Binomial Heaps

Outline for this Week

- ***Binomial Heaps (Today)***
  - A simple, flexible, and versatile priority queue.
- ***Lazy Binomial Heaps (Today)***
  - A powerful building block for designing advanced data structures.
- ***Fibonacci Heaps (Thursday)***
  - A heavyweight and theoretically excellent priority queue.

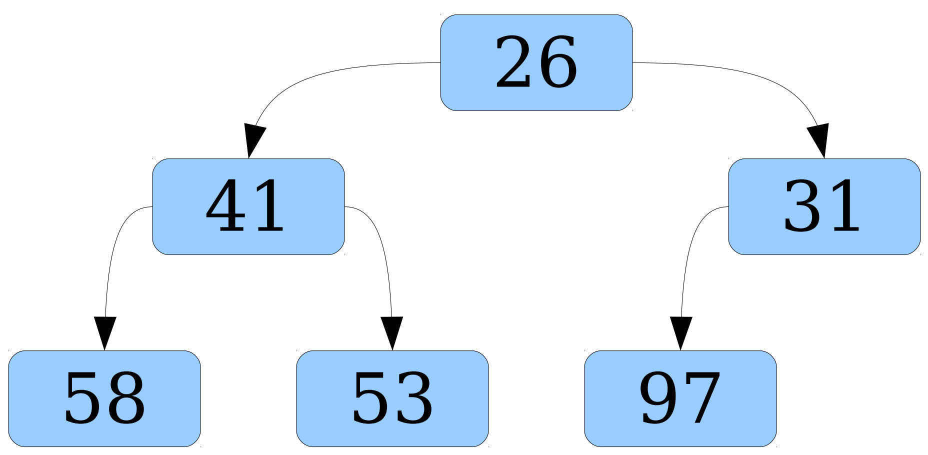
Review: Priority Queues

Priority Queues

- A ***priority queue*** is a data structure that stores a set of elements annotated with totally-ordered *keys* and allows efficient extraction of the element with the least key.
- More concretely, supports these operations:
  - *pq*.***enqueue***(*v*, *k*), which enqueues element *v* with key *k*;
  - *pq*.***find-min***(), which returns the element with the least key; and
  - *pq*.***extract-min***(), which removes and returns the element with the least key,

Binary Heaps

- Priority queues are frequently implemented as ***binary heaps***.
- ***enqueue*** and ***extract-min*** run in time O(log *n*); ***find-min*** runs in time O(1).
- We're not going to cover binary heaps this quarter; I assume you've seen them before.



Priority Queues in Practice

- Many graph algorithms directly rely priority queues supporting extra operations:
  - ***meld***(pq_1, pq_2): Destroy pq_1 and pq_2 and combine their elements into a single priority queue.
  - *pq*.***decrease-key***(v, k'): Given a pointer to element v already in the queue, lower its key to have new value k'.
  - *pq*.***add-to-all***(Δk): Add Δk to the keys of each element in the priority queue (typically used with ***meld***).
- In lecture, we'll cover binomial heaps to efficiently support ***meld*** and Fibonacci heaps to efficiently support ***meld*** and ***decrease-key***.
- You'll design a priority queue supporting efficient ***meld*** and ***add-to-all*** on the problem set.

Meldable Priority Queues

- A priority queue supporting the ***meld*** operation is called a ***meldable priority queue***.
- ***meld***(pq_1, pq_2) destructively modifies pq_1 and pq_2 and produces a new priority queue containing all elements of pq_1 and pq_2.

Meldable Priority Queues

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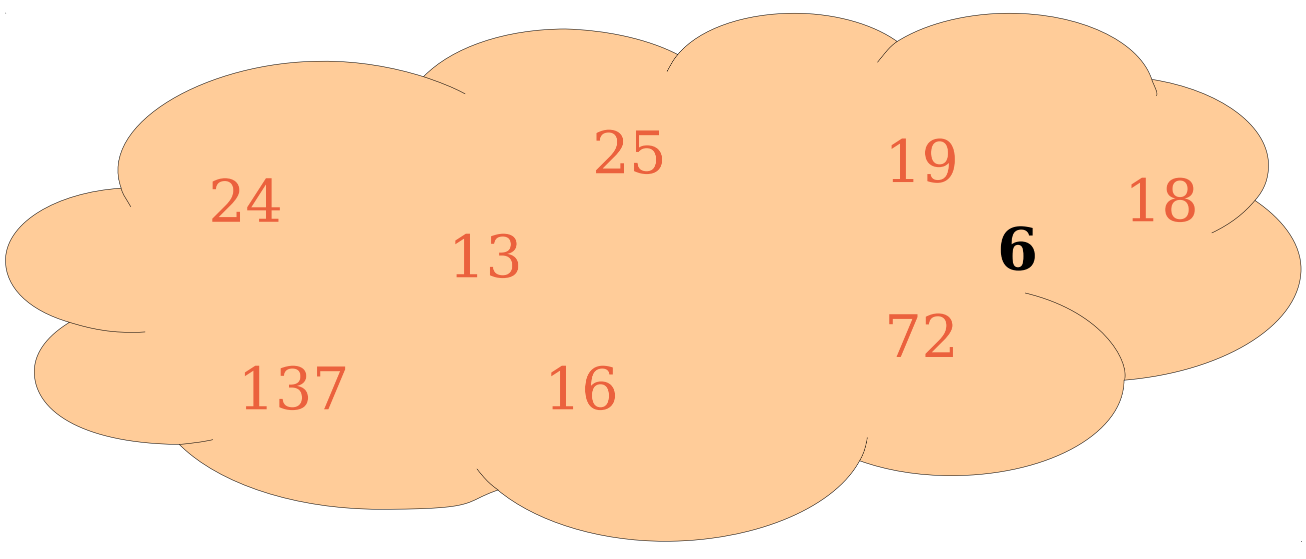
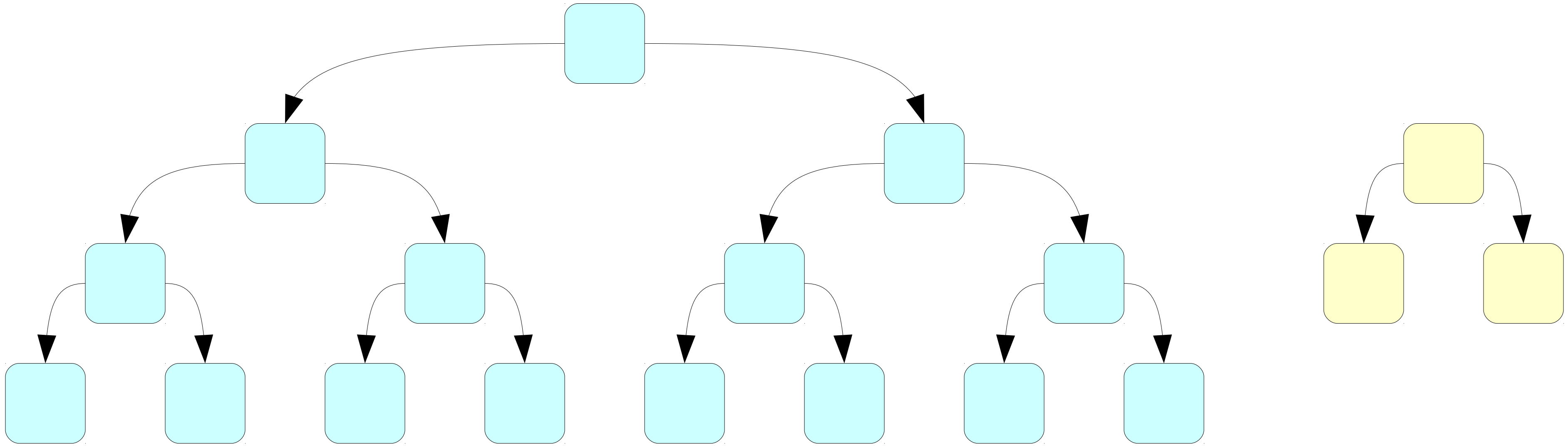


