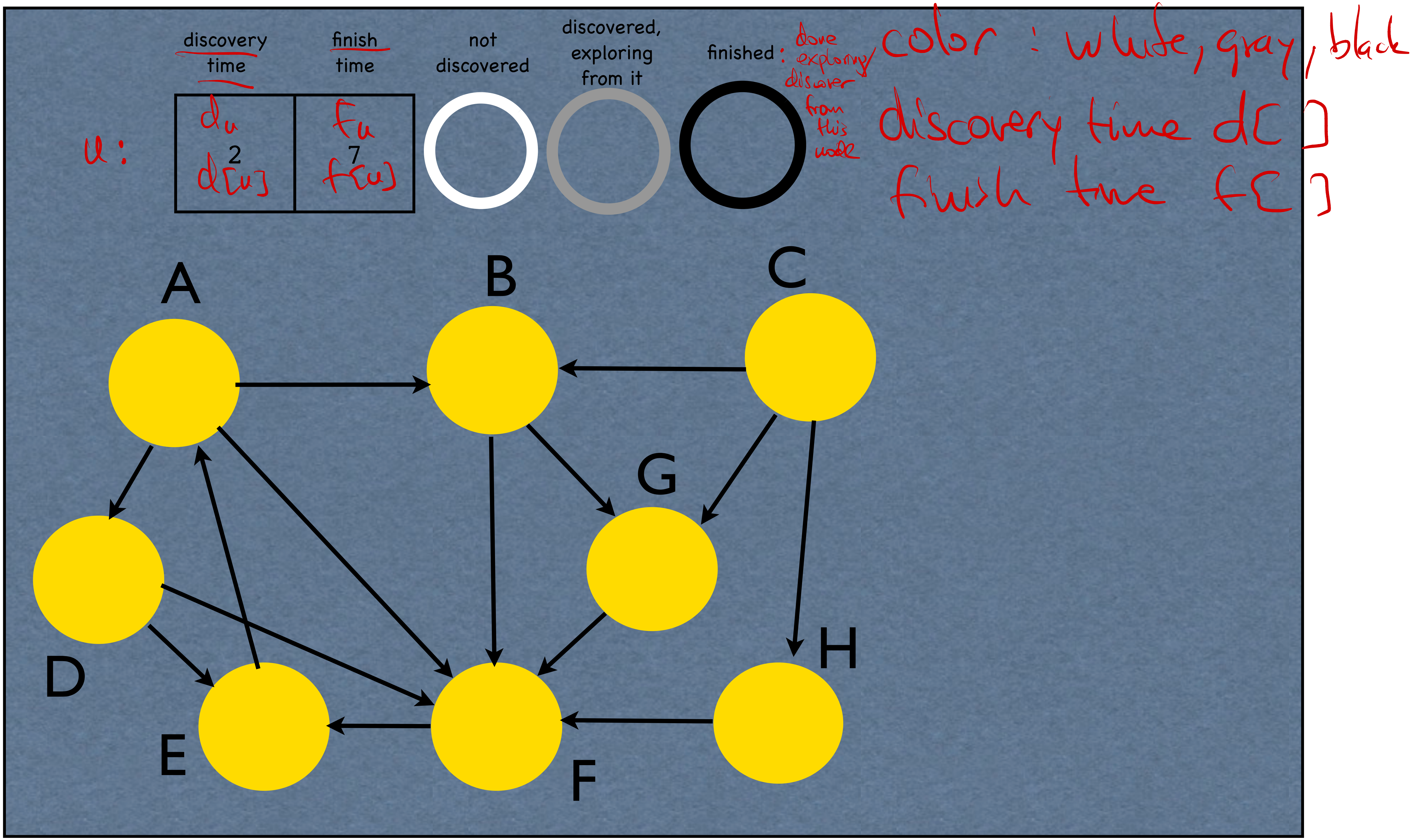
```
DFS-wrapper(V,E)
    foreach vertex u∈V {color[u] = white} end for //color all nodes white
    foreach vertex u∈V
        if (color[u]==white) then DFS-Visit(u)
    end for

DFS-Visit(u) //recursive function
    color[u] = gray; //gray means "exploring from this node"
    time++; discover_time[u] = time;//discover time
    for each v ∈ Adj[u]
        if (color[v]==white) then DFS-Visit(v) //explore from u
    end for
    color [u] = black; finish_time[u]=time; //finish time
```

v = discovered from u

v closed before u

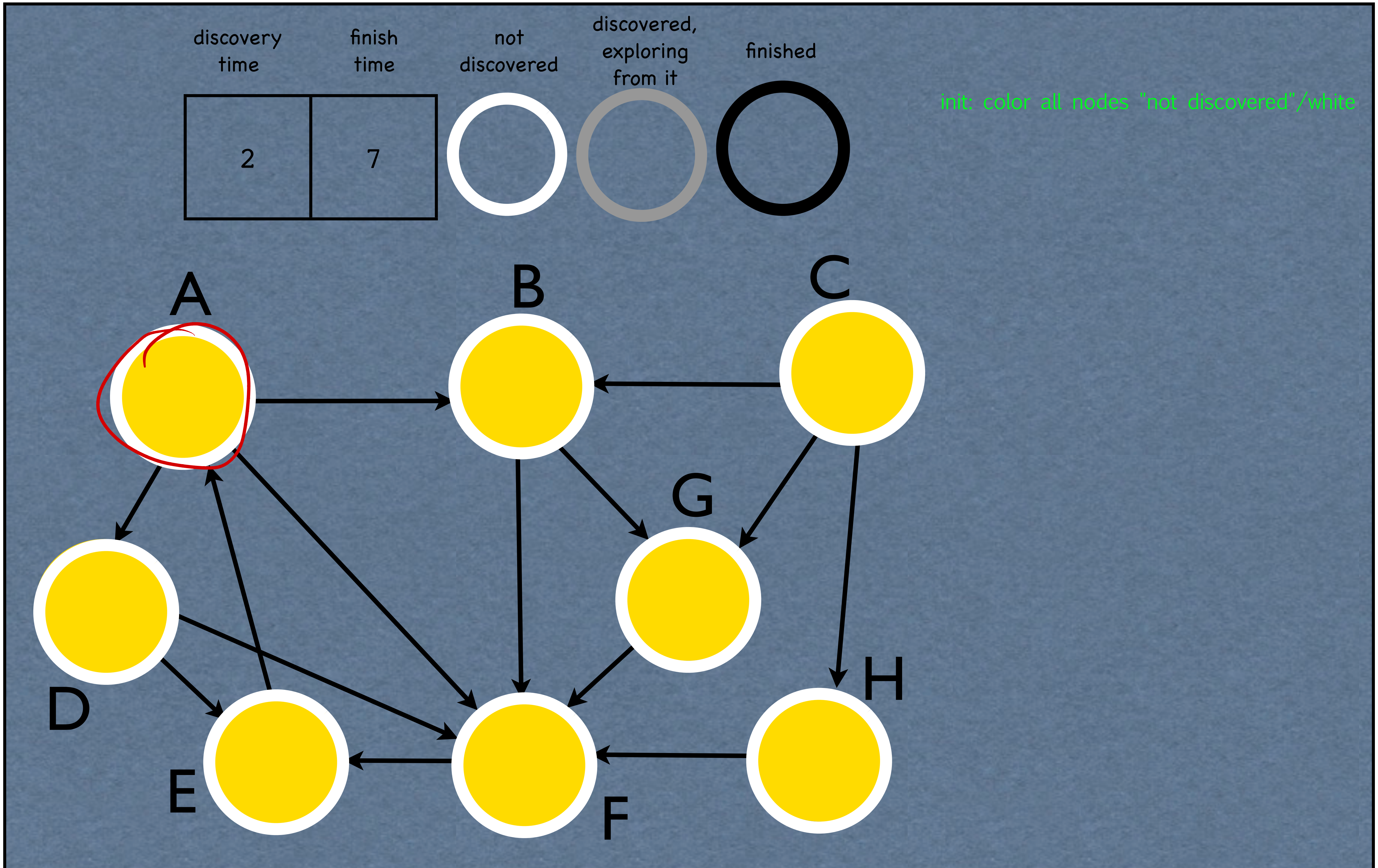
$f_v < f_u$

DFS



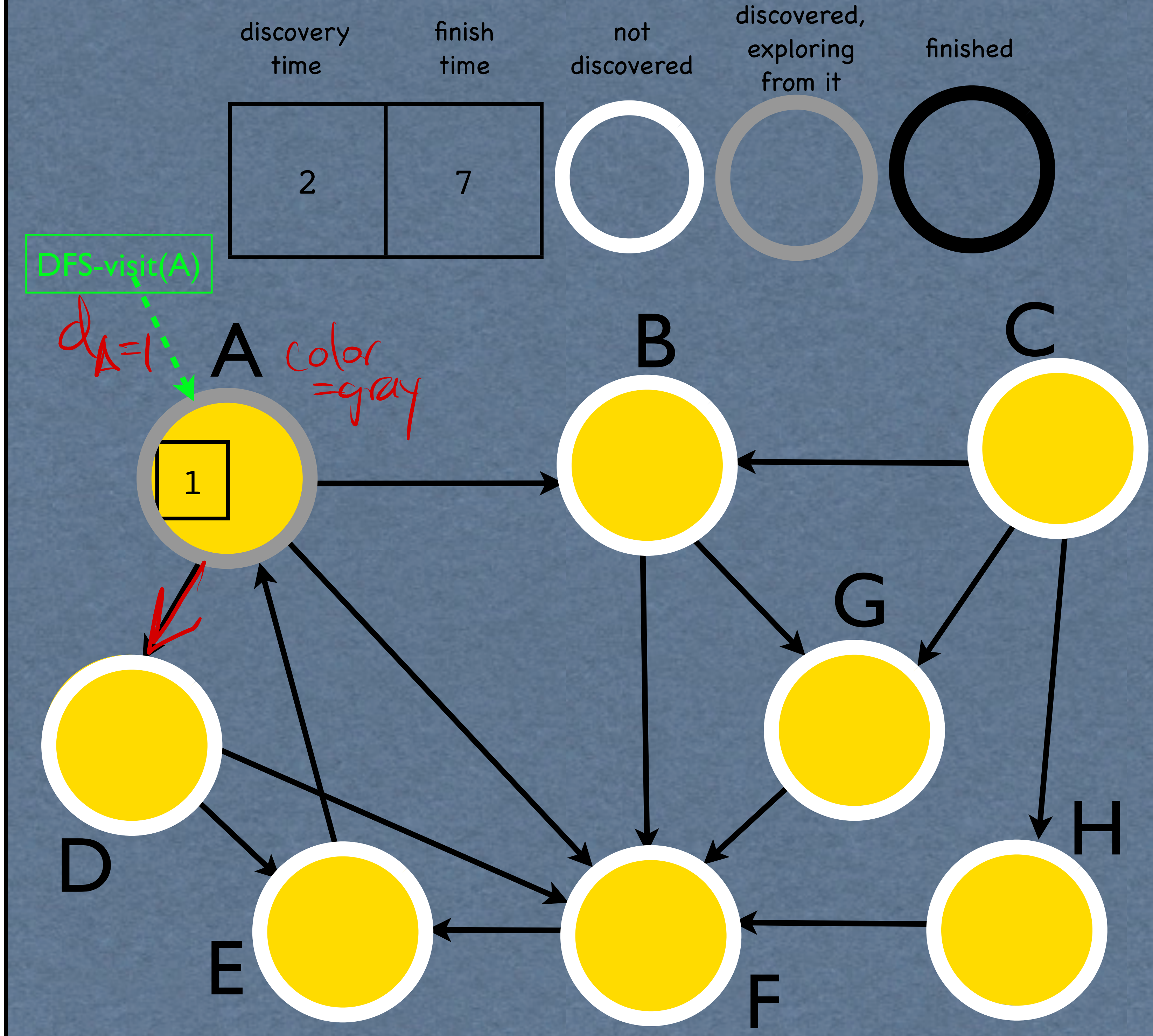
discovery time
finish time
not discovered
discovered, exploring from it
finished: done exploring/discover from this node
u:
d_u
2
d[u]
f_u
7
f[u]
color: white, gray, black
discovery time d[ ]
finish time f[ ]
A
B
C
D
E
F
G
H

DFS



discovery time
finish time
not discovered
discovered, exploring from it
finished
2
7
init: color all nodes "not discovered"/white
A
B
C
D
E
F
G
H

DFS

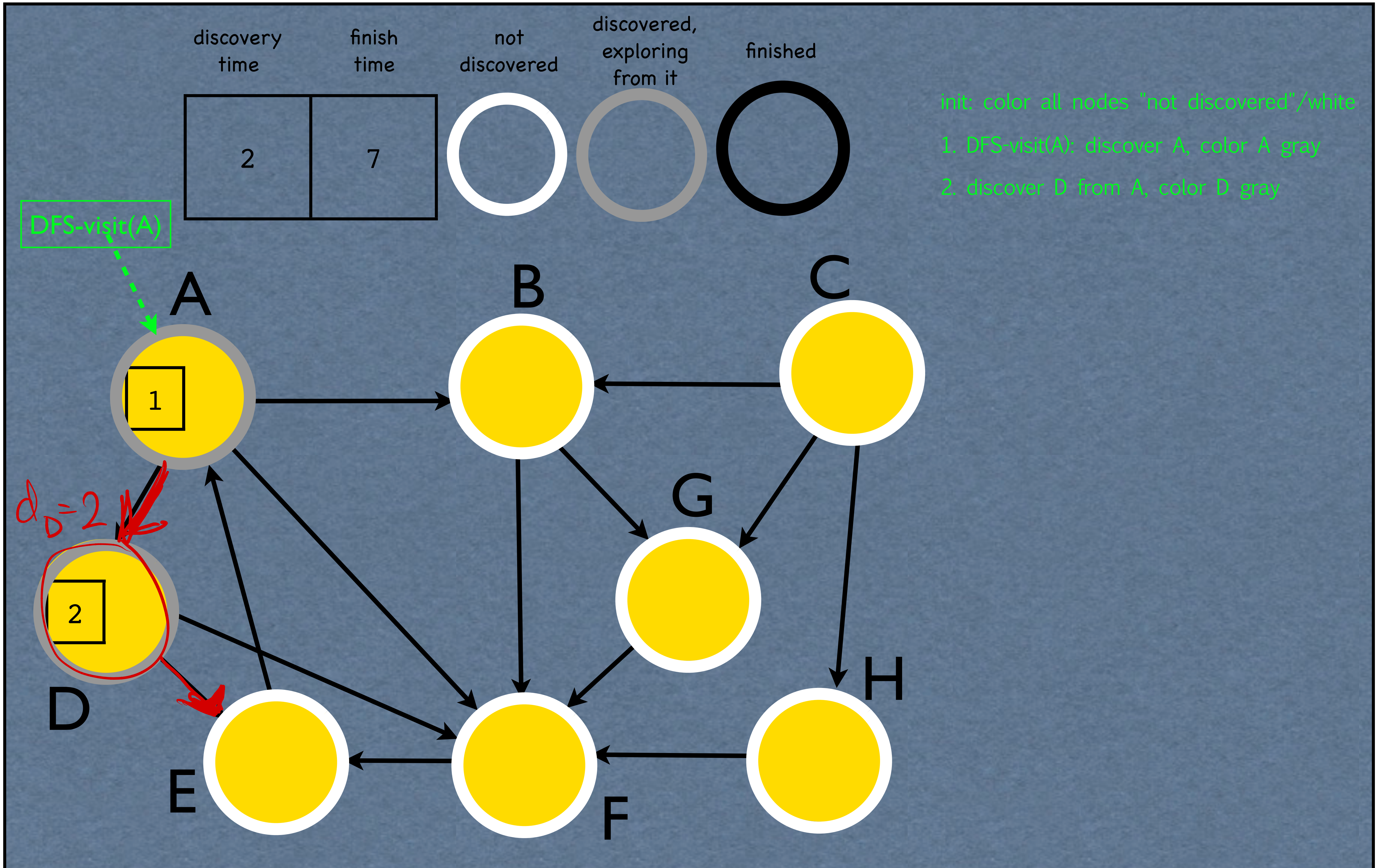


init: color all nodes "not discovered"/white

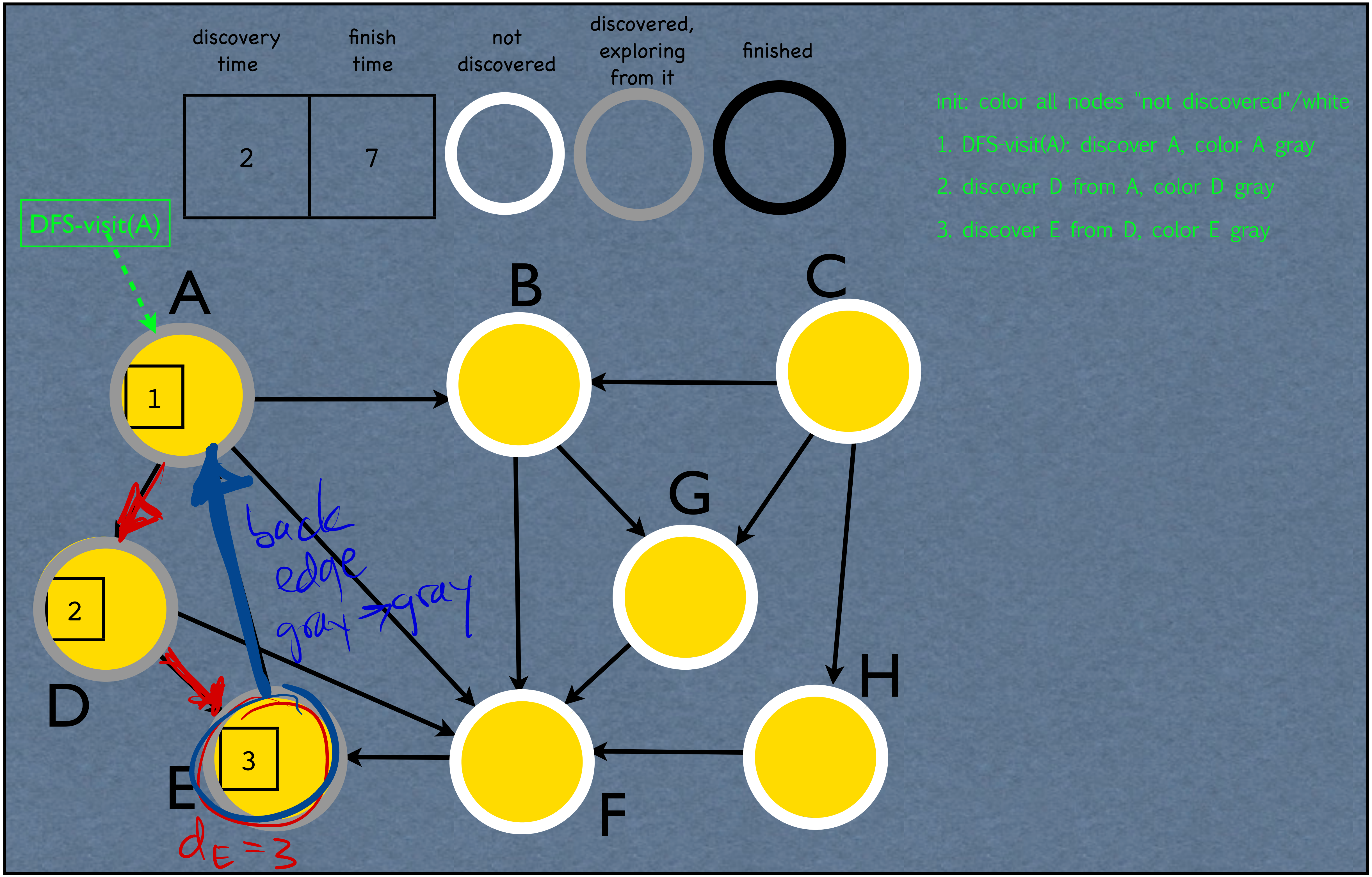
1. DFS-visit(A): discover A, color A gray

advance (BFS, DFS)
on an edge
$u \rightarrow v$
discover v from u
$v.parent = u$

DFS

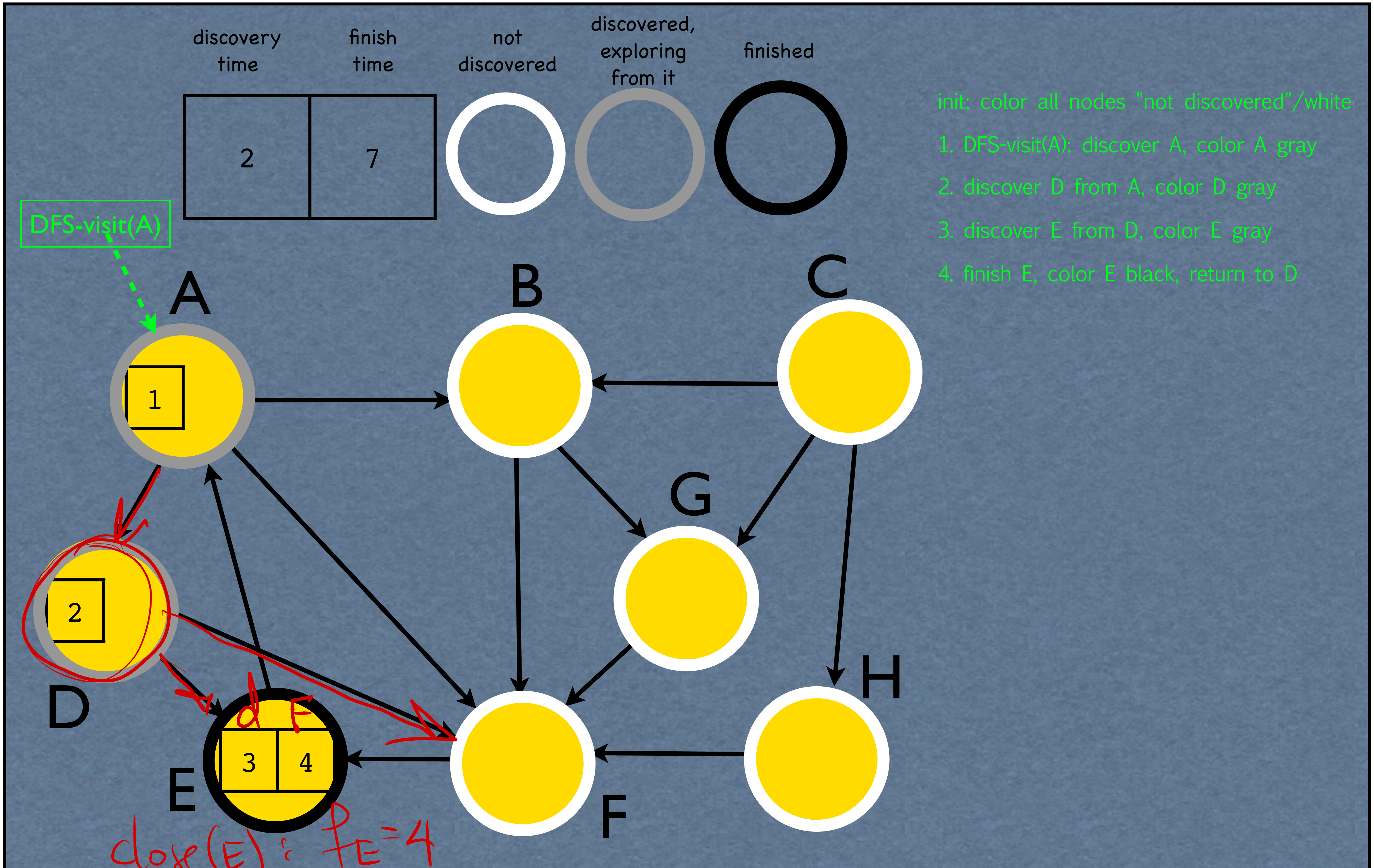

discovery time
finish time
not discovered
discovered, exploring from it
finished
2
7
DFS-visit(A)
A
B
C
G
D
E
F
H
1
2
$d_D=2$
init: color all nodes "not discovered"/white
1. DFS-visit(A): discover A, color A gray
2. discover D from A, color D gray