Efficiently Meldable Queues

- Standard binary heaps do not efficiently support ***meld***.
- ***Intuition***: Binary heaps are complete binary trees, and two complete binary trees cannot easily be linked to one another.



Binomial Heaps

- The ***binomial heap*** is an priority queue data structure that supports efficient melding.
- We'll study binomial heaps for several reasons:
  - Implementation and intuition is totally different than binary heaps.
  - Used as a building block in other data structures (Fibonacci heaps, soft heaps, etc.)
  - Has a beautiful intuition; similar ideas can be used to produce other data structures.

Supporting Efficient Melding

The Intuition: ***Binary Arithmetic***

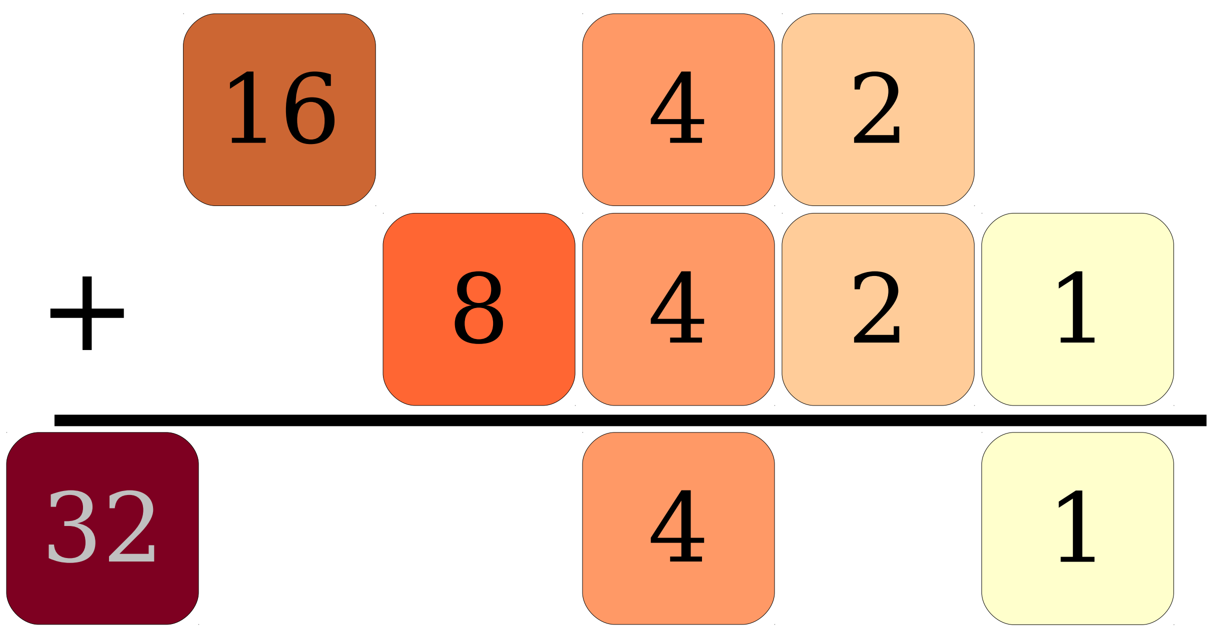
Adding Binary Numbers

- Given the binary representations of two numbers n and m, we can add those numbers in time $\Theta(\max\{\log m, \log n\})$.

1	1	1	1		
	1	0	1	1	0
+		1	1	1	1
1	0	0	1	0	1

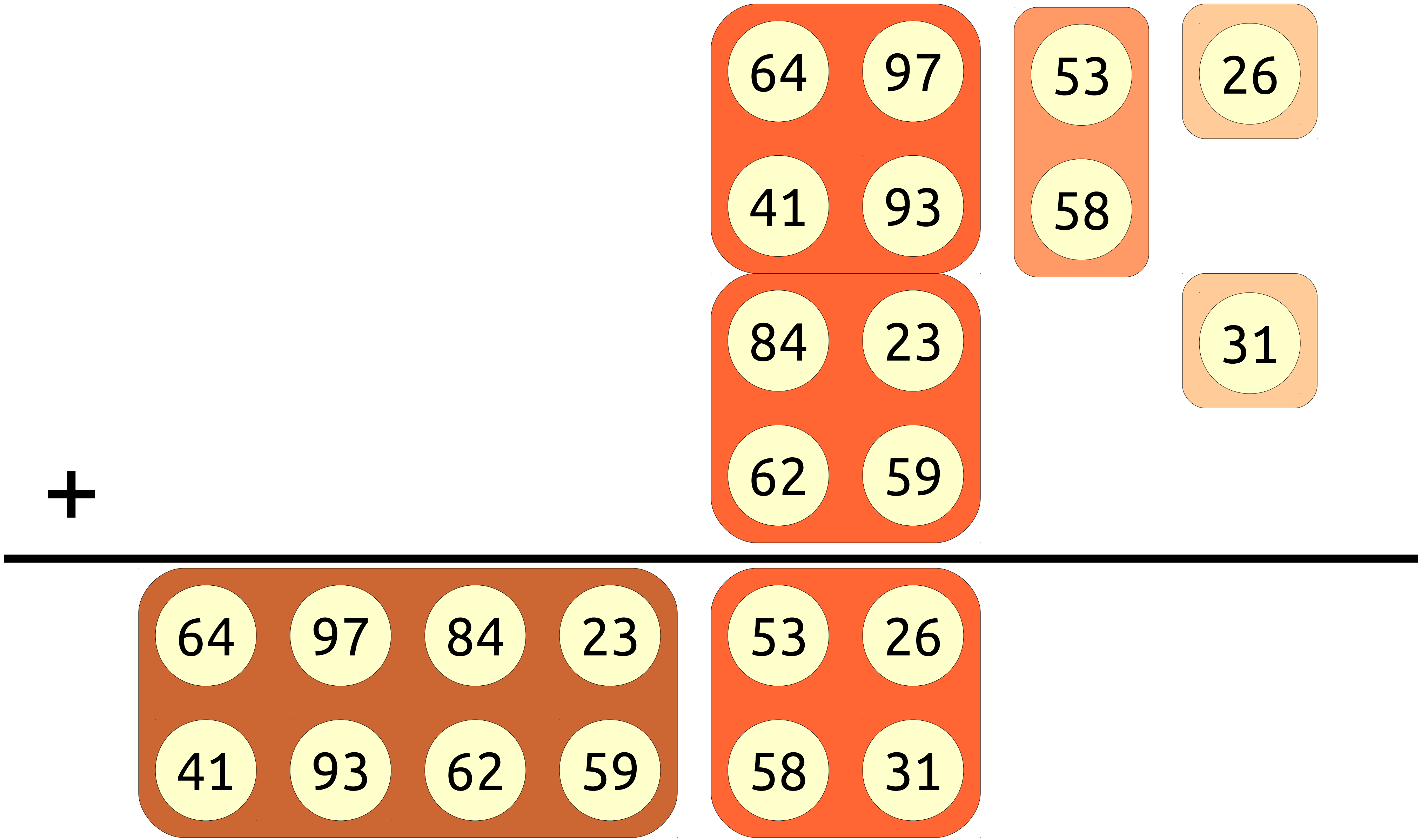
A Different Intuition

- Represent *n* and *m* as a collection of “packets” whose sizes are powers of two.
- Adding together *n* and *m* can then be thought of as combining the packets together, eliminating duplicates



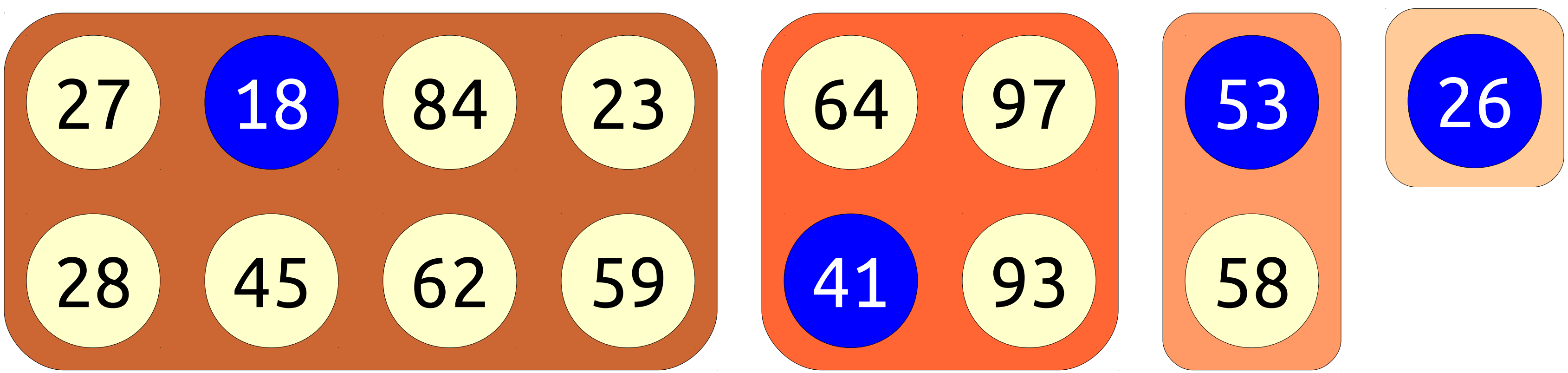
Building a Priority Queue

- ***Idea:*** Adapt this approach to build a priority queue.
- Store elements in the priority queue in "packets" whose sizes are powers of two.
- Store packets in ascending size order.
- We'll choose a representation of a packet so that two packets of the same size can easily be fused together.