DFS

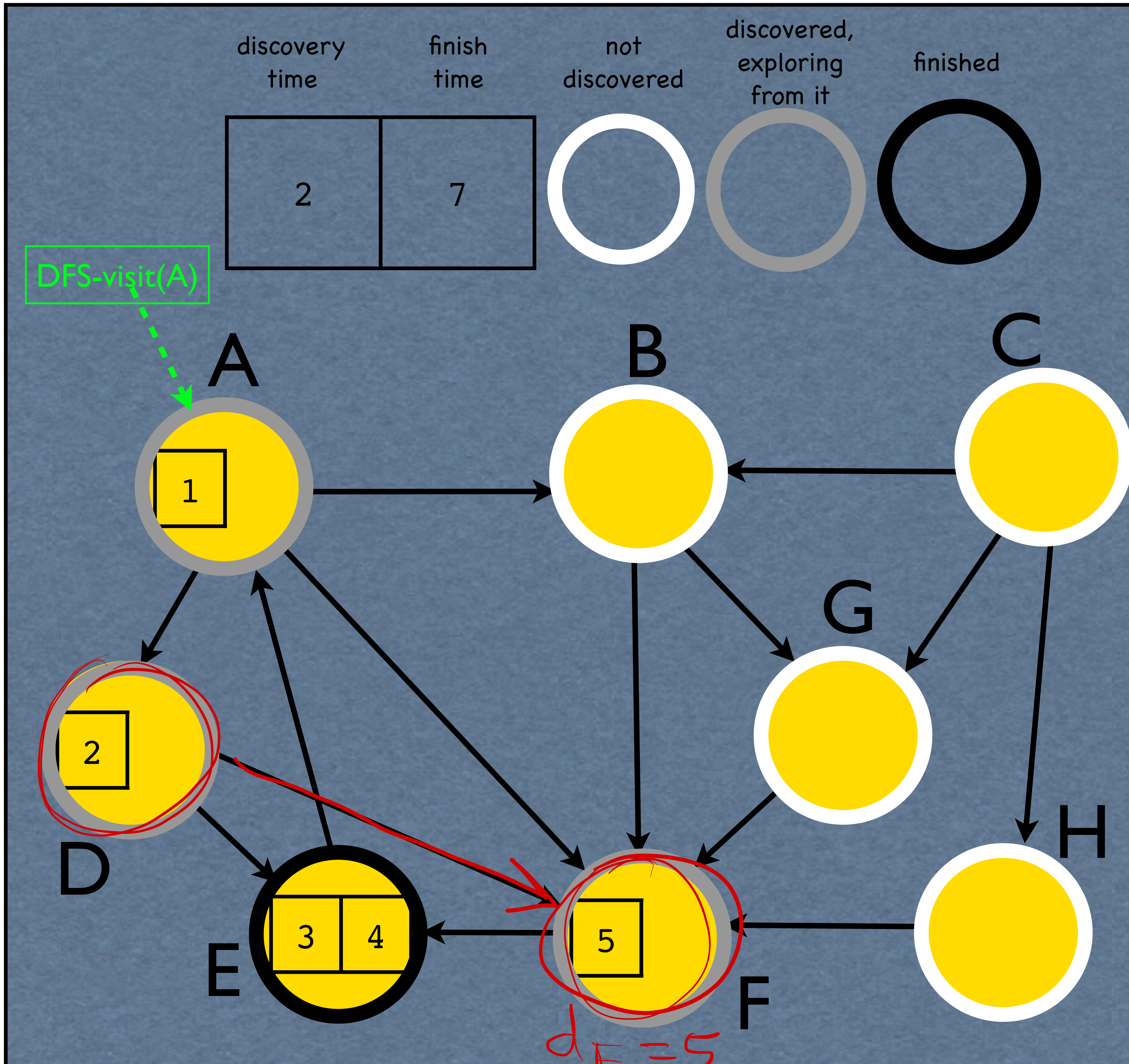


discovery time
finish time
not discovered
discovered, exploring from it
finished
2
7
init: color all nodes "not discovered"/white
1. DFS-visit(A): discover A, color A gray
2. discover D from A, color D gray
3. discover E from D, color E gray
DFS-visit(A)
A
B
C
D
E
F
G
H
1
2
3
back edge gray → gray
$d_E = 3$

DFS


discovery time
finish time
not discovered
discovered, exploring from it
finished
2
7
DFS-visit(A)
A
B
C
D
E
F
G
H
1
2
3
4
d
f
close(E) : $f_E = 4$
init: color all nodes "not discovered"/white
1. DFS-visit(A): discover A, color A gray
2. discover D from A, color D gray
3. discover E from D, color E gray
4. finish E, color E black, return to D

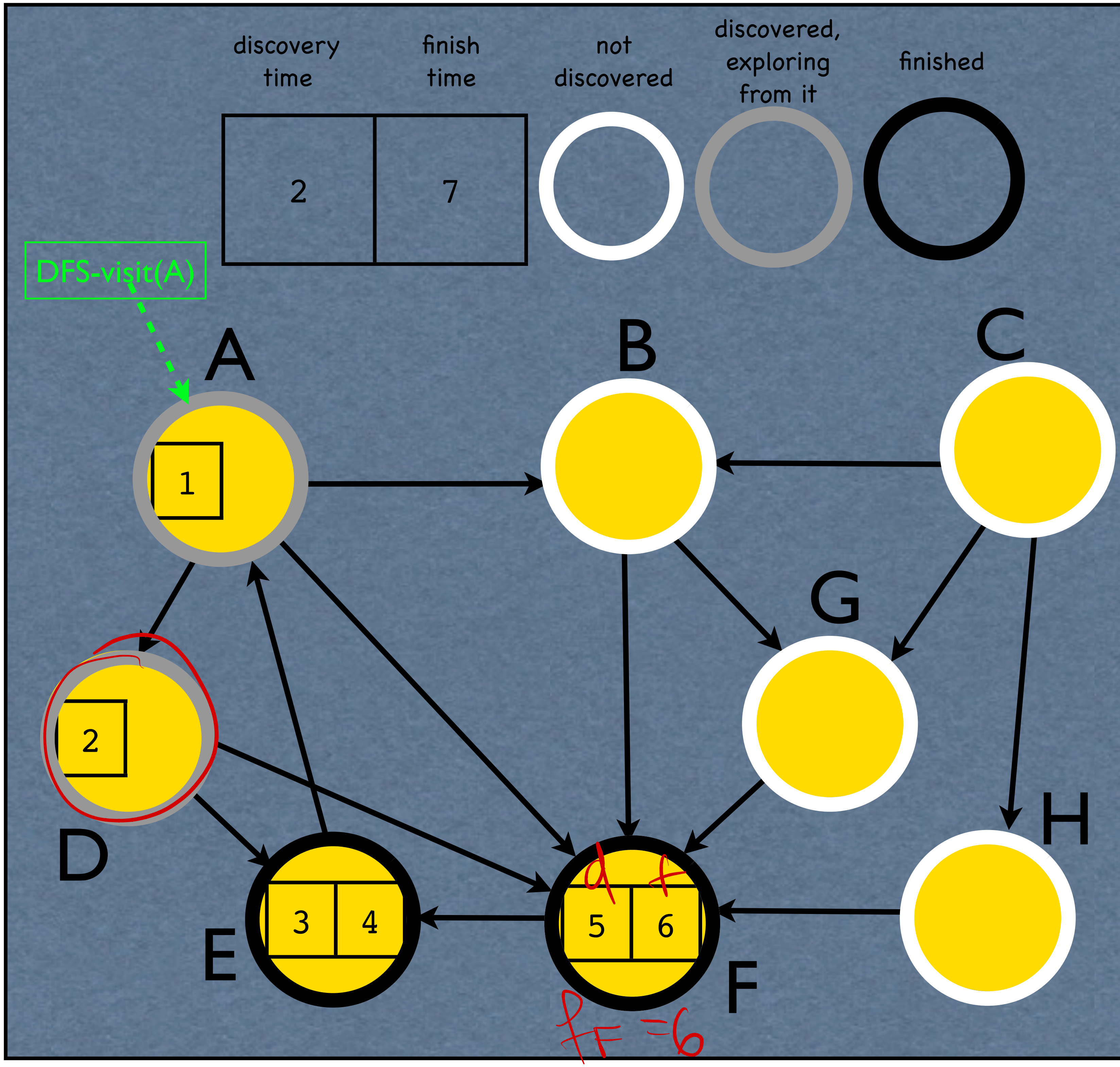
DFS



init: color all nodes "not discovered"/white

1. DFS-visit(A): discover A, color A gray
2. discover D from A, color D gray
3. discover E from D, color E gray
4. finish E, color E black, return to D
5 discover F from D, color F gray

DFS



init: color all nodes "not discovered"/white

1. DFS-visit(A): discover A, color A gray

2. discover D from A, color D gray

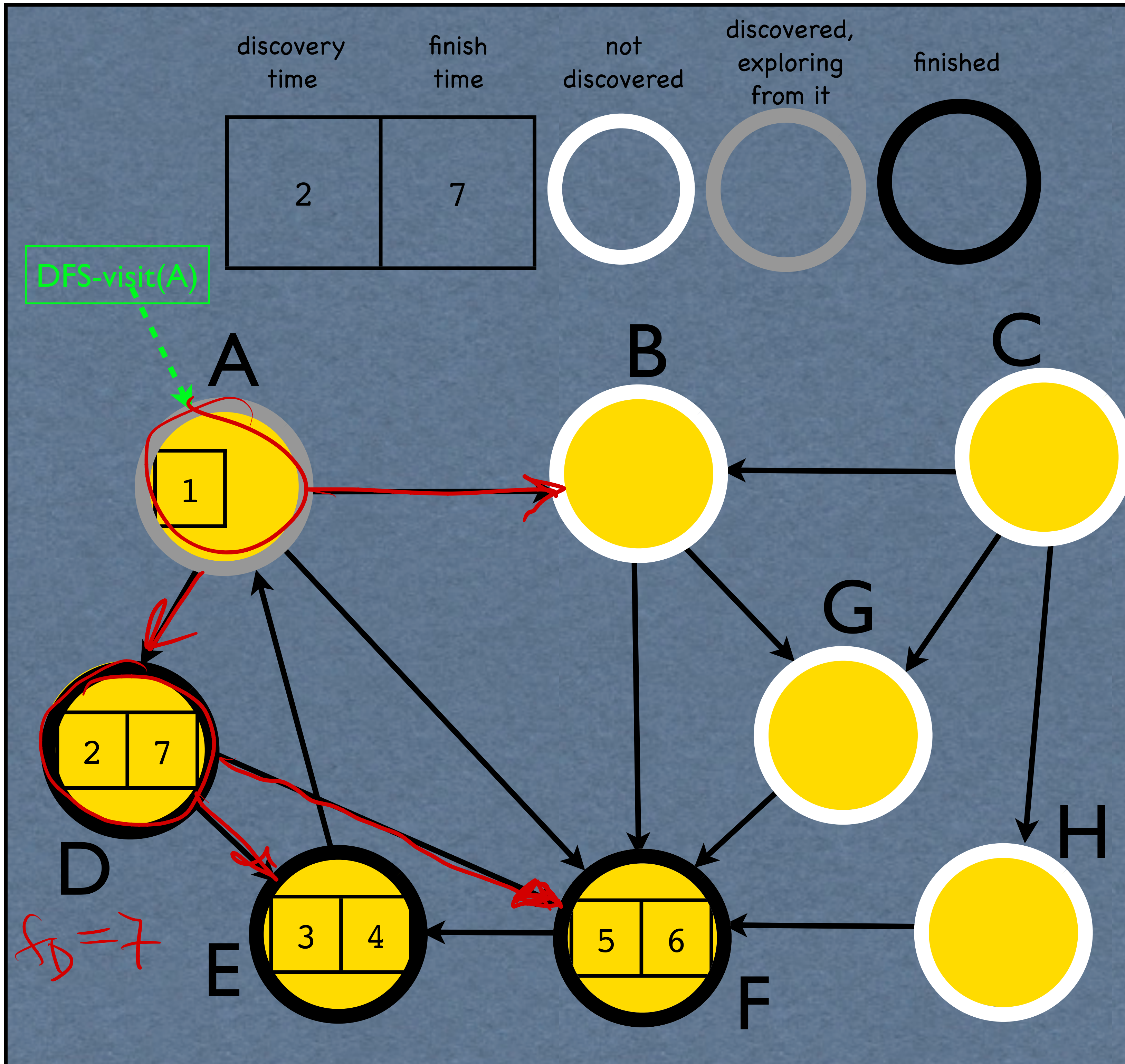
3. discover E from D, color E gray

4. finish E, color E black, return to D

5 discover F from D, color F gray

6 finish F, color F black, return to D

DFS



init: color all nodes "not discovered"/white

1. DFS-visit(A): discover A, color A gray

2. discover D from A, color D gray

3. discover E from D, color E gray

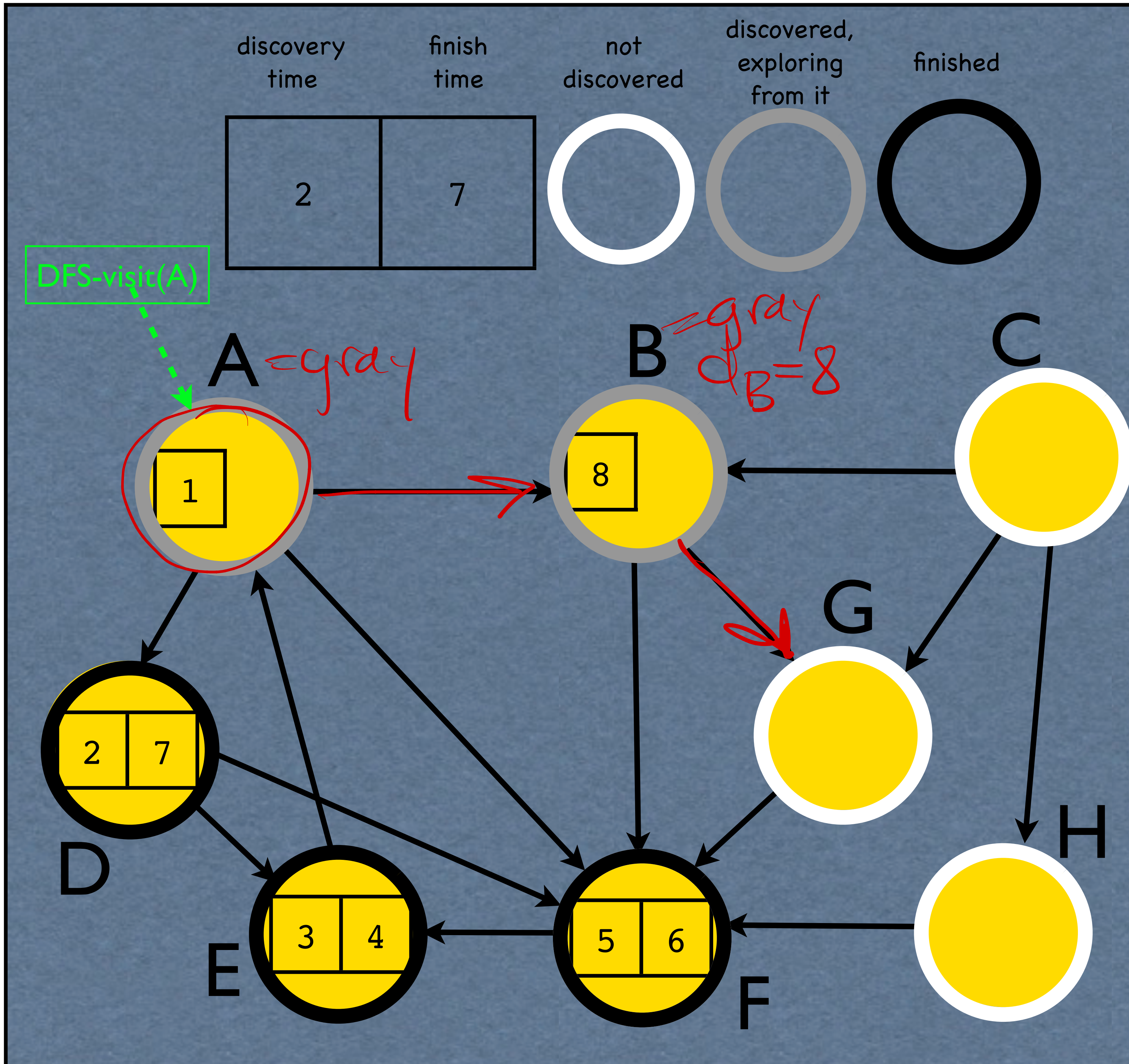
4. finish E, color E black, return to D

5 discover F from D, color F gray

6 finish F, color F black, return to D

7 finish D, color D black, return to A

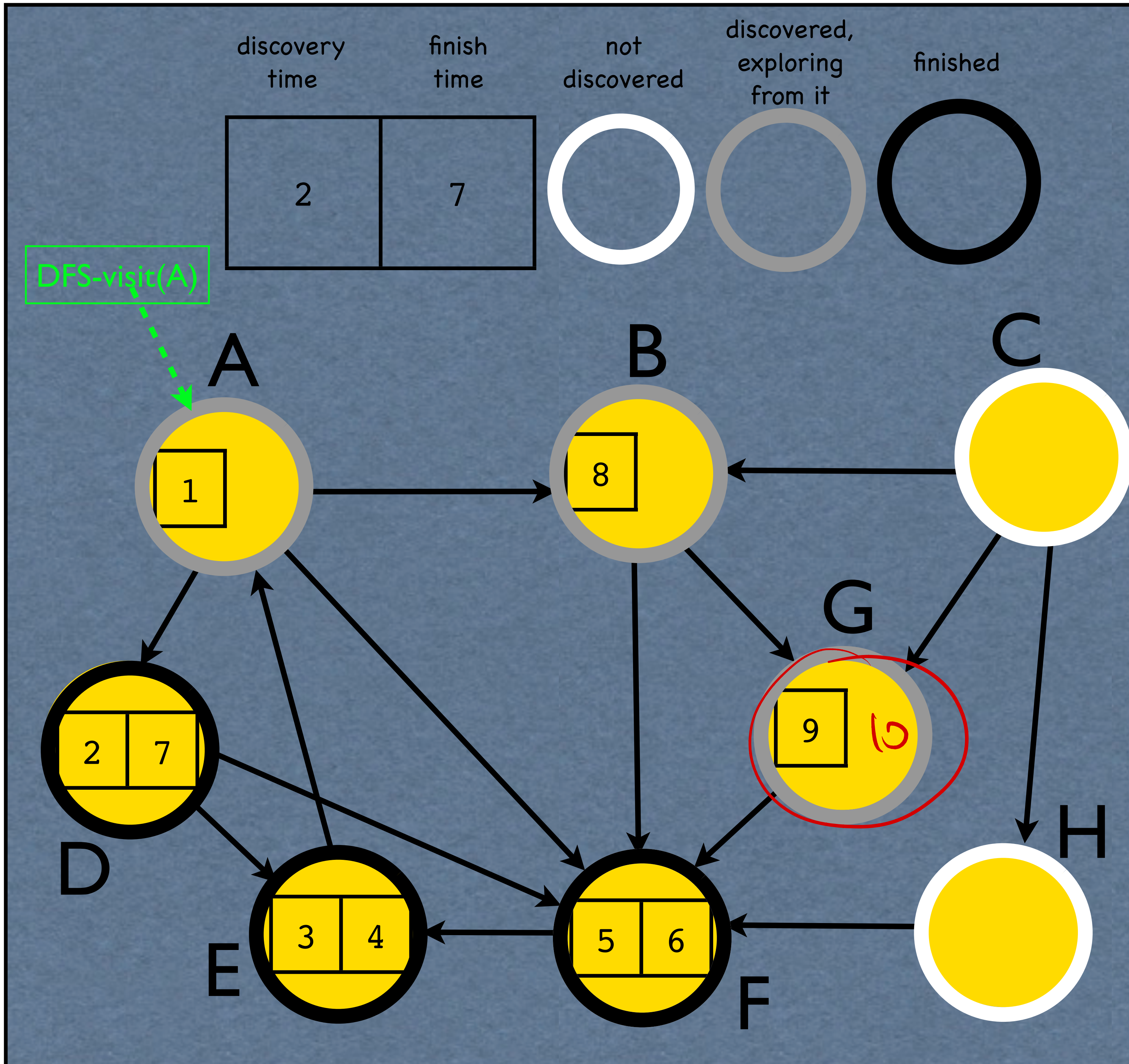
DFS



init: color all nodes "not discovered"/white

1. DFS-visit(A): discover A, color A gray
2. discover D from A, color D gray
3. discover E from D, color E gray
4. finish E, color E black, return to D
5 discover F from D, color F gray
6 finish F, color F black, return to D
7 finish D, color D black, return to A
8 discover B from A, color B gray

DFS



init: color all nodes "not discovered"/white

1. DFS-visit(A): discover A, color A gray

2. discover D from A, color D gray

3. discover E from D, color E gray

4. finish E, color E black, return to D

5 discover F from D, color F gray

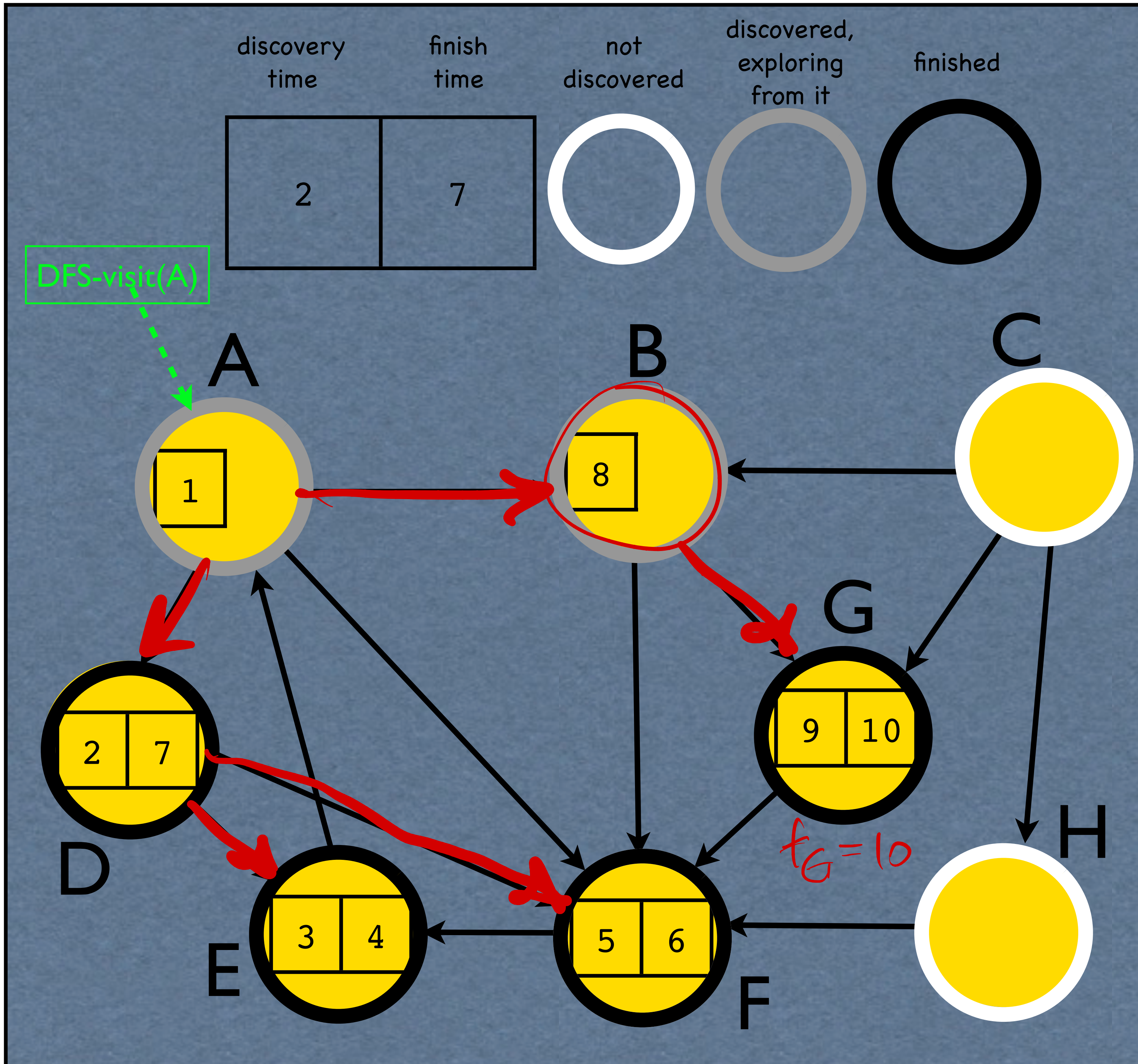
6 finish F, color F black, return to D

7 finish D, color D black, return to A

8 discover B from A, color B gray

9 discover G from B, color G gray

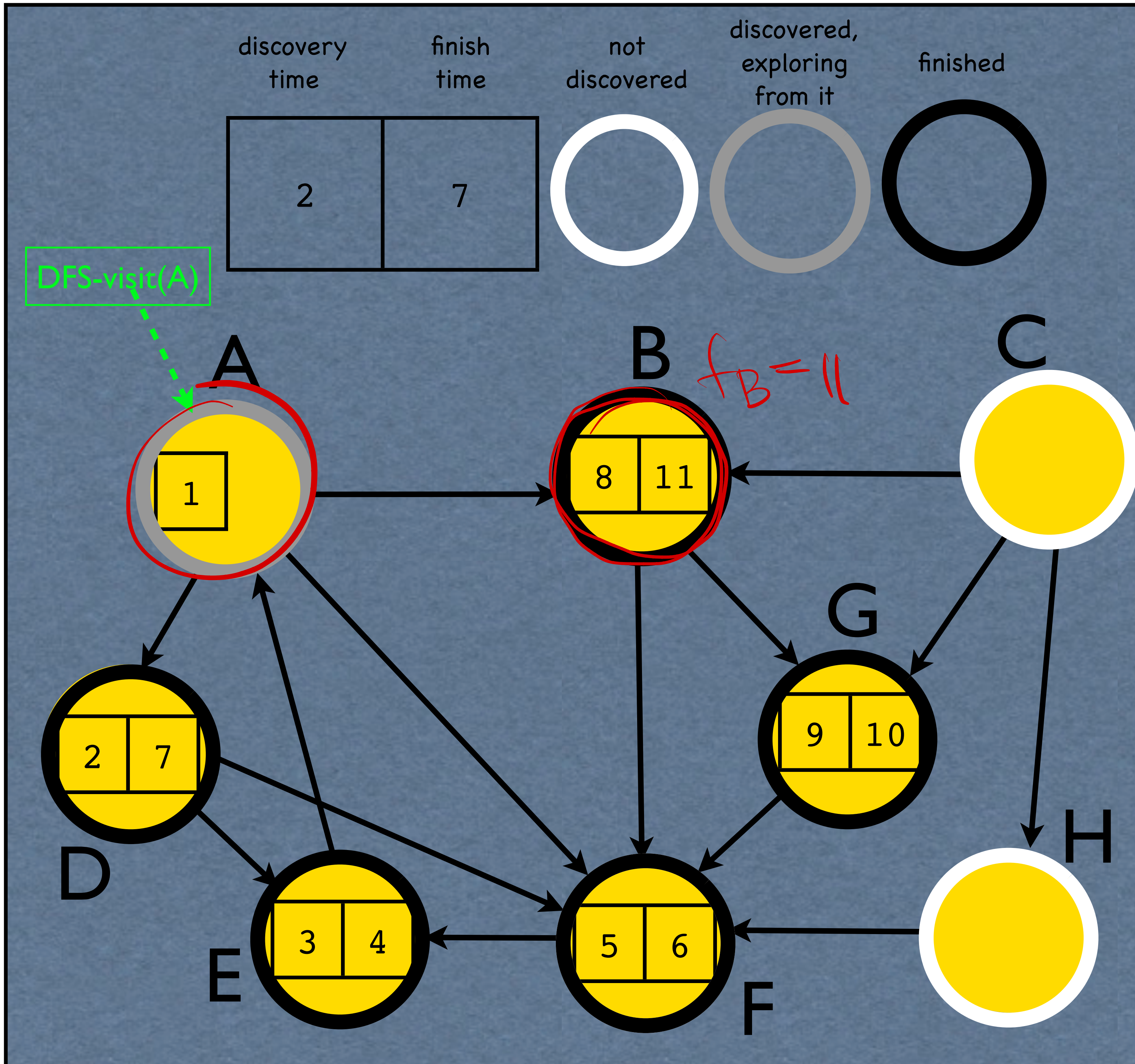
DFS



init: color all nodes "not discovered"/white

1. DFS-visit(A): discover A, color A gray
2. discover D from A, color D gray
3. discover E from D, color E gray
4. finish E, color E black, return to D
5 discover F from D, color F gray
6 finish F, color F black, return to D
7 finish D, color D black, return to A
8 discover B from A, color B gray
9 discover G from B, color G gray
10 finish G, color G black, return to B

DFS



init: color all nodes "not discovered"/white

1. DFS-visit(A): discover A, color A gray

2. discover D from A, color D gray

3. discover E from D, color E gray

4. finish E, color E black, return to D

5 discover F from D, color F gray

6 finish F, color F black, return to D

7 finish D, color D black, return to A

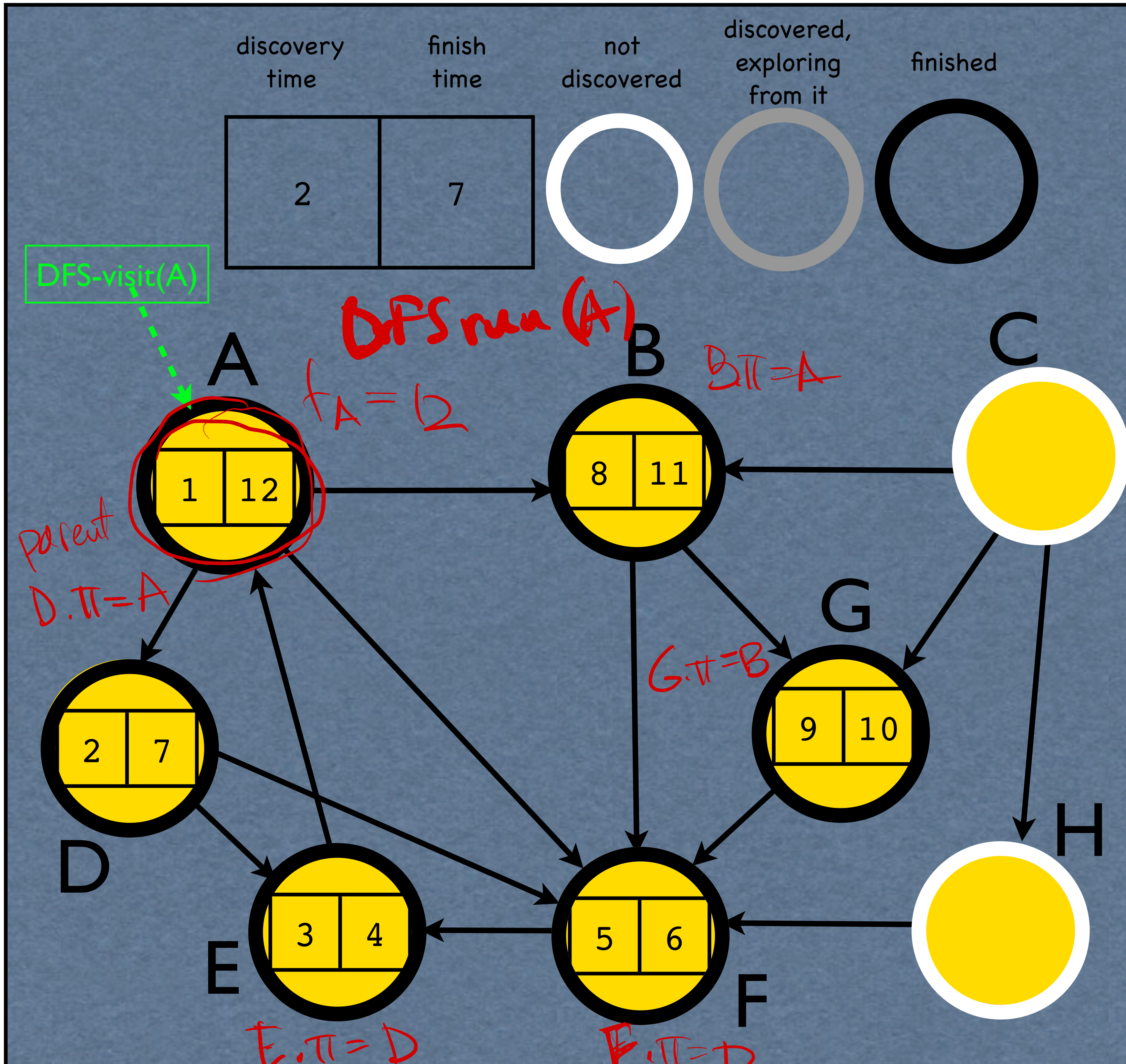
8 discover B from A, color B gray

9 discover G from B, color G gray

10 finish G, color G black, return to B

11 finish B, color B black, return to A

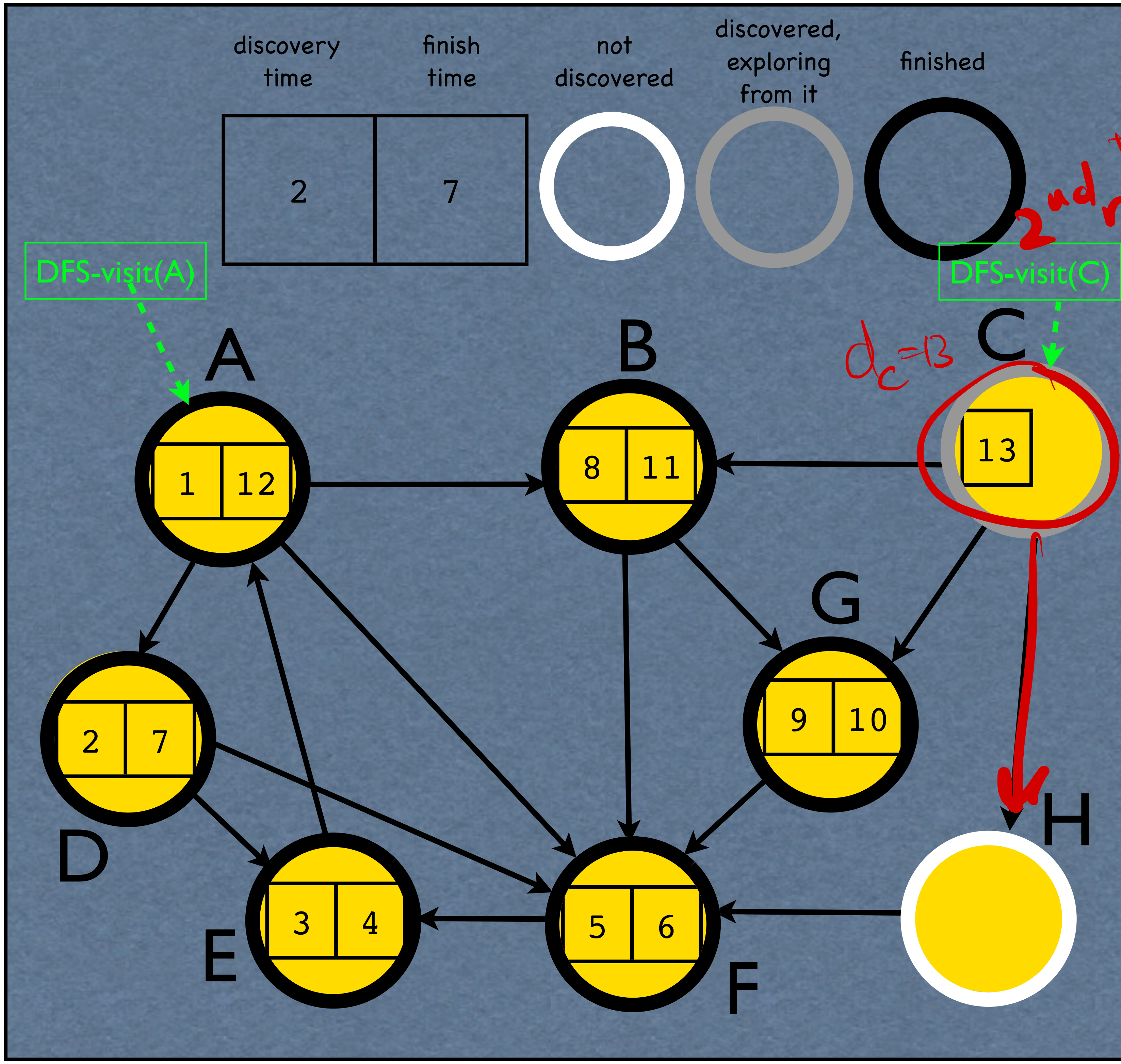
DFS



init: color all nodes "not discovered"/white

1. DFS-visit(A): discover A, color A gray
2. discover D from A, color D gray
3. discover E from D, color E gray
4. finish E, color E black, return to D
5. discover F from D, color F gray
6. finish F, color F black, return to D
7. finish D, color D black, return to A
8. discover B from A, color B gray
9. discover G from B, color G gray
10. finish G, color G black, return to B
11. finish B, color B black, return to A
12. finish A, color A black, done DFS-visit(A)

DFS



init: color all nodes "not discovered"/white

1. DFS-visit(A): discover A, color A gray

2. discover D from A, color D gray

3. discover E from D, color E gray

4. finish E, color E black, return to D

5 discover F from D, color F gray

6 finish F, color F black, return to D

7 finish D, color D black, return to A

8 discover B from A, color B gray

9 discover G from B, color G gray

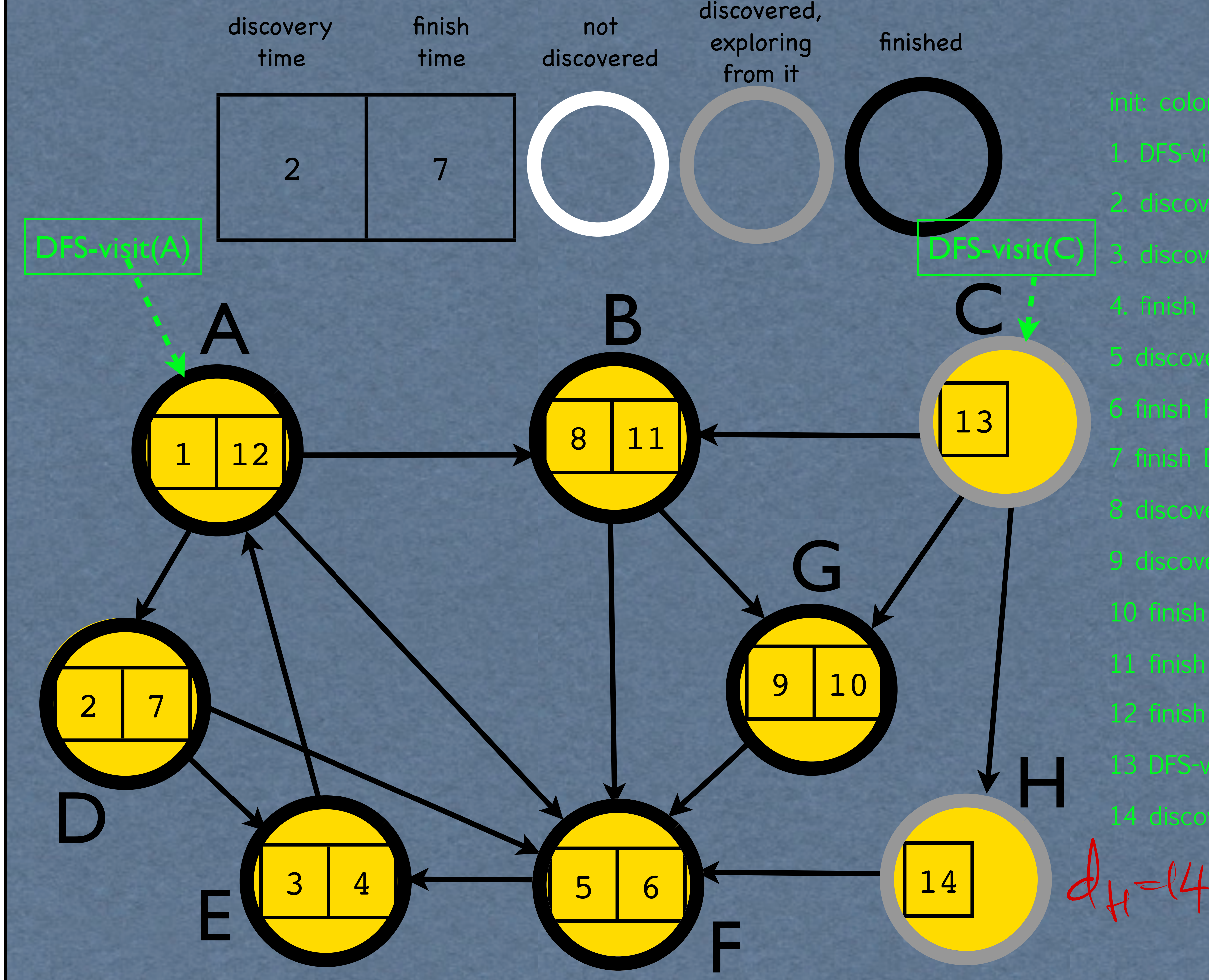
10 finish G, color G black, return to B

11 finish B, color B black, return to A

12 finish A, color A black, done DFS-visit(A)