64 97
41 93
84 23
62 59
53
58
26
31
+
64 97 84 23
41 93 62 59
53 26
58 31

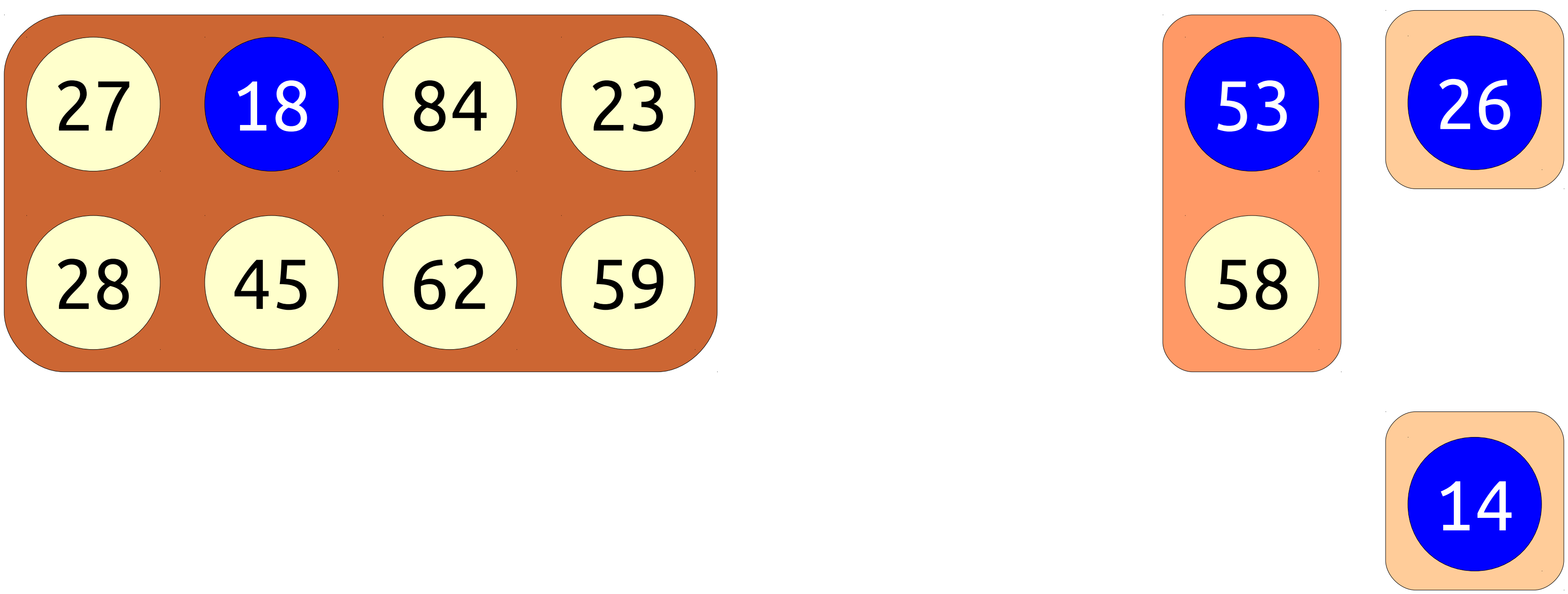
Building a Priority Queue

- What properties must our packets have?
  - Sizes must be powers of two.
  - Can efficiently fuse packets of the same size.
  - Can efficiently find the minimum element of each packet.



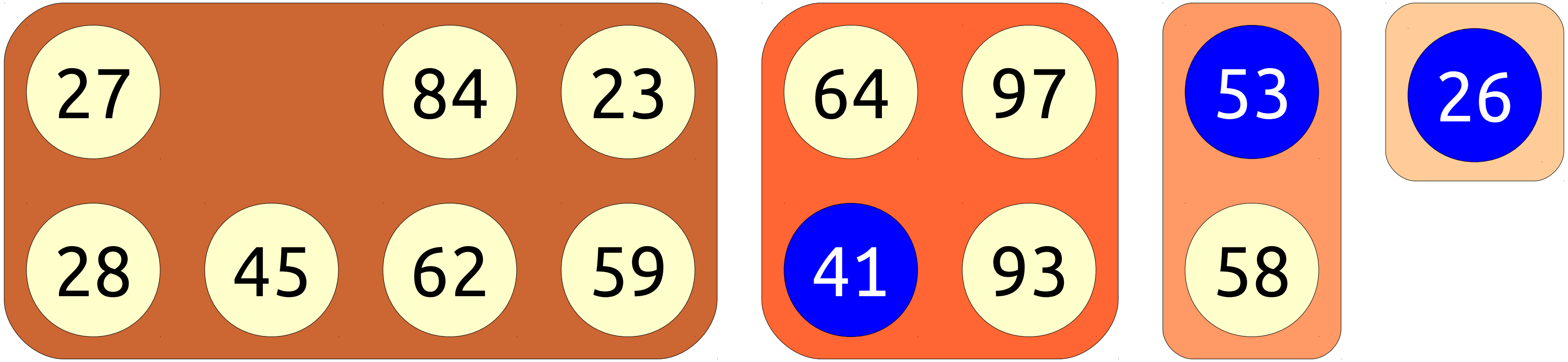
Inserting into the Queue

- If we can efficiently meld two priority queues, we can efficiently enqueue elements to the queue.
- ***Idea:*** Meld together the queue and a new queue with a single packet.



Deleting the Minimum

- Our analogy with arithmetic breaks down when we try to remove the minimum element.
- After losing an element, the packet will not necessarily hold a number of elements that is a power of two.

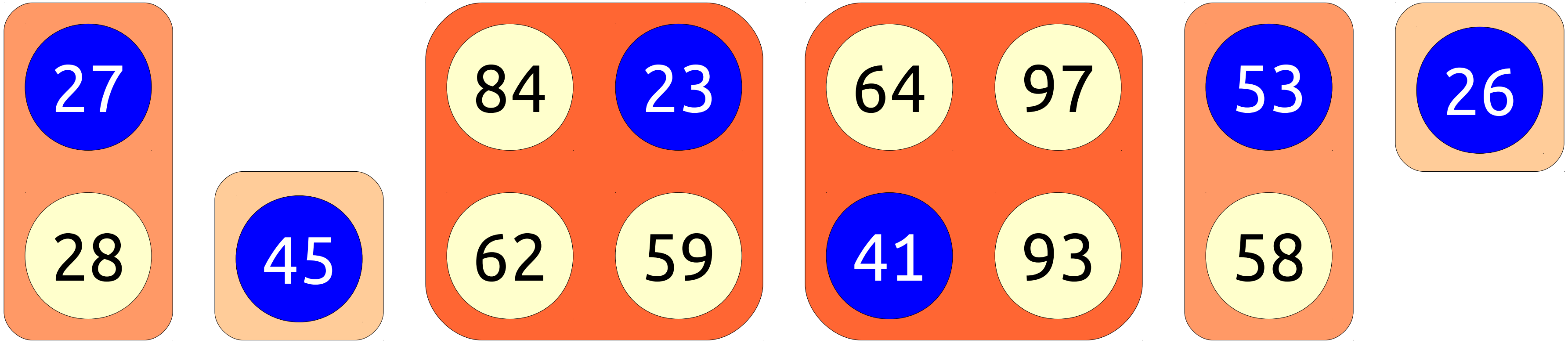


Fracturing Packets

- If we have a packet with 2^k elements in it and remove a single element, we are left with $2^k - 1$ remaining elements.
- ***Fun fact***: $2^k - 1 = 1 + 2 + 4 + \ldots + 2^{k-1}$.
- ***Idea***: "Fracture" the packet into $k - 1$ smaller packets, then add them back in.