13 DFS-visit (C), discover C, color C gray

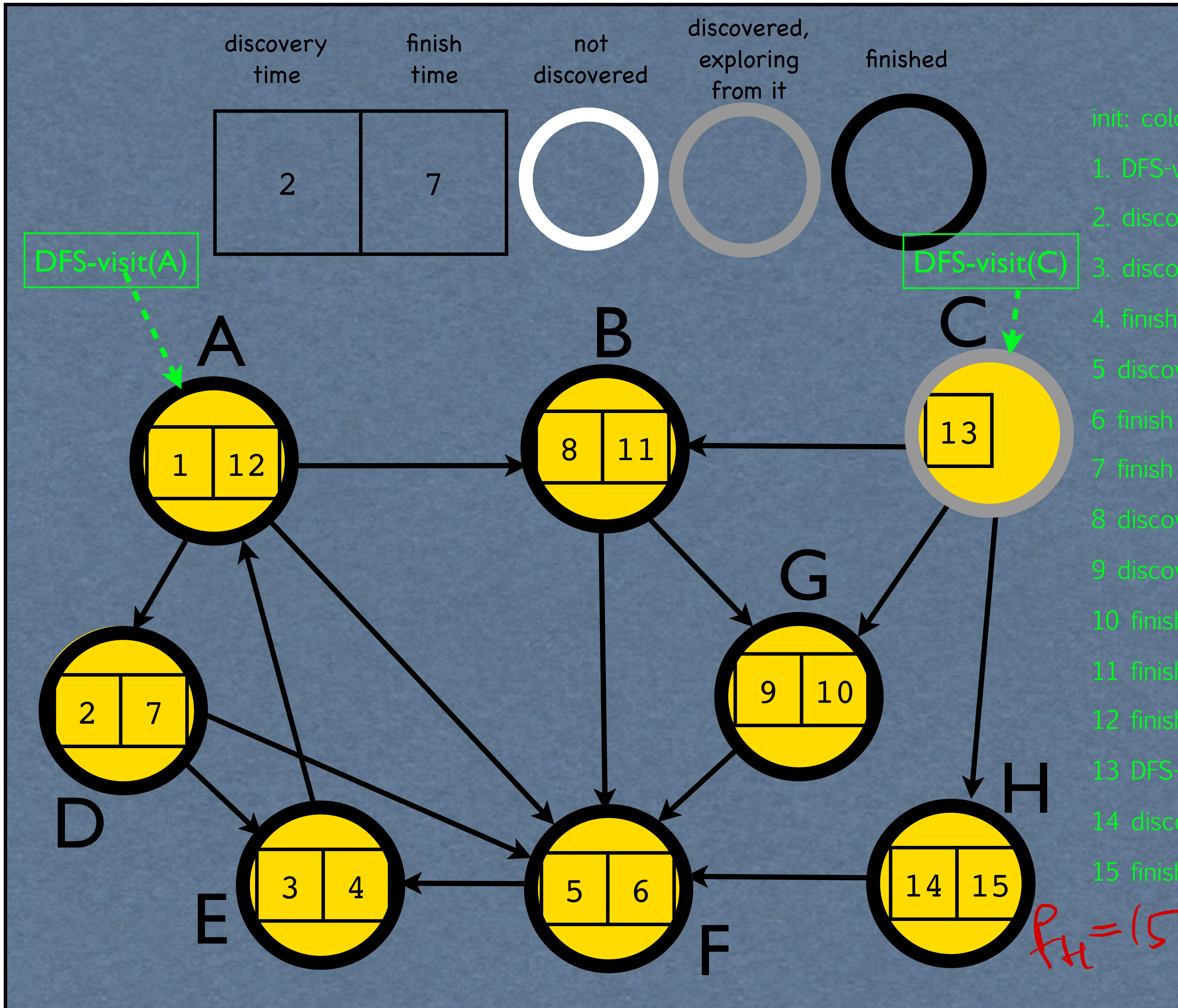
DFS



init: color all nodes "not discovered"/white

1. DFS-visit(A): discover A, color A gray
2. discover D from A, color D gray
3. discover E from D, color E gray
4. finish E, color E black, return to D
5 discover F from D, color F gray
6 finish F, color F black, return to D
7 finish D, color D black, return to A
8 discover B from A, color B gray
9 discover G from B, color G gray
10 finish G, color G black, return to B
11 finish B, color B black, return to A
12 finish A, color A black, done DFS-visit(A)
13 DFS-visit (C), discover C, color C gray
14 discover H from C, color H gray

DFS



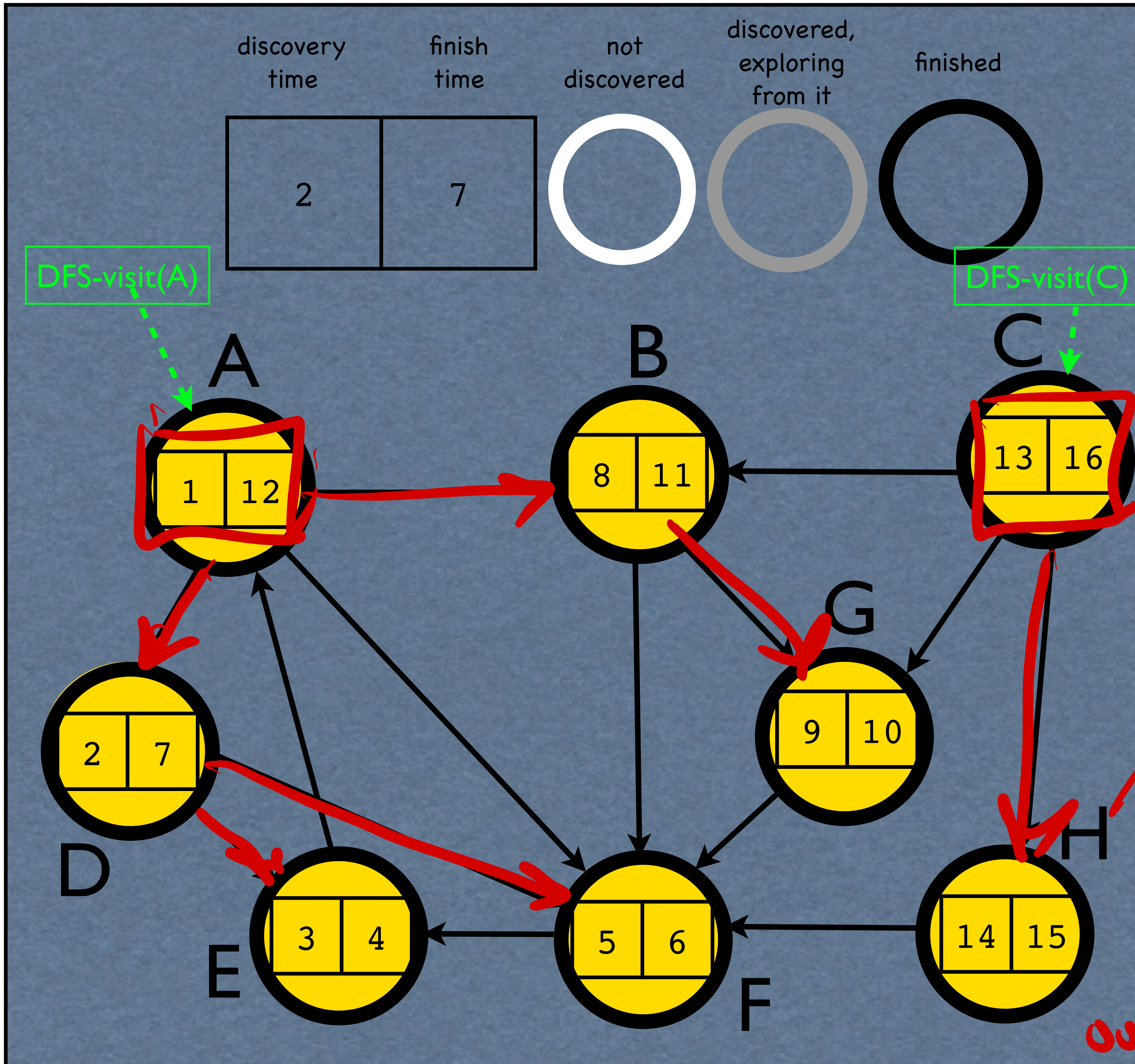
init: color all nodes "not discovered"/white

1. DFS-visit(A): discover A, color A gray
2. discover D from A, color D gray
3. discover E from D, color E gray
4. finish E, color E black, return to D
5 discover F from D, color F gray
6 finish F, color F black, return to D
7 finish D, color D black, return to A
8 discover B from A, color B gray
9 discover G from B, color G gray
10 finish G, color G black, return to B
11 finish B, color B black, return to A
12 finish A, color A black, done DFS-visit(A)
13 DFS-visit (C), discover C, color C gray
14 discover H from C, color H gray
15 finish H, color H black, return to C

DFS

$O(E)$

$O(E+V)$



$x < t < v < y$

d / f

DFS tree

init: color all nodes "not discovered"/white

1. DFS-visit(A): discover A, color A gray
2. discover D from A, color D gray
3. discover E from D, color E gray
4. finish E, color E black, return to D

5 discover F from D, color F gray

6 finish F, color F black, return to D

7 finish D, color D black, return to A

8 discover B from A, color B gray

9 discover G from B, color G gray

10 finish G, color G black, return to B

11 finish B, color B black, return to A

12 finish A, color A black, done DFS-visit(A)

13 DFS-visit (C), discover C, color C gray

14 discover H from C, color H gray

15 finish H, color H black, return to C

16 finish C, color C black, finish DFS-visit(C)

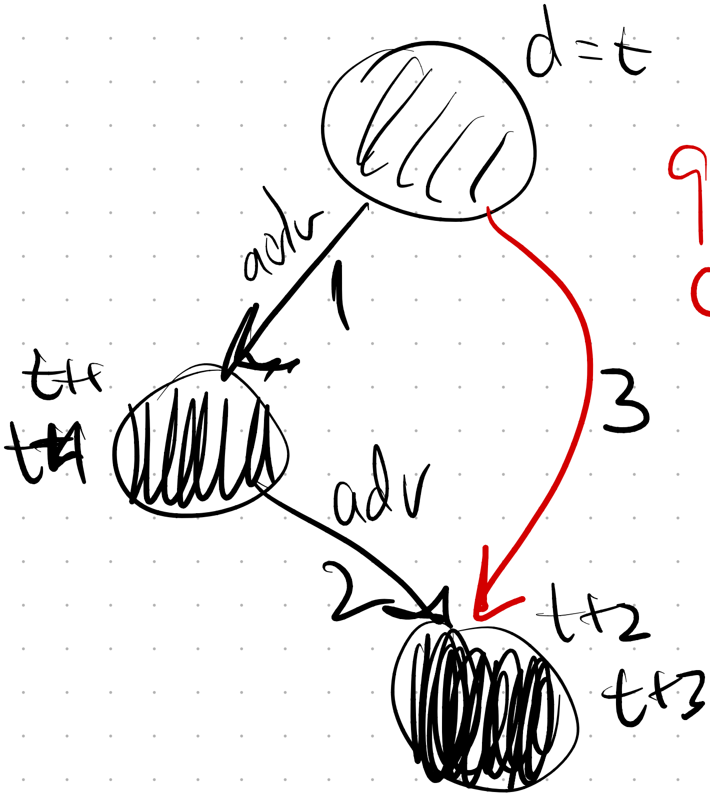
output: d, f arrays
DFS tree(s)

DFS edge classification

DFS, advance

- "tree" edge : from vertices gray to white
  - a tree edge advances the graph exploration/traversal
- "back" edge : from vertices gray to gray
  - a back edge points to a cycle within the current exploration nodes
- "forward" edge : from vertices a(gray) to b(black), if a discovered first
  - discovery_time[a] < discovery_time[b]
  - points to a different part of the tree, already explored from a
- "cross" edge : from vertices a(gray) to b(black), if b discovered first
  - discovery_time[a] > discovery_time[b]
  - points to a different part of the tree, explored before discovering a

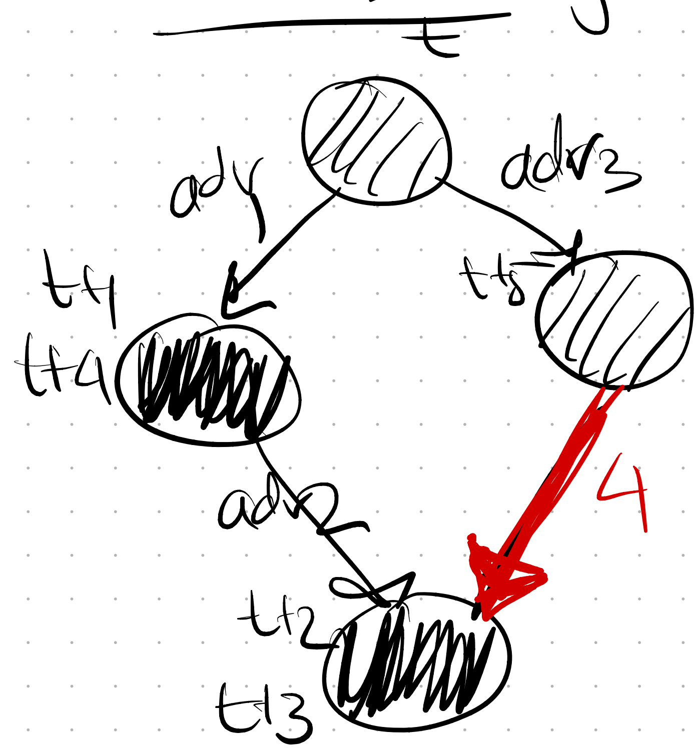
Forward edge



gray → black

d[gray] < d[black]

Cross edge

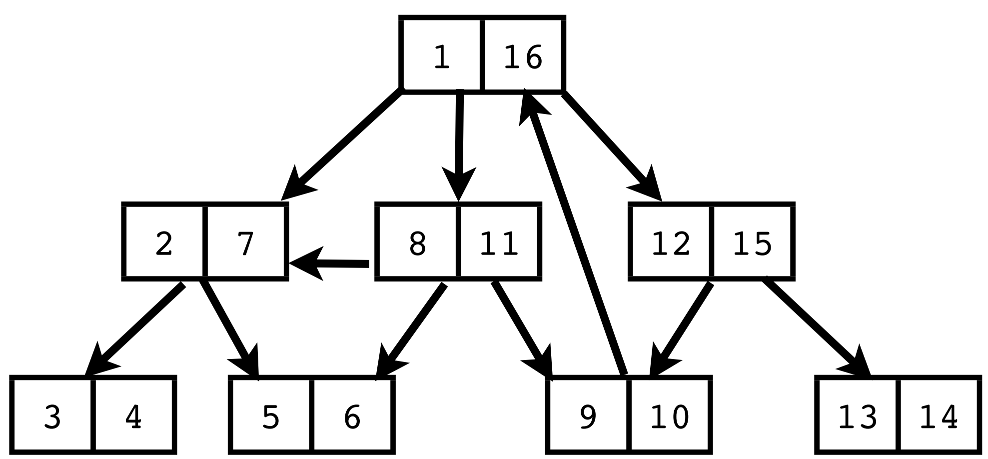


4: gray → black

d[gray] > d[black]

Checkpoint

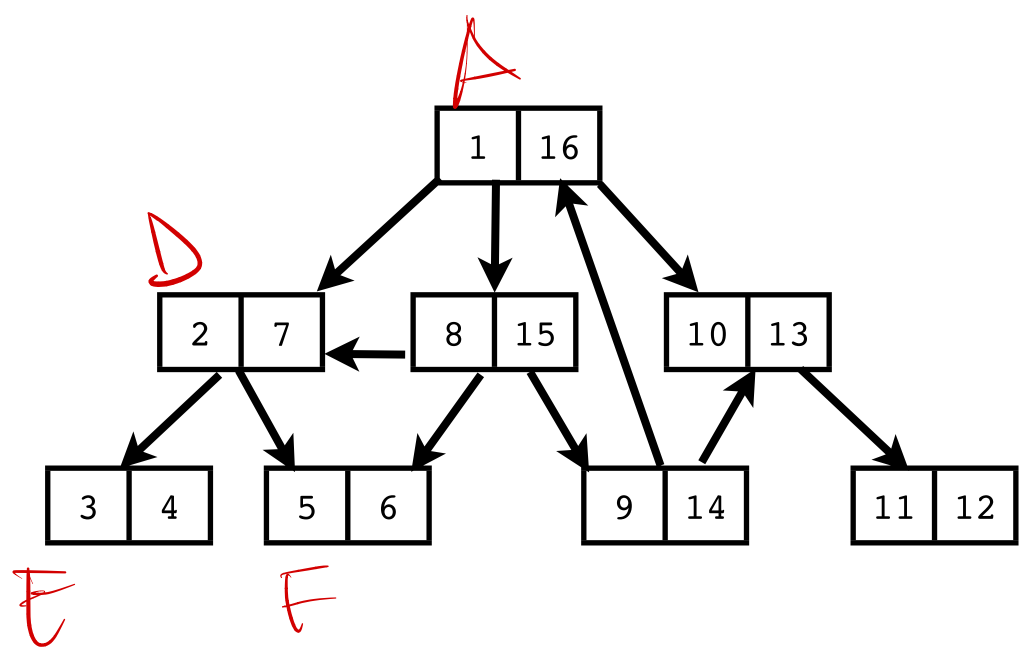
- on the animated example, label each edge as "tree","back", "cross", or "forward"
- do the same on the following example (DFS discovery and finish times marked for each node)



Checkpoint

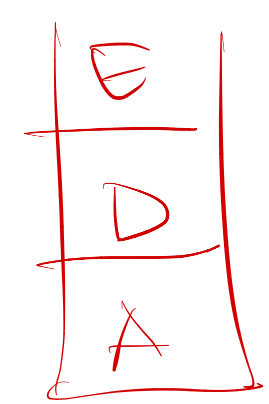
- almost same example, with a small modification: one edge was reversed

DFS with stack, no rec calls.



- Stack = gray nodes in d[] order

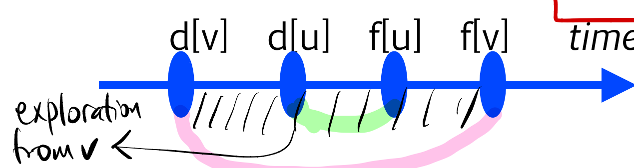
time 3 stack:

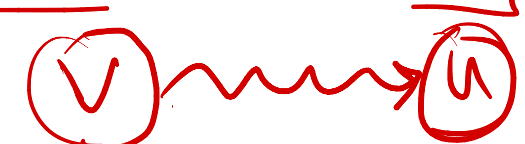


DFS observations

- Running time O(V+E), same as BFS
- vertex v is gray between times discover[v] and finish[v]
- gray time intervals (discover[v], finish[v]) are inclusive of each other
  - (d[v], f[v]) can include (d[u], f[u]) : d[v] < d[u] < f[u] <f[v]

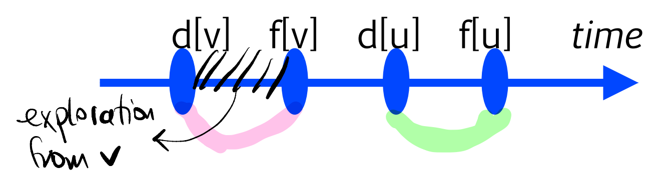
discover u from anchestr v

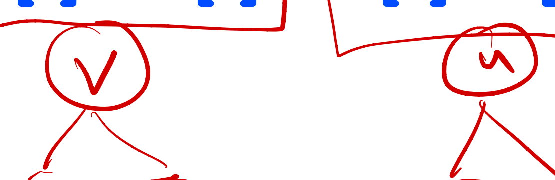




  - (d[v], f[v]) can separate from (d[u], f[u]) : d[v] < f[v] < d[u] <f[u]

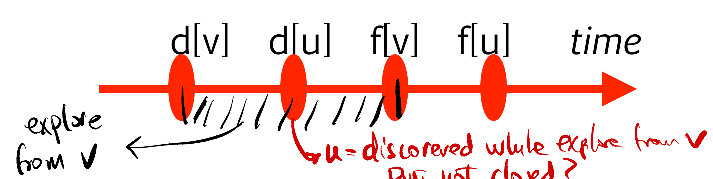
u not reach from v discovered later





  - (d[v], f[v]) cannot intersect (d[u], f[u]) : d(v) < d(u) < f[v] <f[u]

discore u while v=gray (from v)



- graph G=(V,E) is acyclic (does not have cycles) if DFS does not find any "back" edge

Theorem $\left.\begin{array}{l} f[u] > f[v] \\ \boxed{\text{no path } u \rightsquigarrow v} \end{array}\right\} \Rightarrow$ no path $v \rightsquigarrow u$

proof: $f[u] > f[v]$ from previous slide 2 options

? • $d[v] < \underline{f[v]} < d[u] < \underline{f[u]} \Rightarrow$ $u \rightsquigarrow$ path ?

close v before discover u $\Rightarrow$ no path $v \rightsquigarrow u$

OR

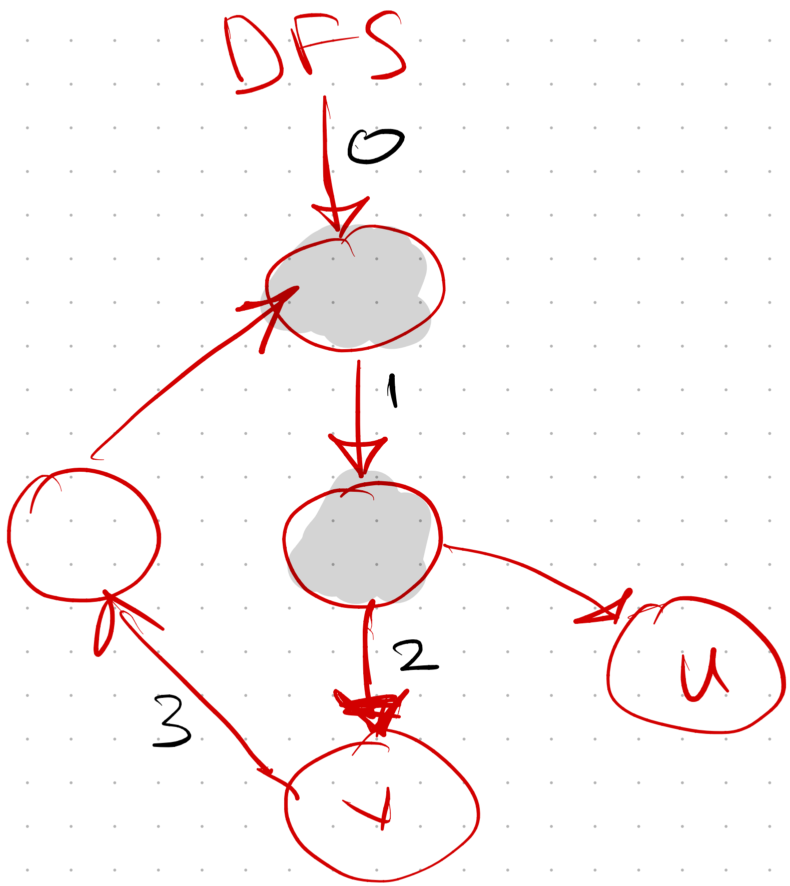
✓ • $d[u] < d[v] < \underline{f[v]} < \underline{f[u]} \Rightarrow$ path $u \rightsquigarrow v$

$\Rightarrow$ contradicts the hypothesis, not possible

$\Rightarrow d[v] < f[v] < d[u] < f[u] \Rightarrow$
no path $v \rightsquigarrow u$ (u discovered after v close)

Look again! $d_v < f_v < d_u < f_u$

white → gray → white



Undirected graphs cycles

- graph G=(V,E) is acyclic (does not have cycles) if DFS does not find any "back" edge
- since G is undirected, no cycles implies $|E| \leq |V|-1$
- running DFS, if we find more than $|V|-1$ edges, there must be a cycle
- Undirected graphs: find-cycles algorithm takes O(V)

Directed graphs cycles

- graph G=(V,E) is acyclic (does not have cycles) if DFS does not find any "back" edge
- for directed graphs, even without cycles they can have more edges, |E| > |V|-1
- algorithm to determine cycles: run DFS, look for back edges - O(V+E) time
- DAG = directed acyclic graph

no cycles

Th ⇒ line sort the DAG: all vertices on line.

all edges left → right

~~Topological~~ sort (DAG)