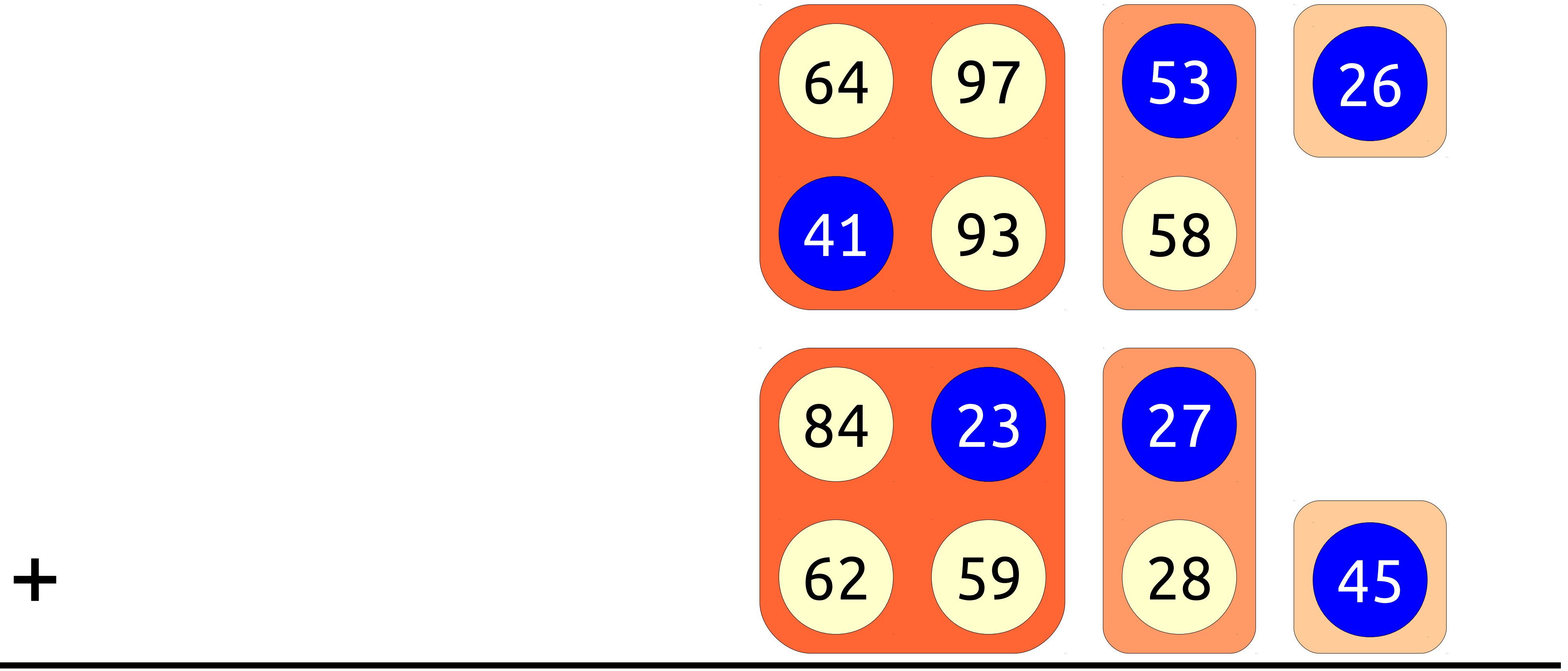
Fracturing Packets

- We can ***extract-min*** by fracturing the packet containing the minimum and adding the fragments back in.



Fracturing Packets

- We can ***extract-min*** by fracturing the packet containing the minimum and adding the fragments back in.
- Runtime is O(log *n*) fuses in ***meld***, plus fragment cost.



Building a Priority Queue

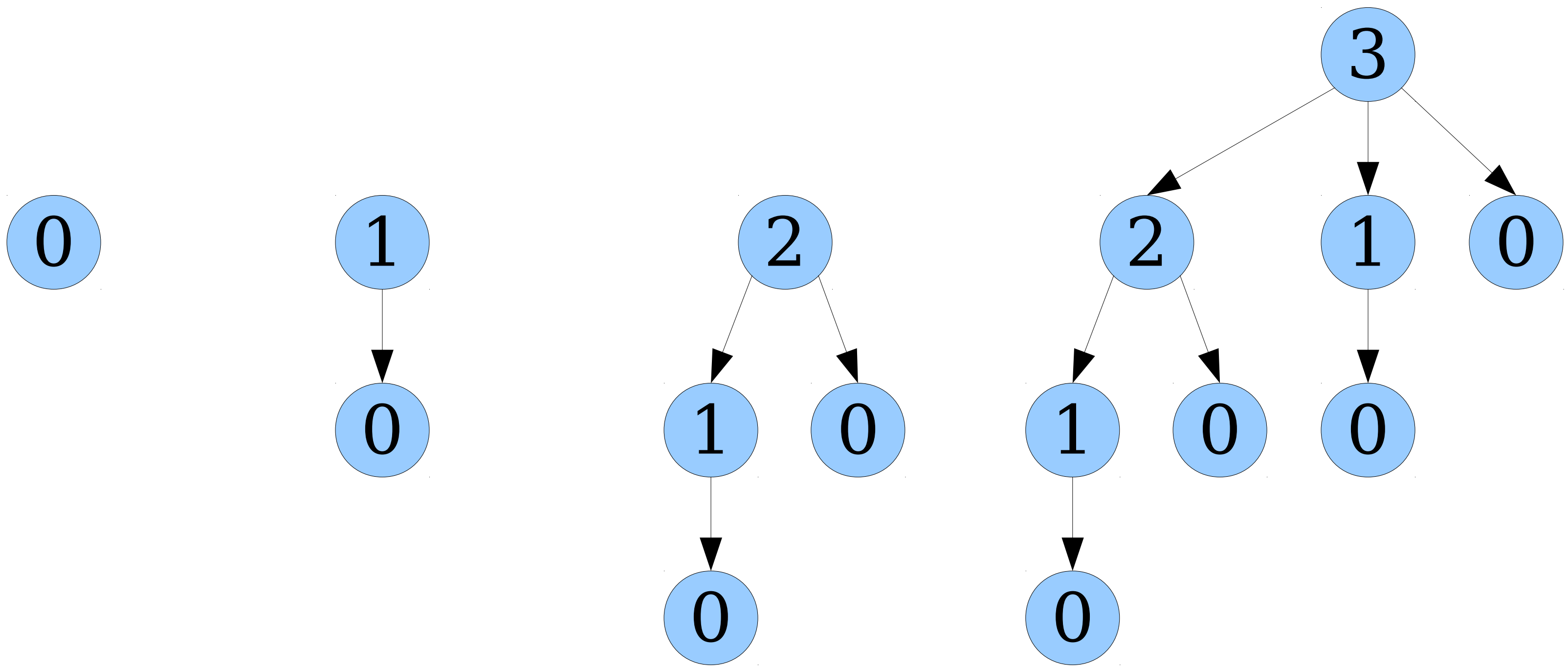
- What properties must our packets have?
  - Size must be a power of two.
  - Can efficiently fuse packets of the same size.
  - Can efficiently find the minimum element of each packet.
  - Can efficiently “fracture” a packet of 2^k nodes into packets of 1, 2, 4, 8, …, 2^{k-1} nodes.
- What representation of packets will give us these properties?

Binomial Trees

- A ***binomial tree of order k*** is a type of tree recursively defined as follows:

  A binomial tree of order k is a single node whose children are binomial trees of order 0, 1, 2, …, k – 1.