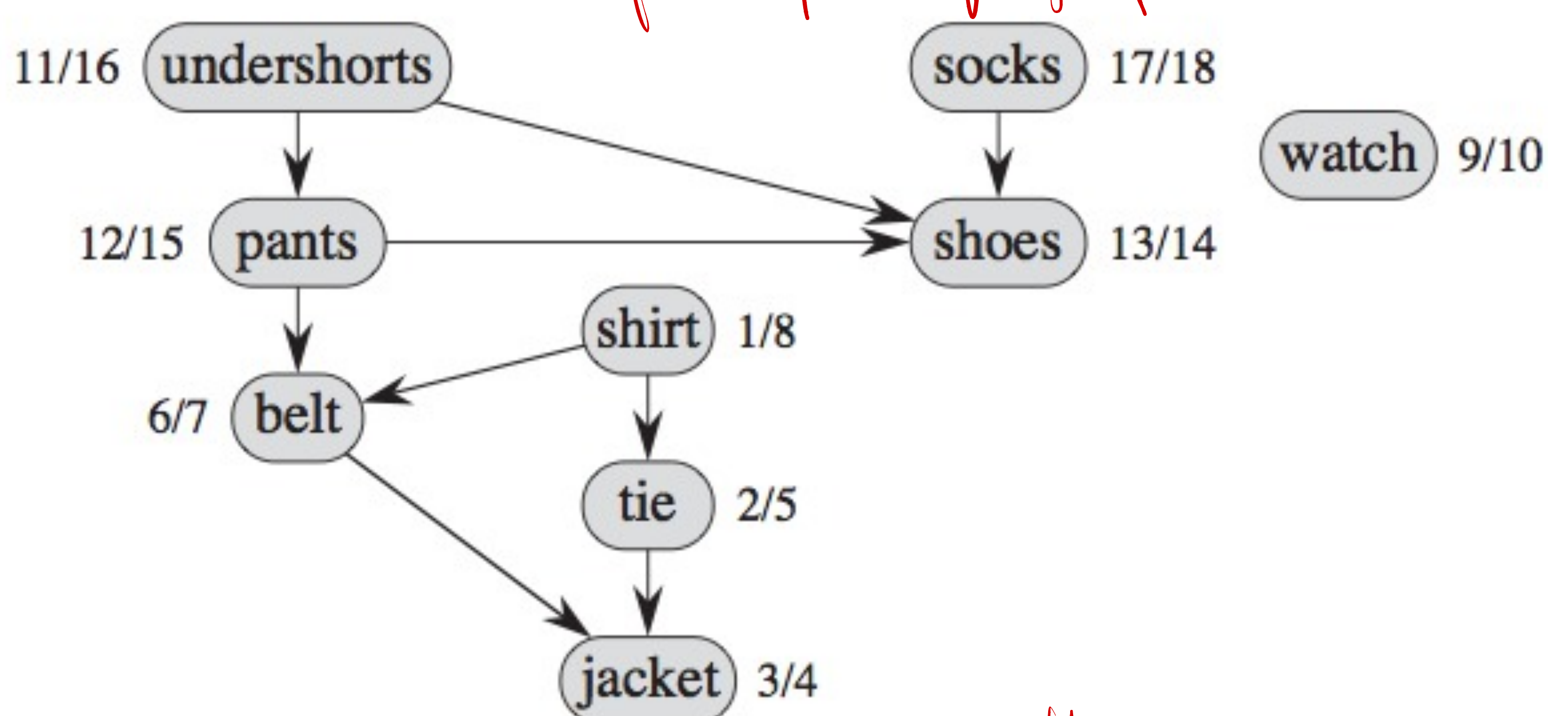
LINE

no cycles

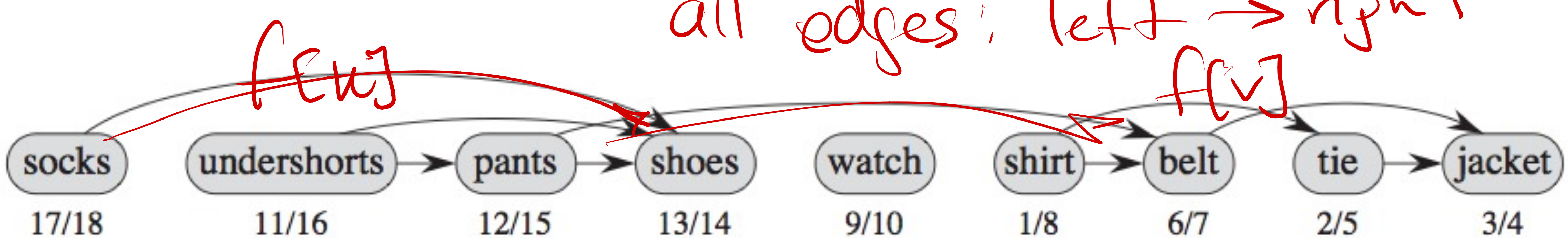
- DAG admits topological sort: all vertices "sorted" on a line, such that all edges point from left to right–no cycles – 2 graphs below are the same–
- to do this: algorithm: run DFS, time O(V+E). Output vertices in reverse order given by finishing time

f[ ]

Example: prereq graphs



all edges: left → right

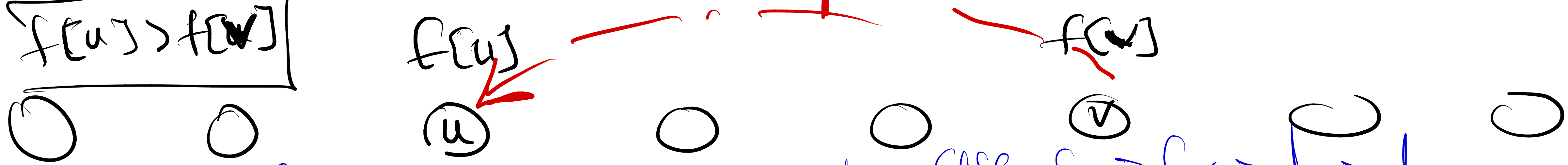


Check Point

sort by f[ ] descending

want : no edge
$v \to u$

$\boxed{f[u] > f[v]}$



case $f_u > d_u > f_v > d_v$
with edge $v \to u \Rightarrow$ u discover from v $\boxed{\text{CONTRAD}}$

case $f_u > f_v > d_v > d_u$
$\Rightarrow \exists$ path $u \rightsquigarrow v$ + edge $(v \to u)$
$\Rightarrow$ cycle $\Rightarrow$ CONTRAD

- how can we use DFS to determine if there is a path from u to v ?

- prove that by sorting vertices in the reverse order of finishing times, we obtained a topological sort

  max f( ) left ; min f( ) right side

  - assuming no cycles

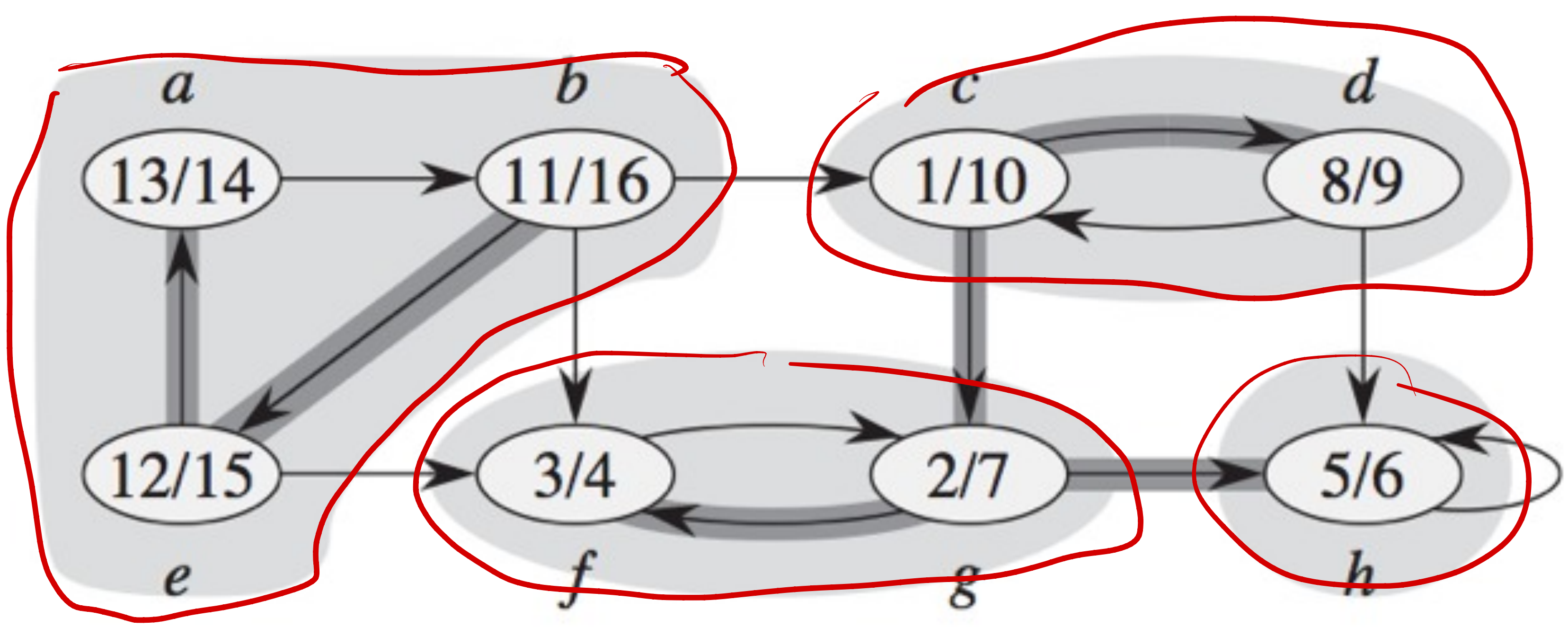
  - in other words, all edges point in the same direction

Thu Blue edges are NOT part of cycles!
$\Rightarrow$ SCC-s form a DAG

blue edges (u,v) : edges across SCC-s.

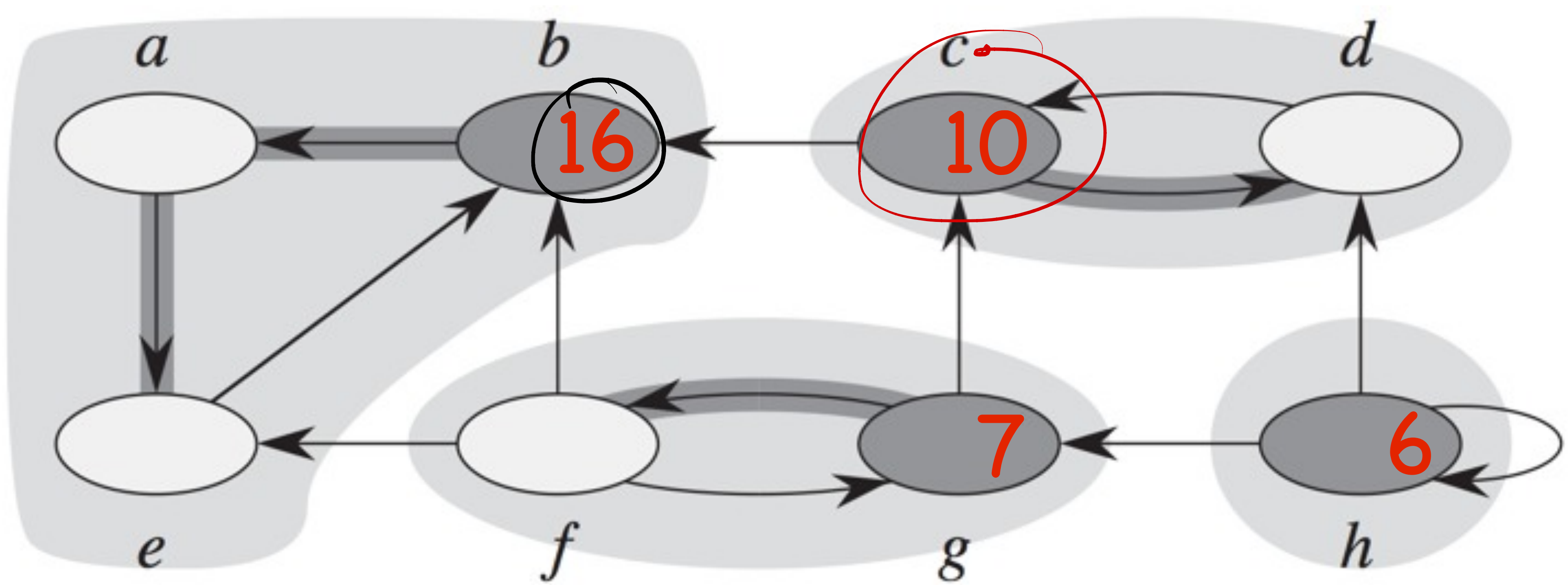
Strongly connected components

- SCC = a set of vertices S⊂V, such that for any two (u,v)∈S, graph G contains a path u⇝v and a path v⇝u
- trivial for undirected graphs
  - all connected vertices are in fact strongly connected
- tricky for directed graphs
- graph below has the DFS discover/finish times and marked 4 strongly connected components; "tree" edges highlighted
- between two SCC, A and B, there cannot exists paths both ways (A∋u⇝v∈B and B∋v′⇝u′∈A)
  - paths both ways would make A and B a single SCC



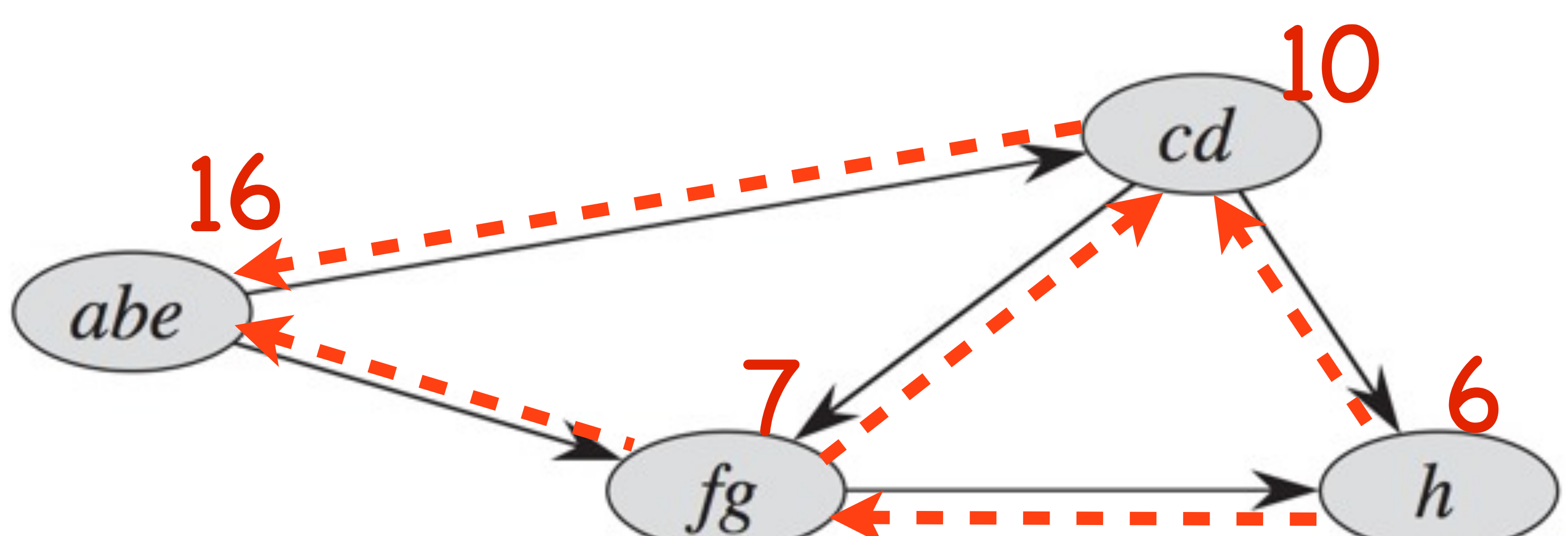
Strongly connected components

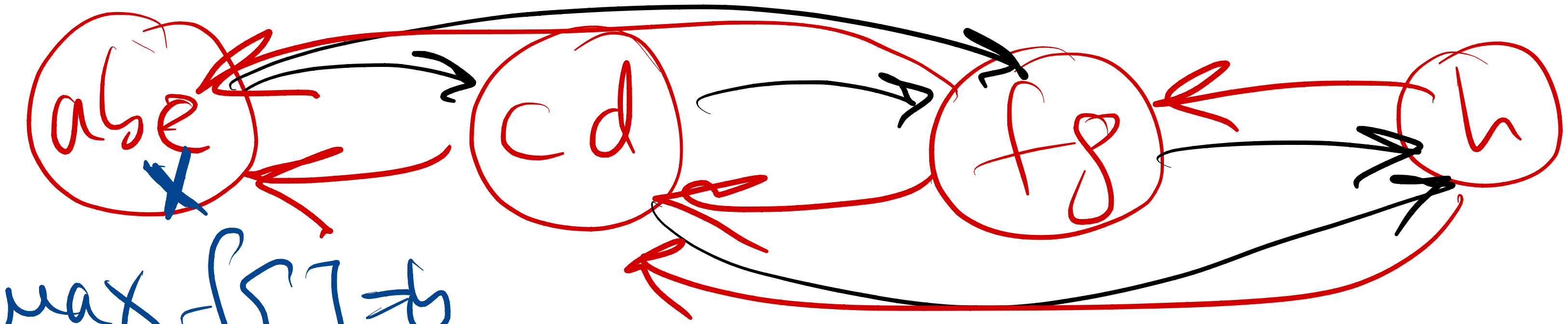
- run 1st DFS on G to get finishing times f[u]
- run 2nd time DFS on G-reversed (all edges reversed -see picture), each DFS-visit in reverse order of f[u]
  DFS-run = SCC(root)
  - finishing times marked in red for the DFS-visit root vertices
- output each tree (vertices reached) obtained by 2nd DFS as an SCC



Strongly connected components

- why 2nd DFS produces precisely the SCC -s?
- SCC-graph of G: collapse all SCC into one SCC-vertex, keep edges between the SCC-vertices
- - SCC graph is a DAG;
  - contradiction argument: a cycle on the SCC-graph would immediately collapse the cycle's SCC-s into one SCC
- reversed edges (shown in red); reversed-SCC-graph also a DAG
- second DFS runs on reversed-edges (red); once it starts at a high-finish-time (like 16) it can only go through vertices in the same SCC (like abe)





$x = \max f[\,] \to b$

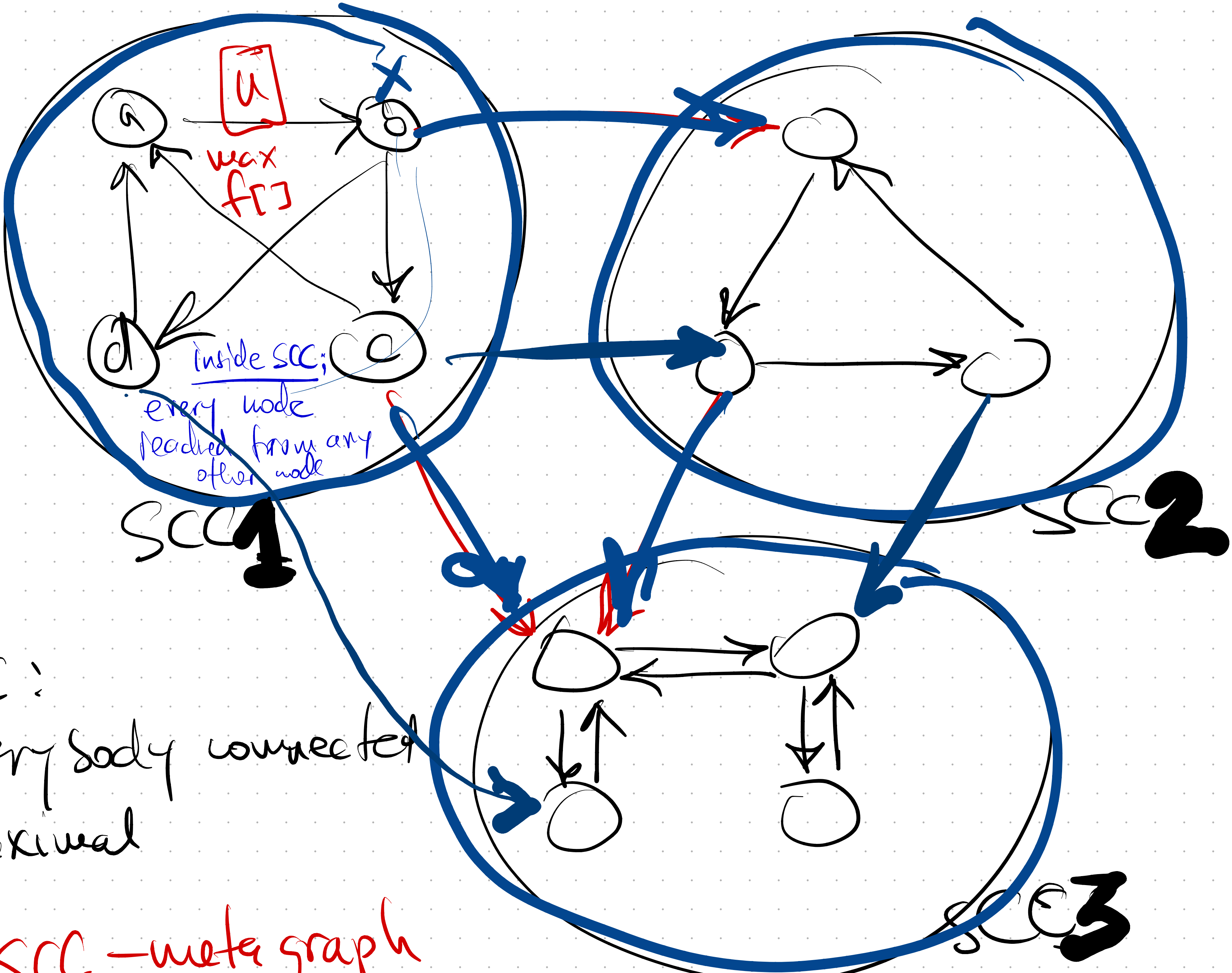
first DFS

Th version 2

$f[u] > f[v]$ (first DFS run)
$u, v \notin$ same SCC
$\Rightarrow$ no path $v \rightsquigarrow u$