- Here are the first few binomial trees:



Binomial Trees

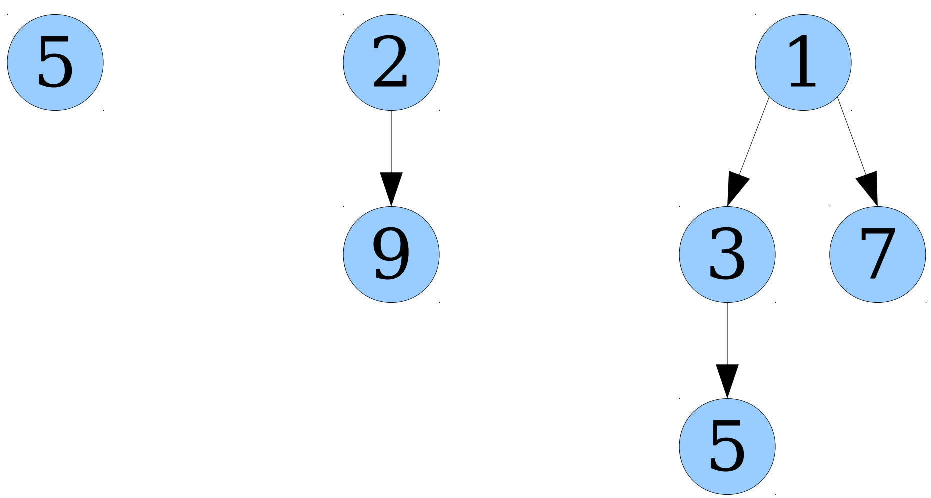
- ***Theorem:*** A binomial tree of order k has exactly 2^k nodes.
- ***Proof:*** Induction on k. Assuming that binomial trees of orders 0, 1, 2, …, $k - 1$ have $2^0, 2^1, 2^2, \ldots, 2^{k-1}$ nodes, then then number of nodes in an order-k binomial tree is

$$2^0 + 2^1 + \ldots + 2^{k-1} + 1 = 2^k - 1 + 1 = 2^k$$

So the claim holds for k as well. ■

Binomial Trees

- A ***heap-ordered binomial tree*** is a binomial tree whose nodes obey the heap property: all nodes are less than or equal to their descendants.
- We will use heap-ordered binomial trees to implement our "packets."



Binomial Trees

- What properties must our packets have?
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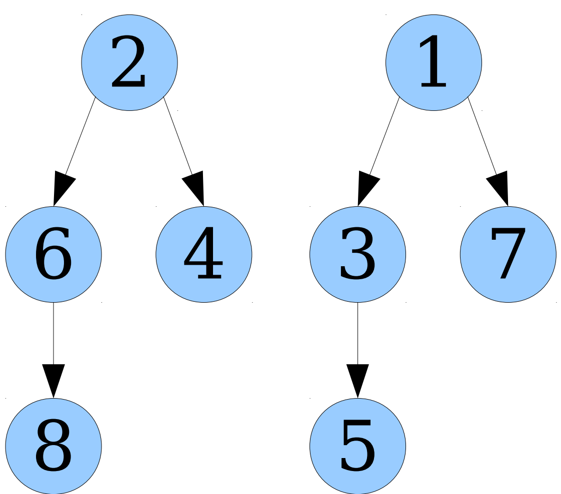

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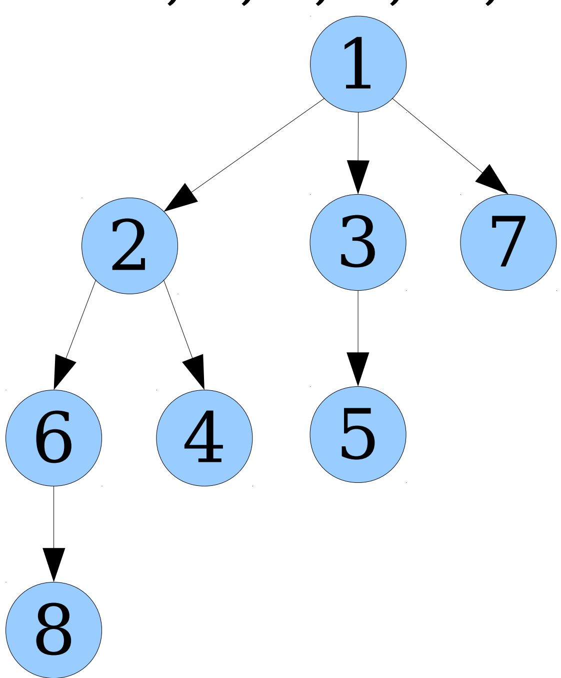
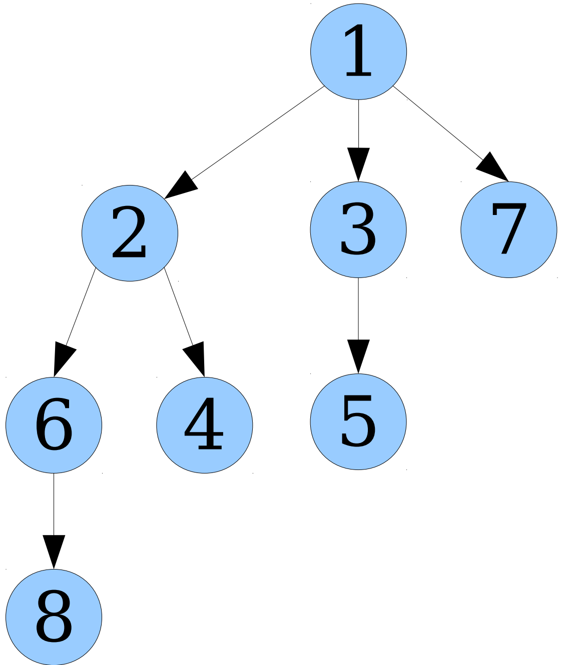


Make the binomial tree with the larger root the first child of the tree with the smaller root.

Binomial Trees

- What properties must our packets have?
  - Size must be a power of two. ✓
  - Can efficiently fuse packets of the same size. ✓
  - Can efficiently find the minimum element of each packet. ✓
  - Can efficiently “fracture” a packet of 2^k nodes into packets of 1, 2, 4, 8, …, 2^{k-1} nodes.



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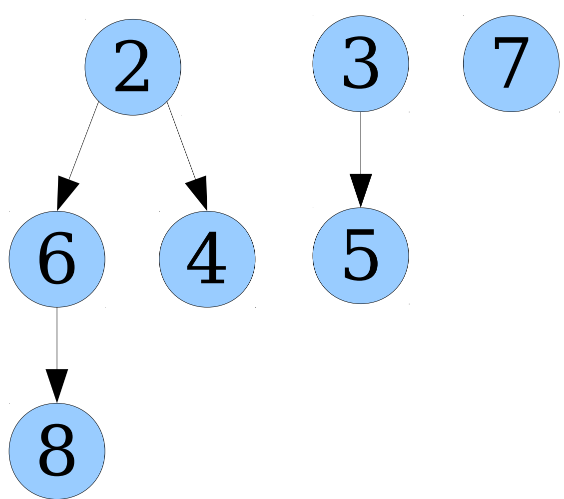
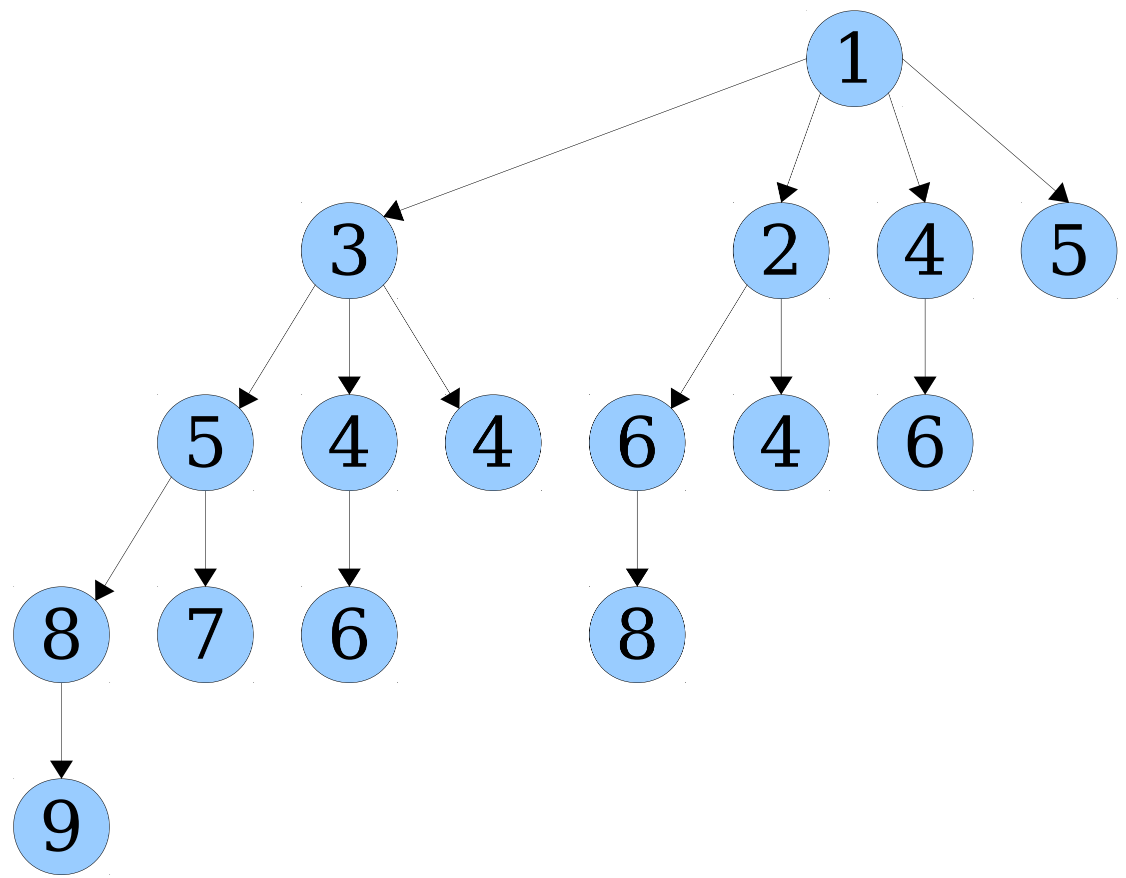


The Binomial Heap

- A ***binomial heap*** is a collection of heap-ordered binomial trees stored in ascending order of size.
- Operations defined as follows:
  - ***meld***(pq_1, pq_2): Use addition to combine all the trees.
    - Fuses O(log n) trees. Total time: O(log n).
  - pq.***enqueue***(v, k): Meld pq and a singleton heap of (v, k).
    - Total time: O(log n).
  - pq.***find-min***(): Find the minimum of all tree roots.
    - Total time: O(log n).
  - pq.***extract-min***(): Find the min, delete the tree root, then meld together the queue and the exposed children.
    - Total time: O(log n).

An Issue of Representation

- Binomial trees are *logically* multiway trees, but are typically *implemented* as binary trees.
- We use the ***left-child/right-sibling*** representation.
- Each node's left pointer points to its first child.
- Each node's right pointer points to its next sibling.