Minimum Spanning Trees

Lesson 2

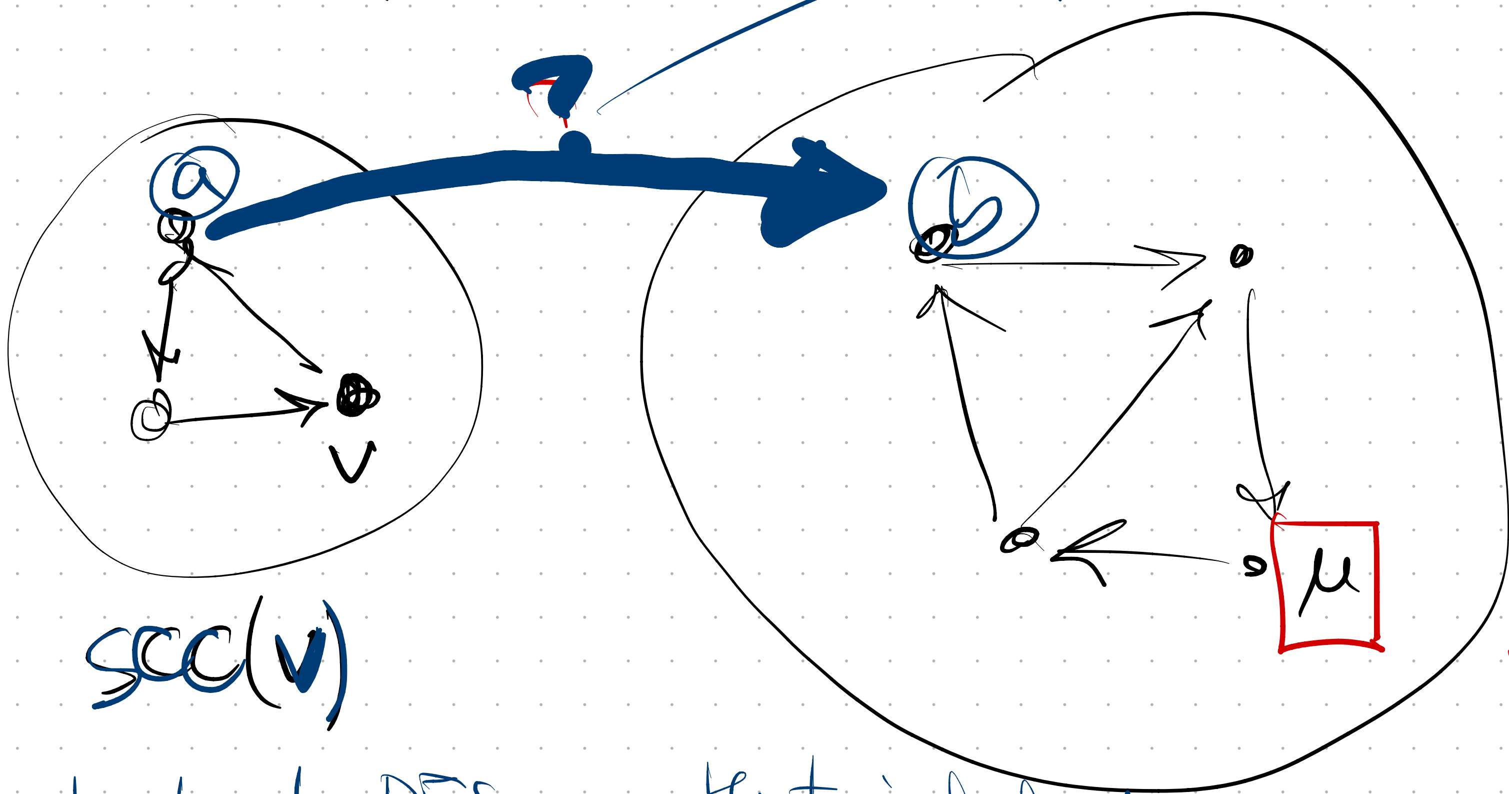


SCC:
- everybody connected
- maximal

Th SCC-metagraph
nodes = SCC
edges = between SCC $\Rightarrow$ DAG

Th: Run DFS look at node u with max f_u

$\Rightarrow$ SCC(u) has no incoming edges from other SCC-s $\rightarrow$ not possible such an edge

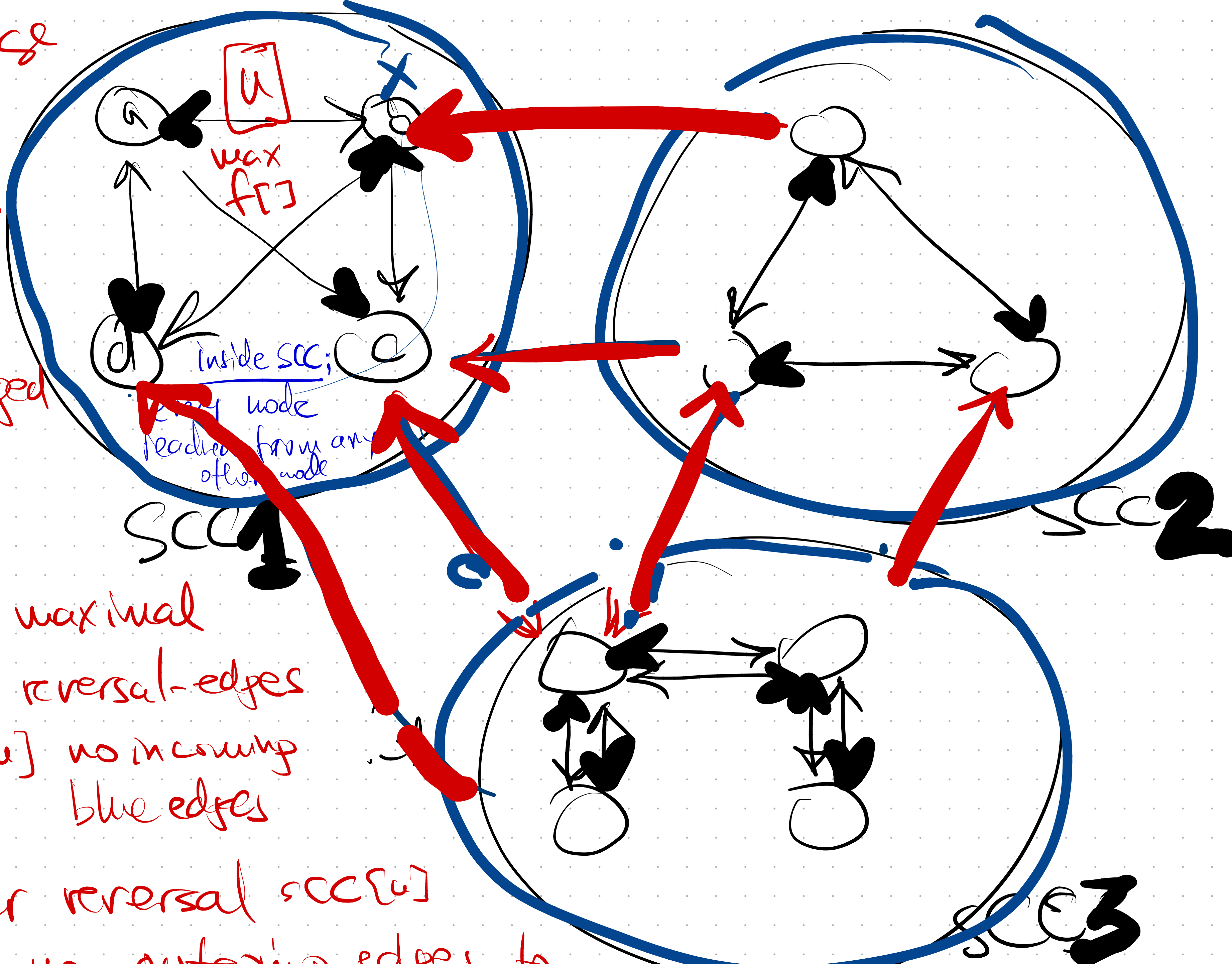


$\boxed{f_u = max}$

Look at DFS run that includes u.

- includes a $\Rightarrow$ $f_a > f_u$ contradict
- does not include a $\Rightarrow$ a is processed in a later DFS run $\Rightarrow$ $f_a > f_u$ contradict.

Reverse all edges
- SCC unchanged



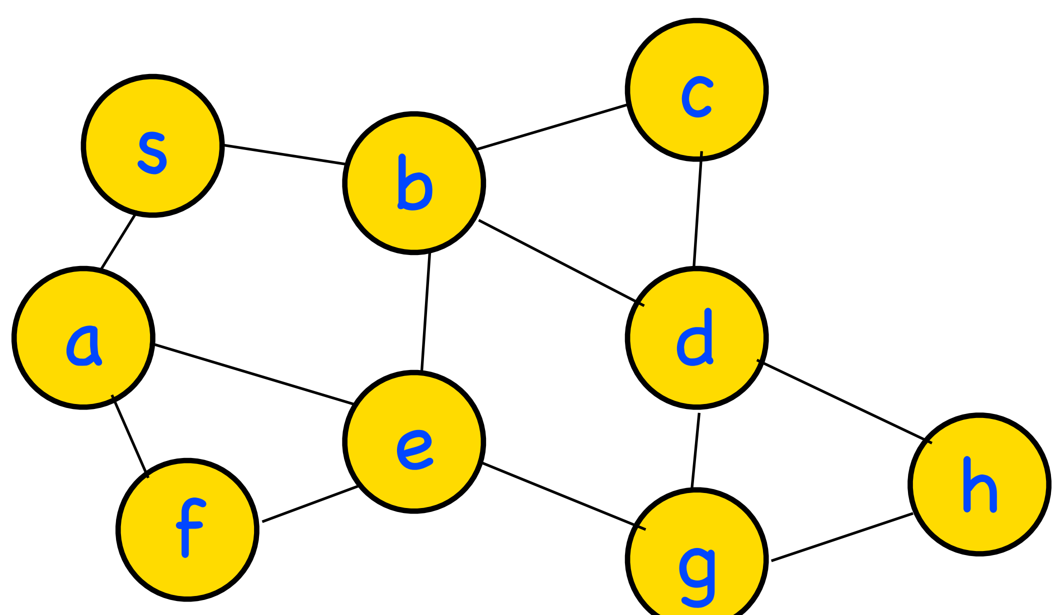
f[u] maximal
- befor reversal - edges SCC[u] no incoming blue edges

=> after reversal SCC[u] has no outgoing edges to other SCC.

=> 2nd DFS(start u) => run 1 SCC[u]

Spanning Trees

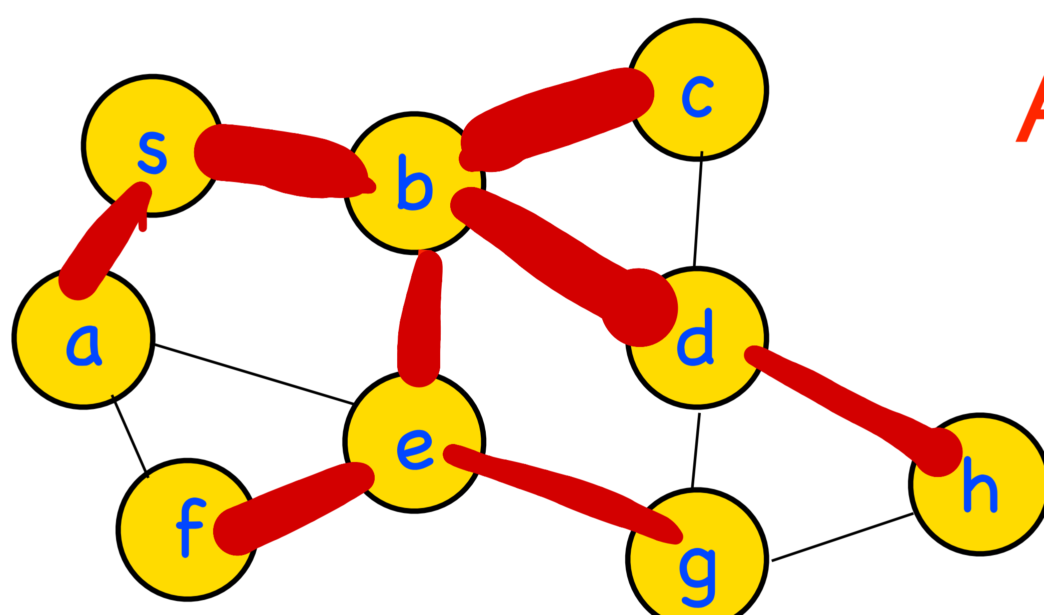
- context : undirected graphs
- a set of edges A that "span" or "touch" all vertices, and forms no cycles
  - necessary this set of edges A has size = $|V|-1$
- spanning tree: the tree formed by the set of spanning edges together with vertex set $T = (V,F)$



Spanning Trees

- context : undirected graphs
- a set of edges A that "span" or "touch" all vertices, and forms no cycles
  - necessary this set of edges A has size = |V|-1
- spanning tree: the tree formed by the set of spanning edges together with vertex set T = (V,F)

no cycles



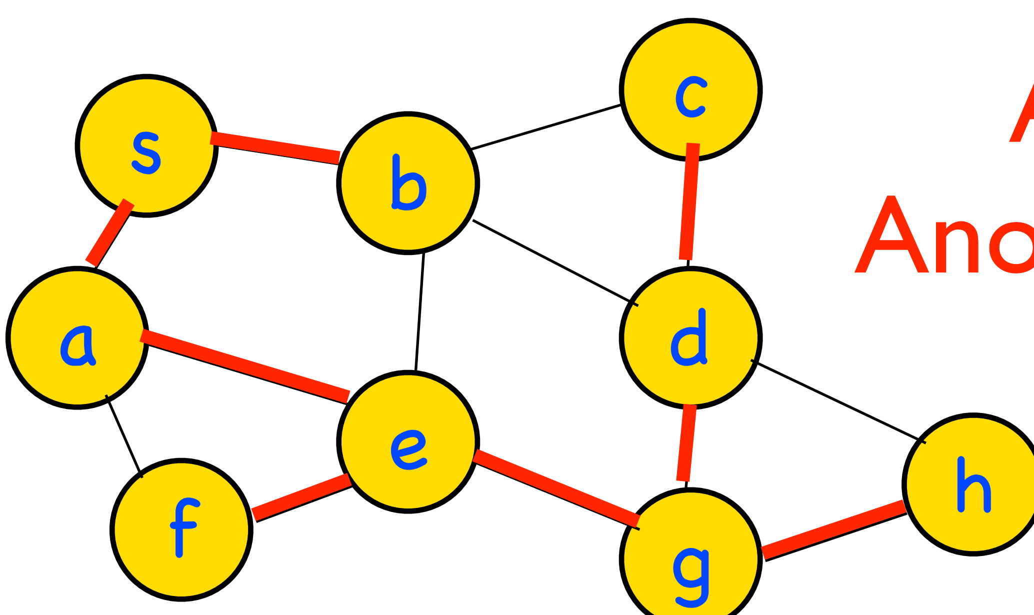
A spanning tree

connects all V nodes

$\Rightarrow$ # tree edges $= V-1$

Spanning Trees

- context : undirected graphs
- a set of edges A that "span" or "touch" all vertices, and forms no cycles
  - necessary this set of edges A has size = |V|-1
- spanning tree: the tree formed by the set of spanning edges together with vertex set T = (V,F)



A spanning tree
Another spanning tree

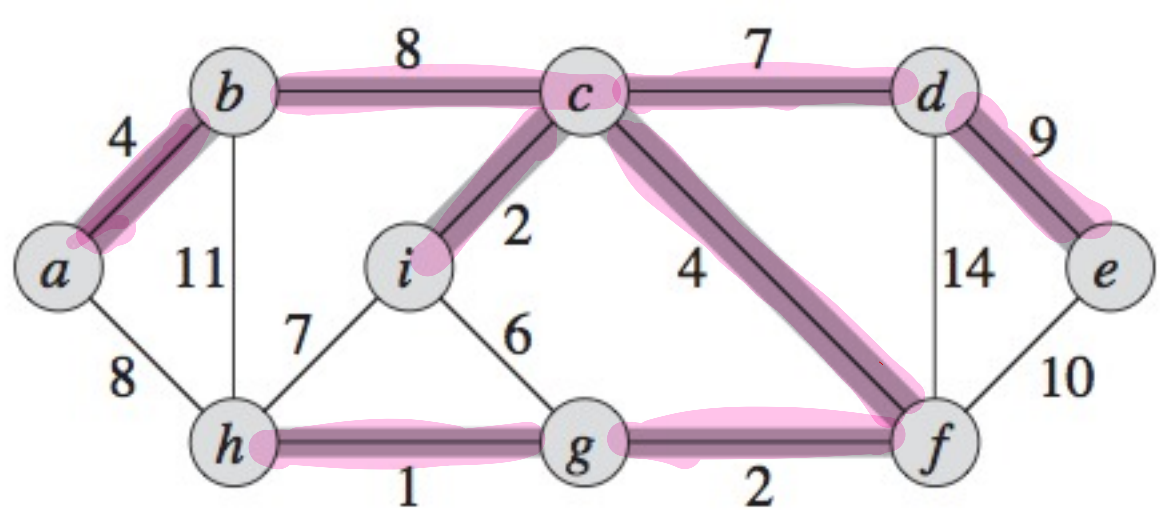
Minimum Spanning Tree (MST)

- context : undirected graph, $\boxed{\text{edges have weights}}$ ~ penalties cost
  - edge $(u,v) \in E$ has weight $w(u,v)$
- MST is a spanning tree of minimum total weight (of its edges)
  - must span all vertices
  - exactly $|V|-1$ edges
  - sum of edges weight be minimum among spanning trees

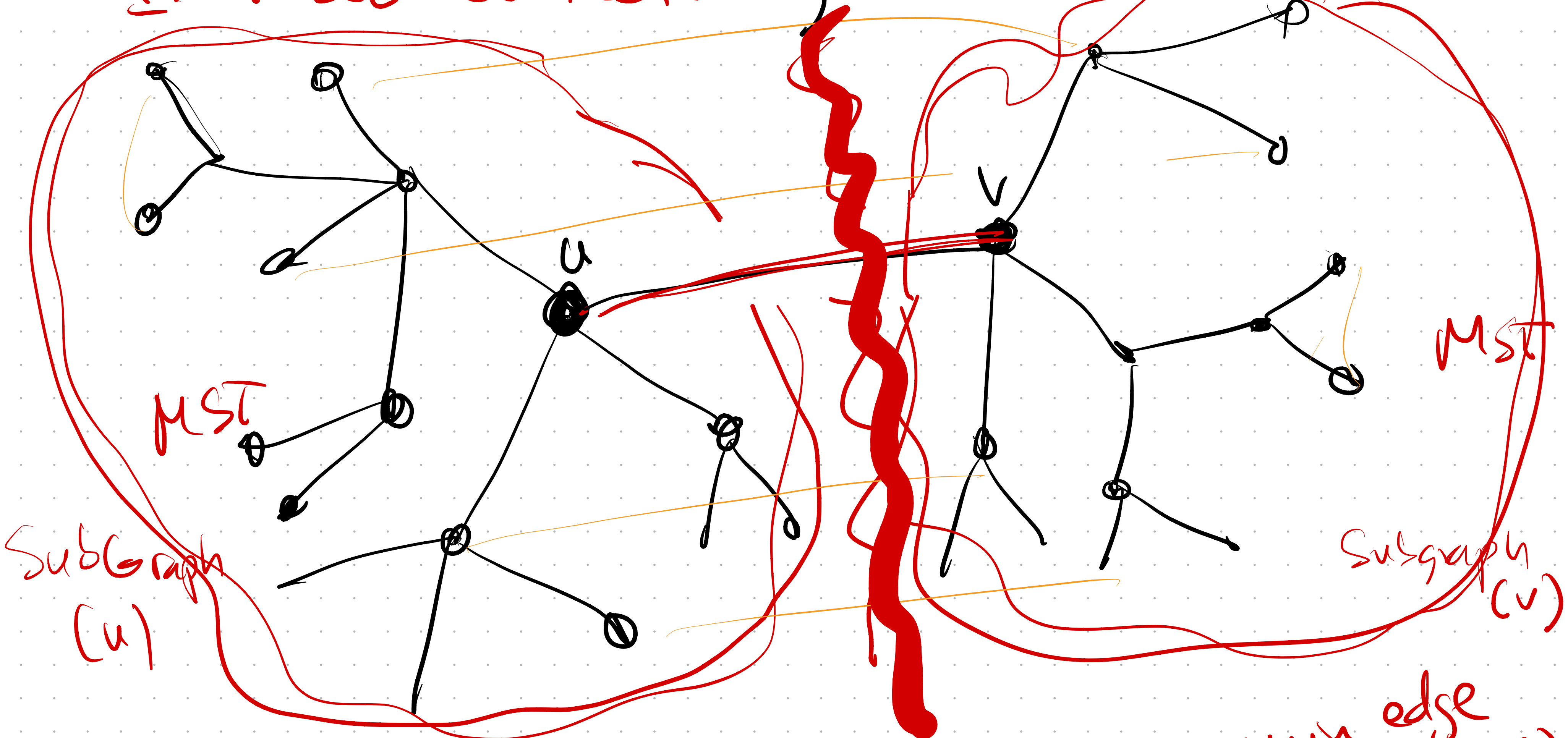
min total cost $= \sum w(\text{edges used})$

edge weights ≥ 0?

Cost $= 4+8+7+9+2+4+1+2$



1. OPTSOL charact. given MST



(u,v) any edge in MST

Th partition G into A and B

min edge across (A,B) in part MST

existing MST edges → in A / → or B

(I know ∈ MST)

not across A → B

Growing Minimum Spanning Trees

- "safe edge" (u,v) for a given set of edges A: there is a MST that uses A and (u,v)
  - that MST may not be unique

- `GENERIC-MST (G)`
- `A = set of tree edges, initially empty`
- `while A does not form a spanning tree` *// meaning while* |A| < |V|-1
  - `find edge (u,v) that is safe for A`
  - `add (u,v) to A`
- `end while`

- how to find a safe edge to a given set of edges A?
  - Prim algorithm
  - Kruskal algorithm

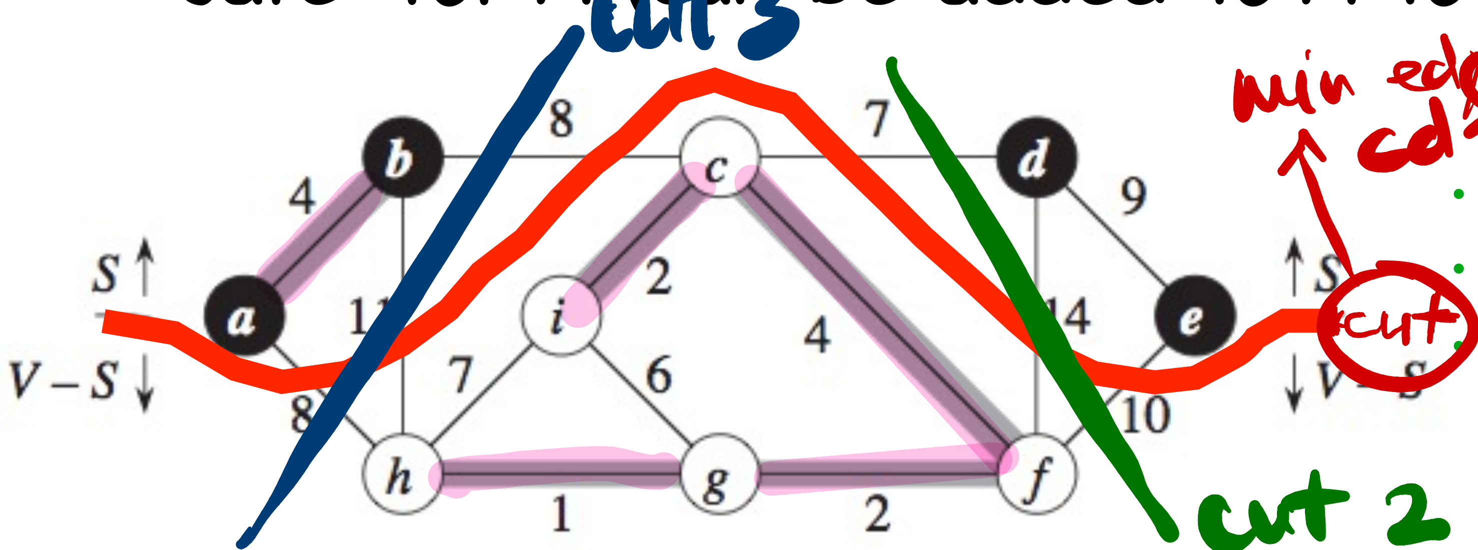
Cuts in the graph

- "cut" is a partition of vertices in two sets : V=S ∪ V-S

A = set of edges already selected to be in MST

- an edge (u,v) crosses the cut (S,V–S) if u and v are on different partitions (one in S the other in V–S)
- cut (S, V–S) respects set of edges A if A has no cross edge
- "min weight cross edge" is a cross edge for the cut, having minimum weight across all cross edges
- Cut Theorem : if A is a set of edges part of some MST, and (S,V–S) a cut respecting A , then a min–weight cross edge is "safe" for A (can be added to A towards an MST)

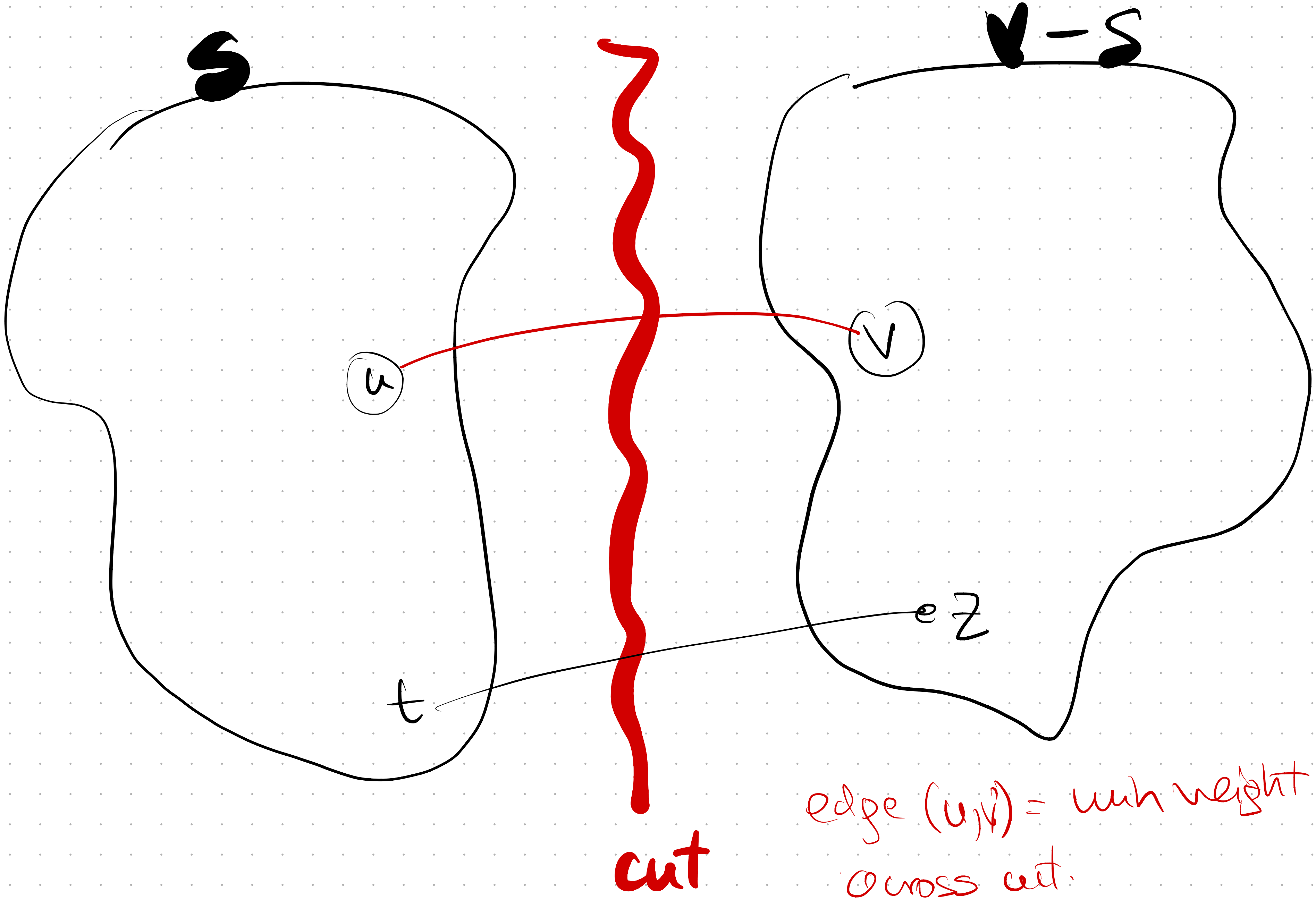
add to set A



min edge cross : cd

- A={ab, ic, cf, hg, fg}
- cut S={a,b,d,e} V-S={h,i,c,g,f} respects A
- safe crossing edge : cd, weight(cd)=7

cut repects edges in A
=> no edge across the cut



edge (u,v) = min weight across cut.

proof idea: MST eventually connects side S w/ side V-S with some edge t-z.

$w(u \to v) < w(t \to z)$

$\Rightarrow$ edge (u,v) is better

Prim algorithm

- grows a single tree A, S = set of vertices in the tree
  - as opposed to a forest of smaller disconnected trees
- add a safe edge at a time
  - connecting one more node to the current tree

x = outside

ESP(x) = closeness to the tree

Prim: grows a tree

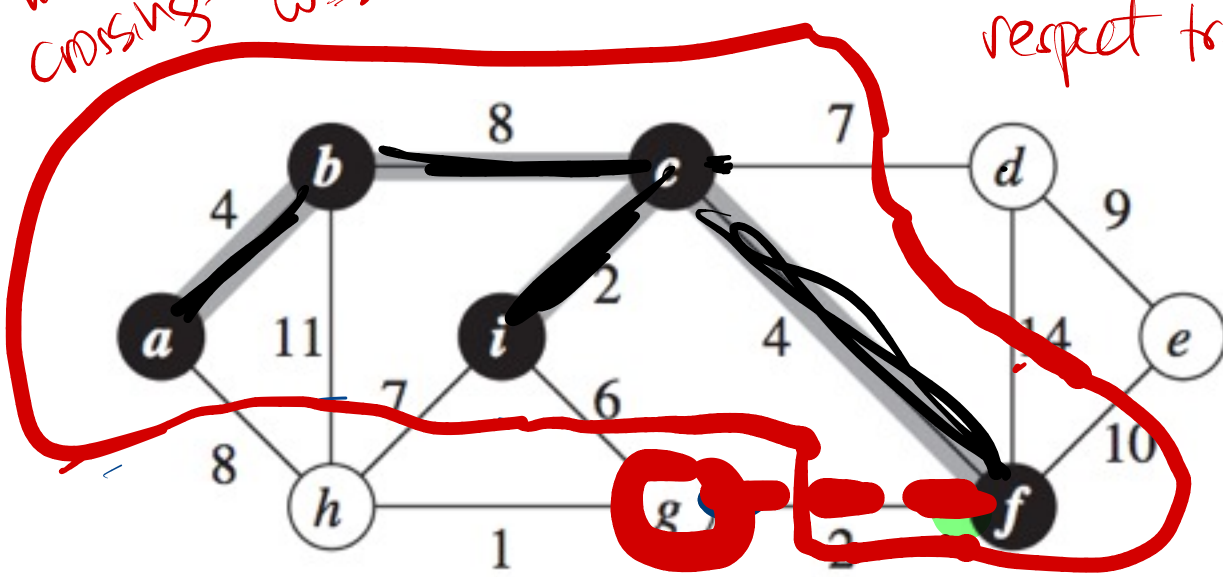
next edge: min edge not forming cycles (connects 2 existing trees)

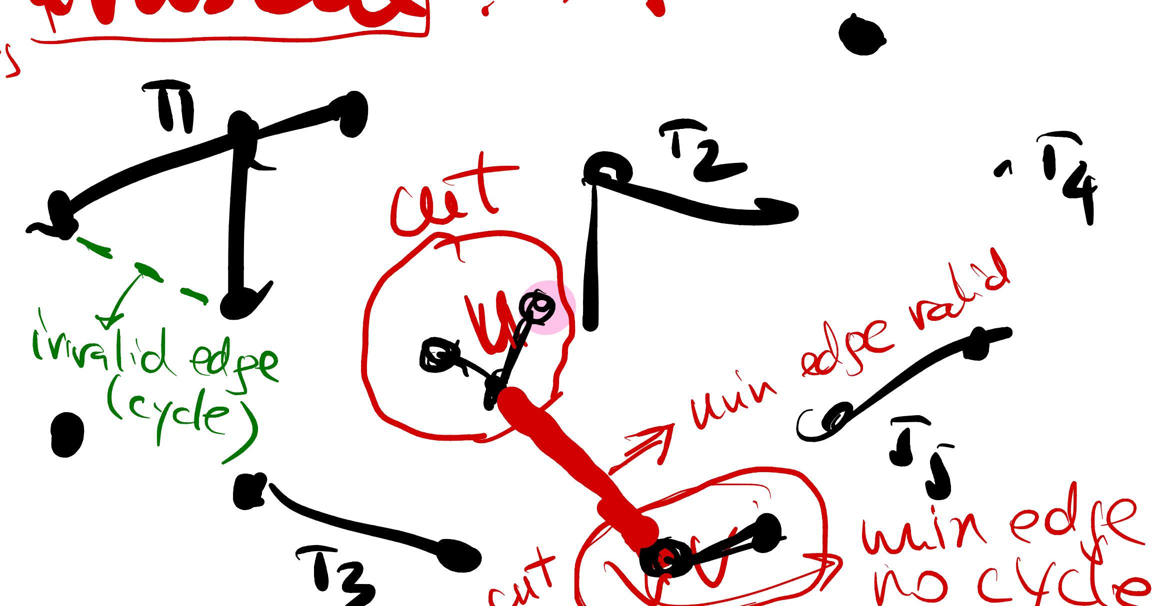
Kruskal :→ forest

min edge crossing-cut: GF w=2

Prim - grows a tree

cut (tree, non-tree) respect tree ex. edges





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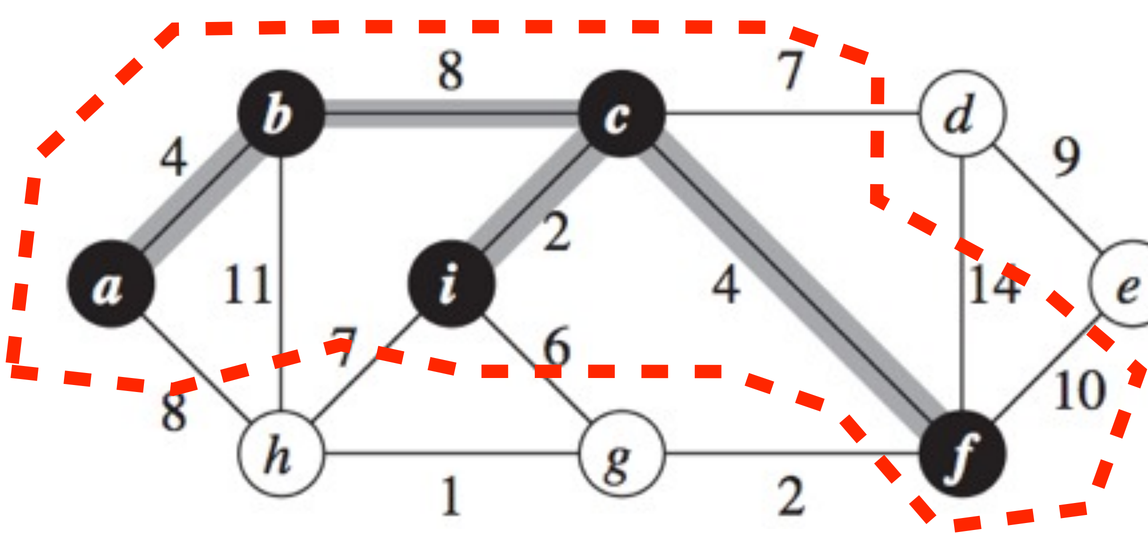

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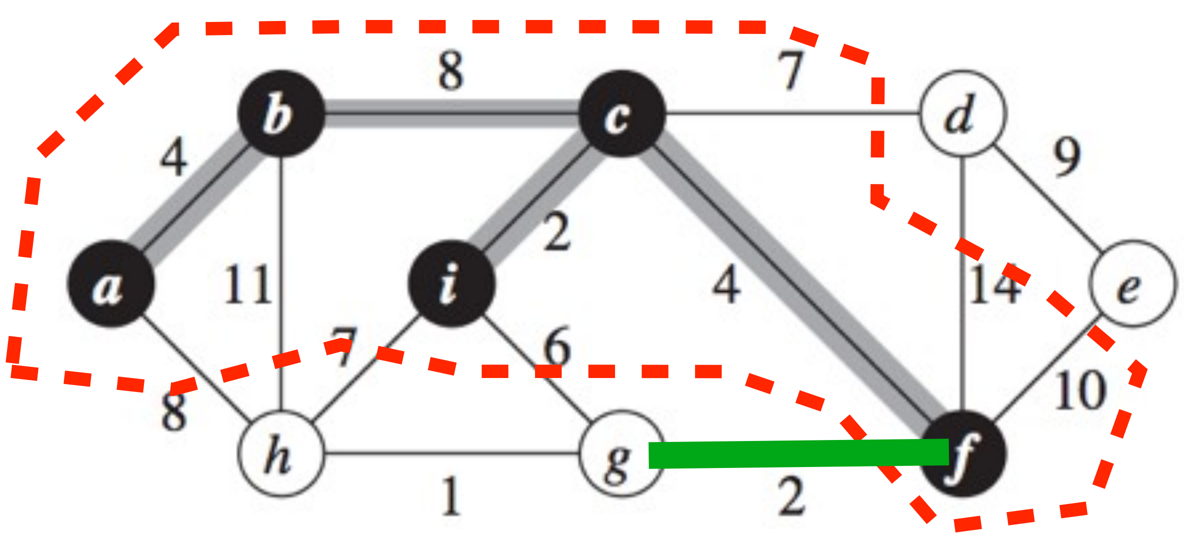

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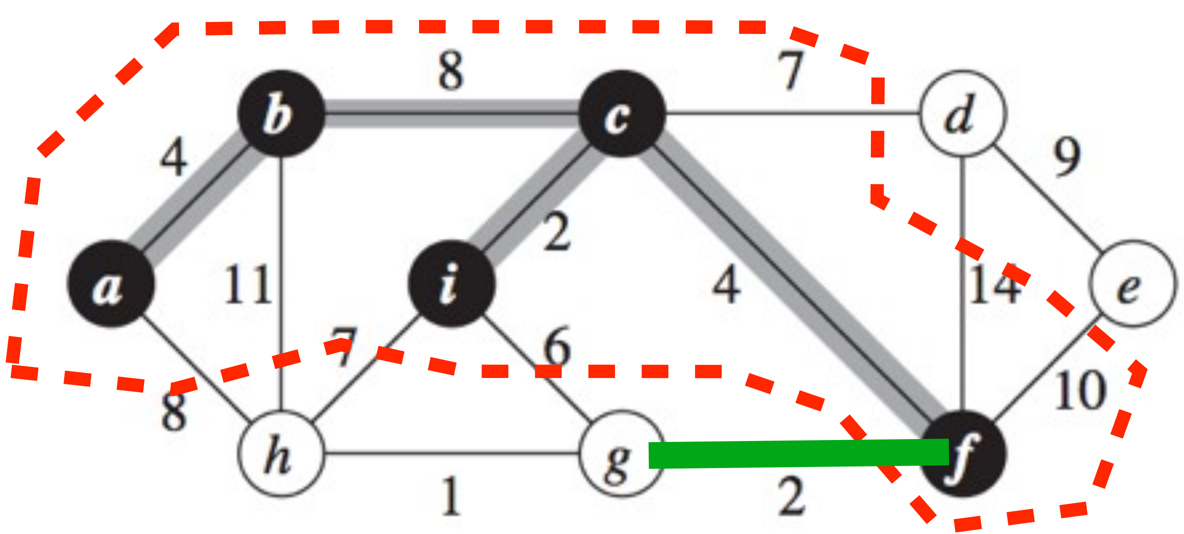

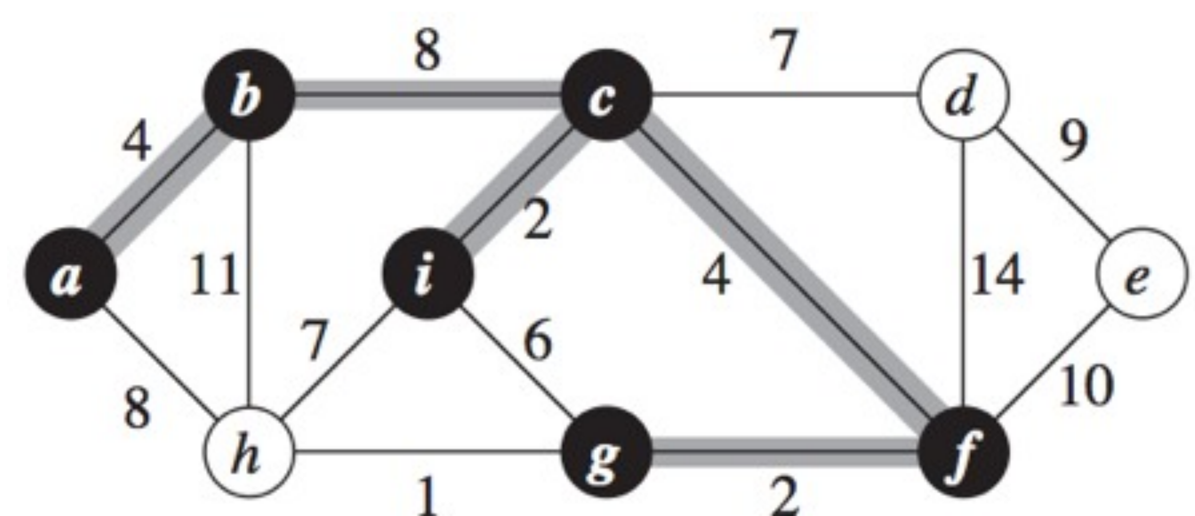

Prim algorithm

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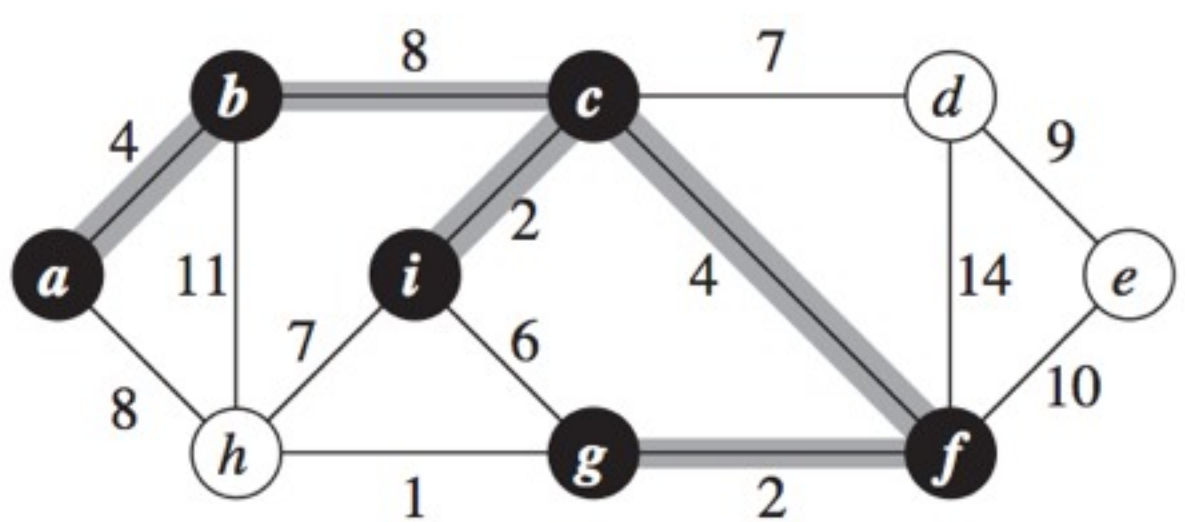

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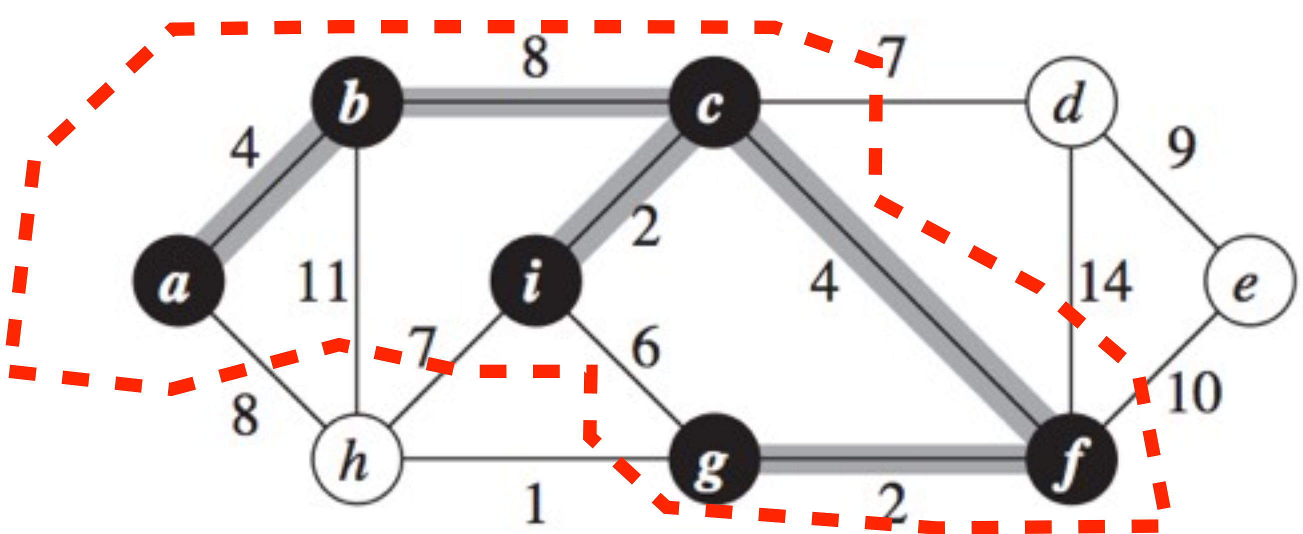

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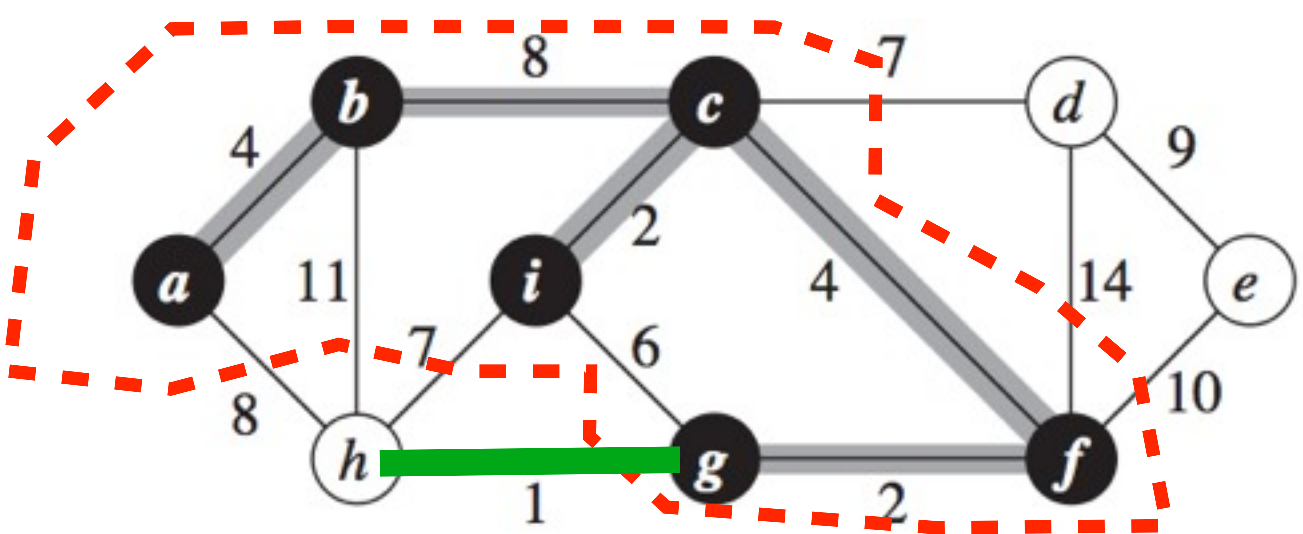

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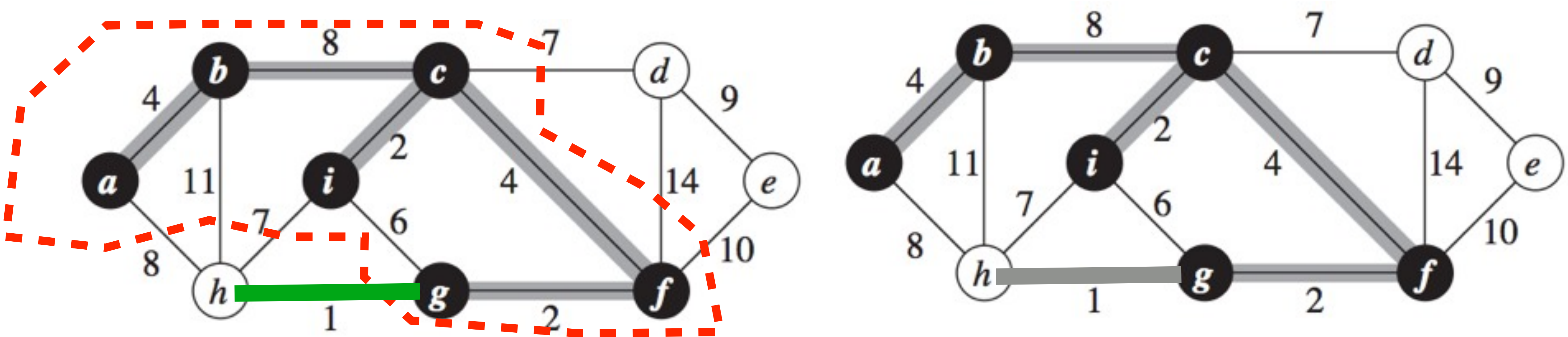


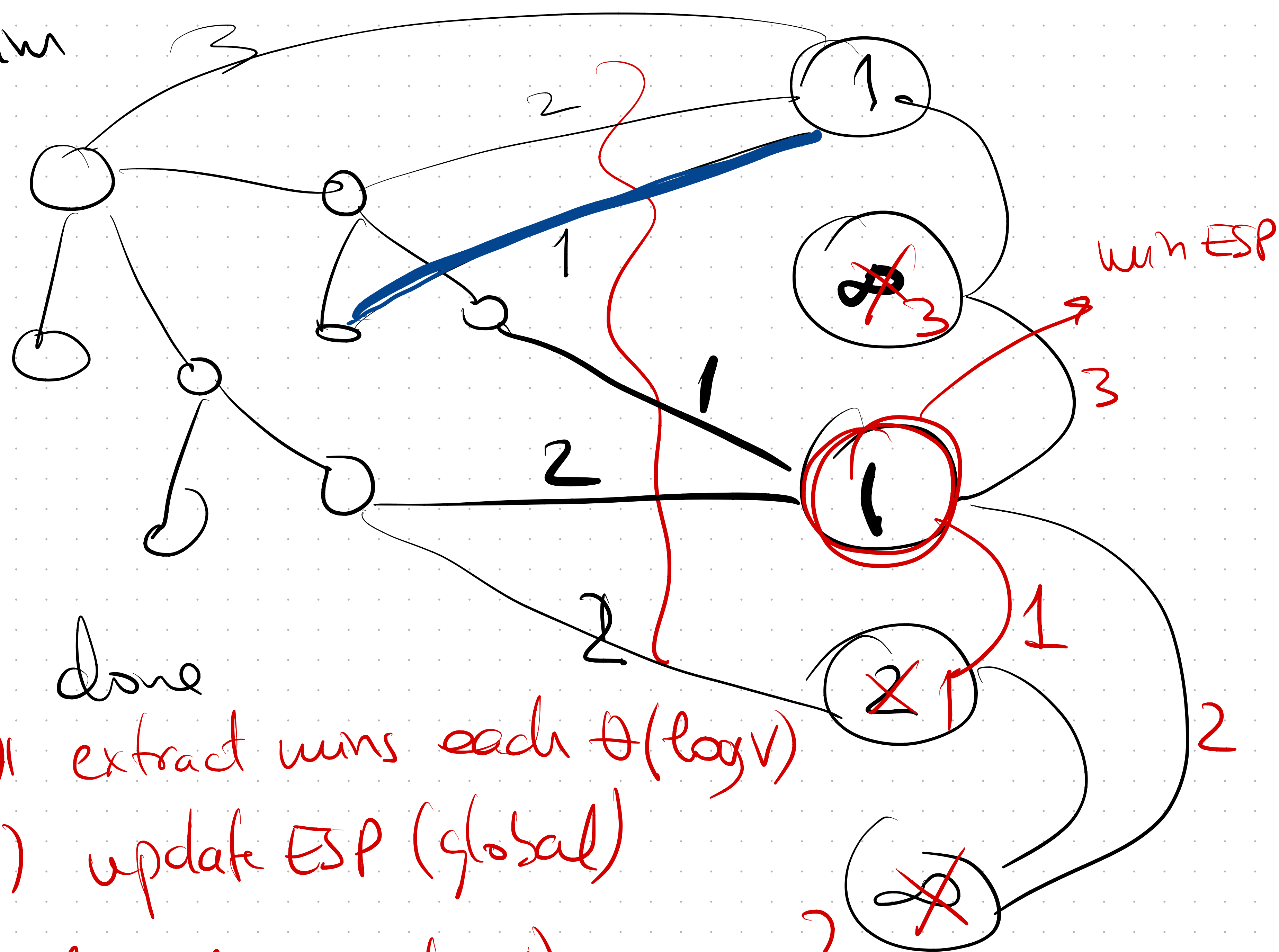
Prim MST algorithm

- Prim simple
  - but implementation a bit tricky
- Running Time depends on implementation of Extract-Min from the Queue
  - best theoretical implementation uses Fibonacci Heaps
  - also the most complicated
  - only makes a practical difference for very large graphs

MST-PRIM(G, w, r)

```
for each u ∈ G.V
    u.key = ∞
    u.π = NIL
r.key = 0
Q = G.V
while Q ≠ ∅
    u = EXTRACT-MIN(Q)
    for each v ∈ G.Adj[u]
        if v ∈ Q and w(u, v) < v.key
            v.π = u
            v.key = w(u, v)
```

Prim



not connected yet

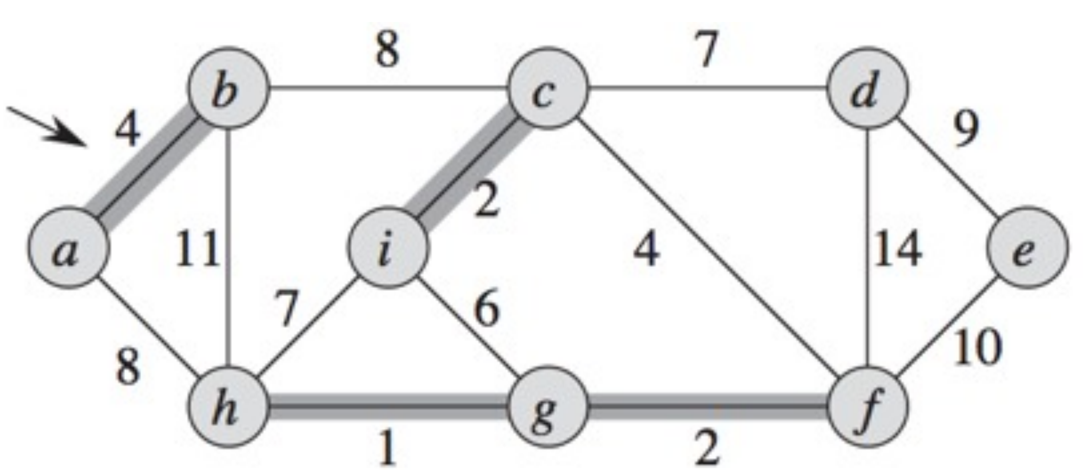
$\Theta(V)$ extract mins each $\Theta(\log V)$

$\Theta(E)$ update ESP (global)

total $\Theta(\underbrace{E}_{\text{update}} + \underbrace{V \log V}_{\text{extract min}})$

Kruskal MST algorithm

- Grows a forest of trees Forrest = (V,A)
  - eventually all connected into a MST
  - initially each vertex is a tree with no edges, and A is empty



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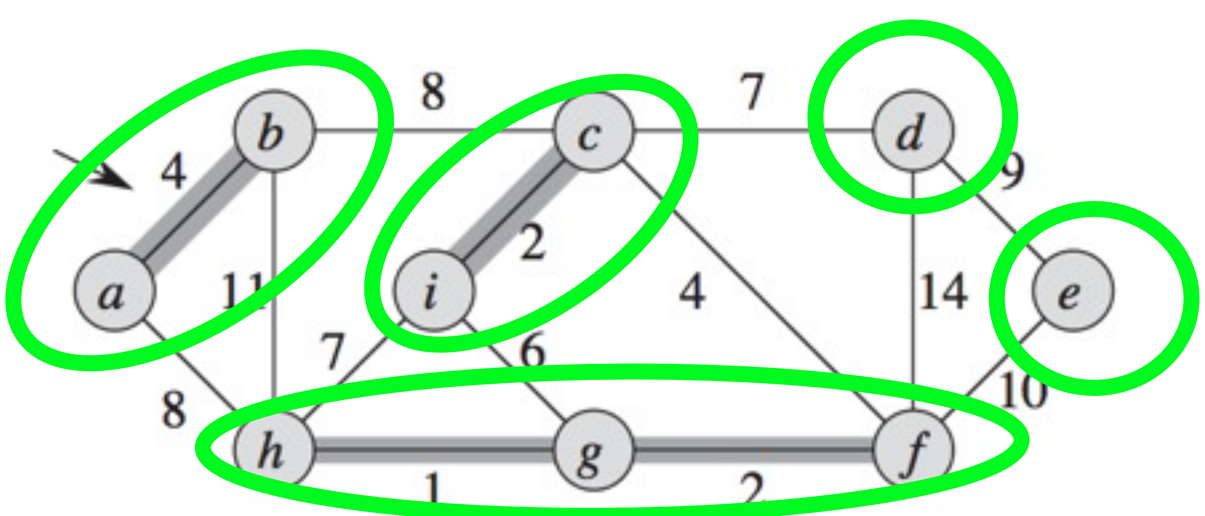

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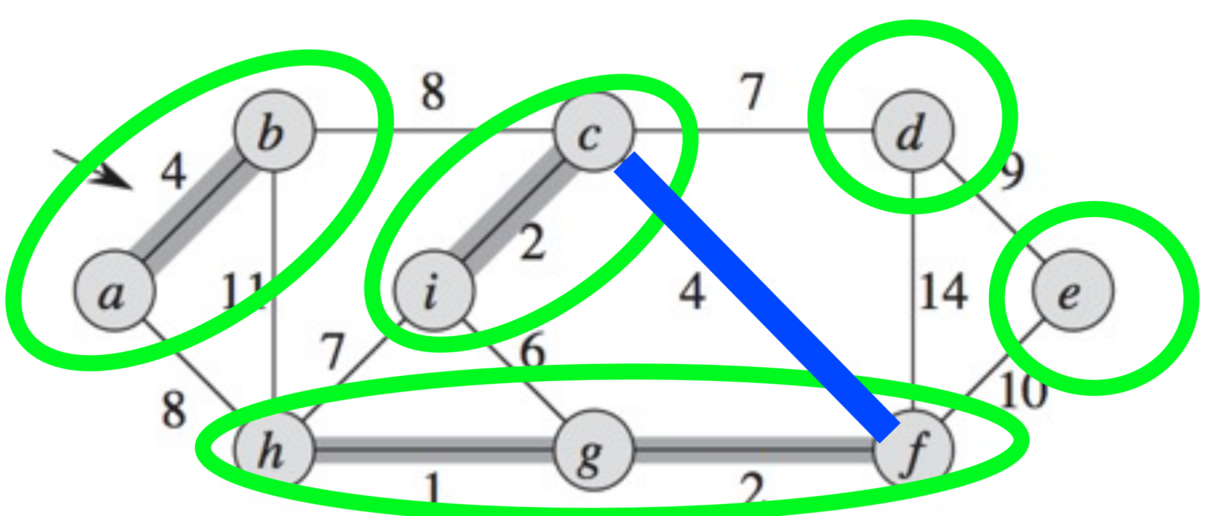

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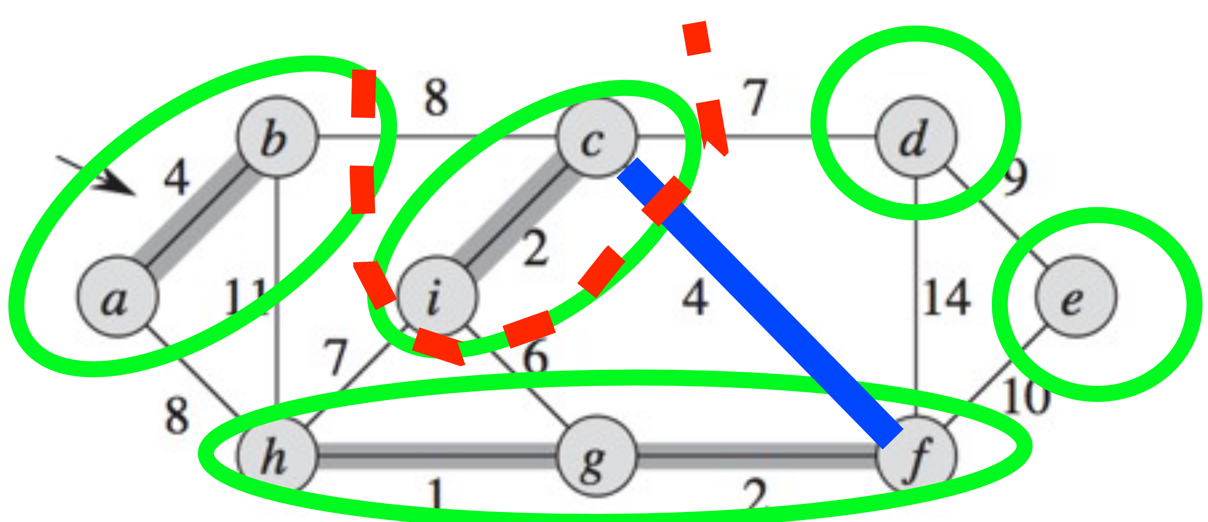

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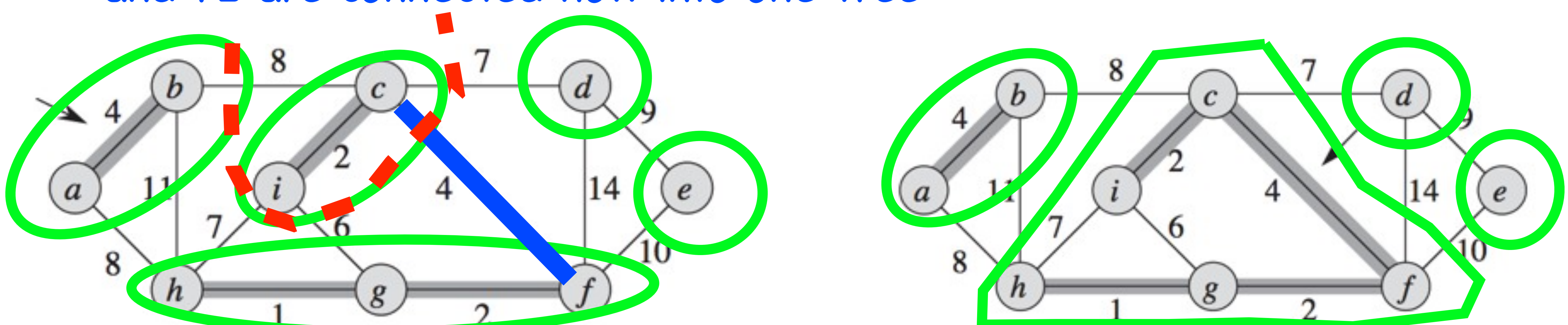
  - define cut (S,V-S); S = vertices of T1 (in red). This cut respects set A
  - edge (u,v) is the minimum cross edge, thus a safe edge to add to A. T1 and T2 are connected now into one tree



Kruskal algorithm

MST-KRUSKAL(G, w)

$A = \emptyset$

for each vertex $v \in G.V$

  MAKE-SET(v)

sort the edges of $G.E$ into nondecreasing order by weight w

for each edge $(u, v) \in G.E$, taken in nondecreasing order by weight

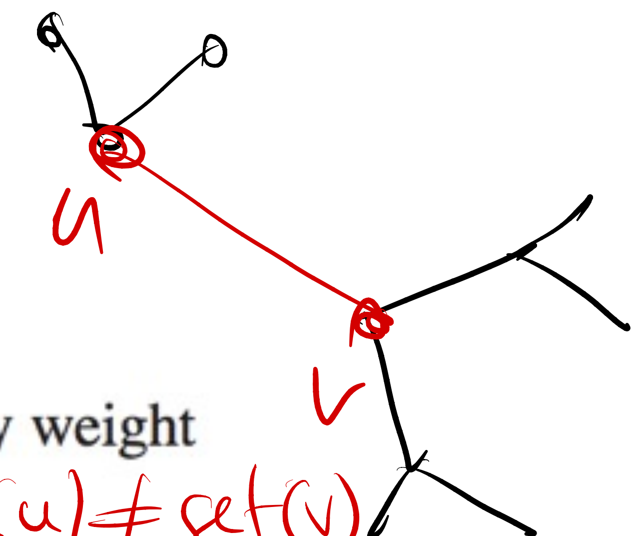
  $O(1)$ ← **if** $\boxed{\text{FIND-SET}(u) \neq \text{FIND-SET}(v)}$ → check set(u) ≠ set(v)

    u, v not cycle

    $A = A \cup \{(u, v)\}$

hard $\alpha(V)$ ←   UNION(u, v) |V| nodes

return A



$\boxed{\alpha(V) \leq 5 \text{ in practice}}$

$\alpha(V) \to \infty$ very very slow in theory

- Kruskal is simple
- implementation and running time depend on FIND-SET and UNION operations on the disjoint-set forest.
  - chapter 21 in the book, optional material for this course
- running time O(E logV) naive

$O(\underbrace{E \log E}_{\text{sort edges}} + \underbrace{E\,\alpha(V)}_{\text{selected union}})$

MST algorithm comparison

- if you know graph density (edges to vertices)

	Kruskal	Prim with array implement.	Prim w/ binomial heap	Prim w/ Fibonacci heap	in practice
sparse graph $E=O(V)$	$O(V\log V)$	$O(V^2)$	$O(V\log V)$	$O(V\log V)$	Kruskal, or Prim+binom heap
dense graph $E=\Theta(V^2)$	$O(V^2\log V)$	$O(V^2)$	$O(V^2\log V)$	$O(V^2)$	Prim with array
avg density $E=\Theta(V\log V)$	$O(V\log^2 V)$	$O(V^2)$	$O(V\log^2 V)$	$O(V\log V)$	Prim with Fib heap, if graph is large