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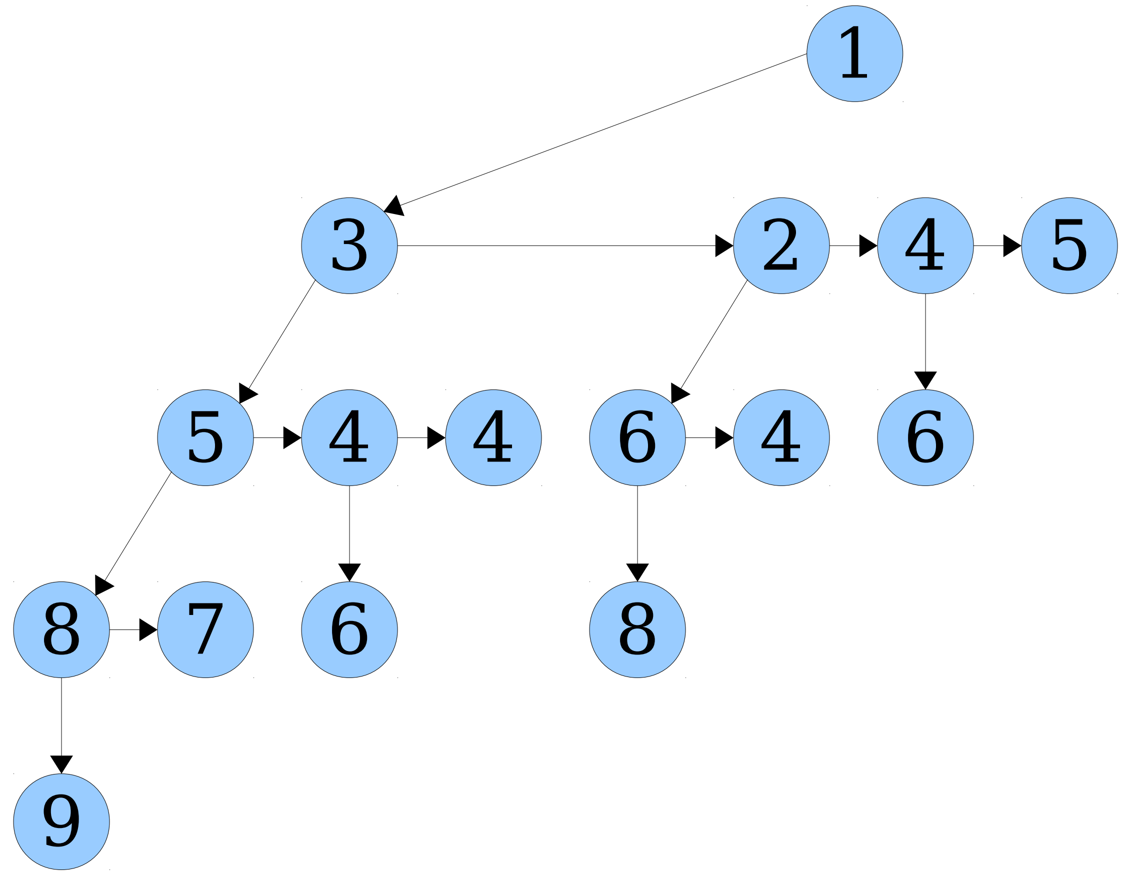
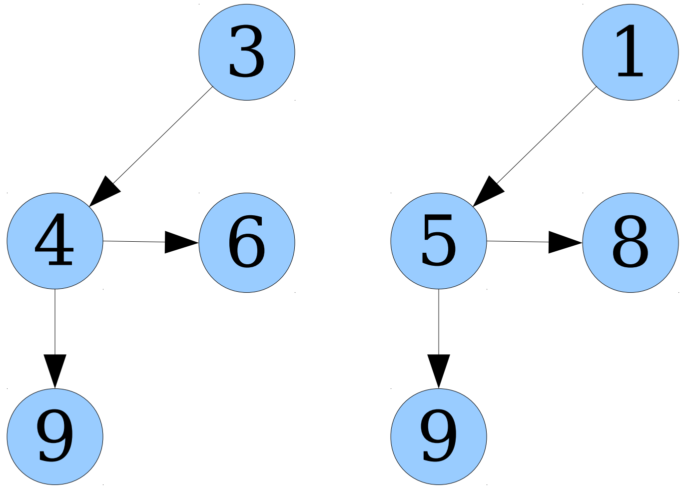


An Issue of Representation

- The LCRS representation of binomial trees improves efficiency.
- Fusion takes time O(1).
- Fracturing takes time O(log n).



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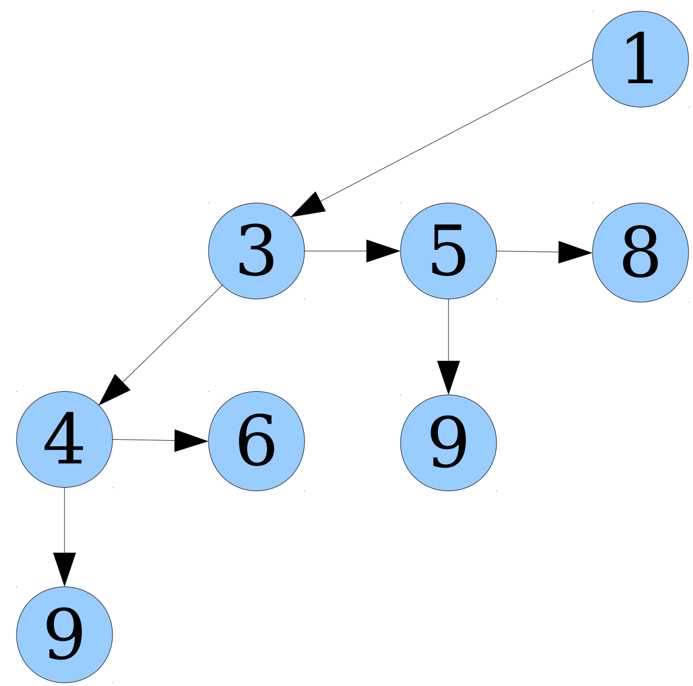

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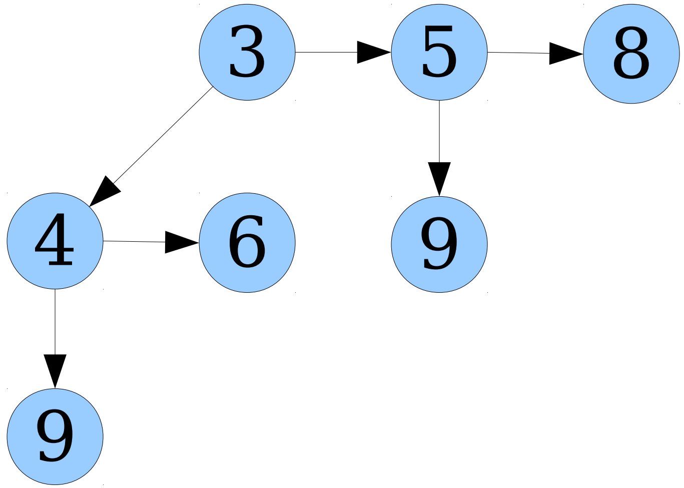


Time-Out for Announcements!

Stanford Women
in Computer Science

Casual Dinner

{w}

Tuesday, May 1st from 6-7 PM at Gates 403

Come mingle with CS professors and
friends with delicious Lotus Thai!

The Final Project

- We've just posted information online (and in hardcopy here) about the CS166 final project.
- The quick summary:
  - Work in teams of two or three.
  - Pick a data structure, algorithm, or technique of your choice.
  - Become experts on it. Put together a writeup and presentation on the topic.
  - Do something "interesting" with it. You have broad latitude how to interpret what "interesting" means – pick something you're excited about!
- Projects and presentations are due in the last week of class. They're usually a highlight of the quarter for everyone involved!

Project Proposals

- Before working on the project, you'll need to submit a proposal about what you'd like to work on.
- Your proposal should consist of a ranked list of ***five*** data structures you'd be interested in exploring, along with some preliminary information about each one.
- The proposal is due next ***Thursday, May 10th*** at 2:30PM.
- We'll do a global matchmaking to assign topics over that weekend.

Problem Sets

- Problem Set Three is due Thursday at 2:30PM.
  - There's plenty of space to ask us questions – let us know what we can do to help out!
- Problem Set Four will go out next Tuesday. You'll have a little gap between those problem sets.
  - We recommend using this gap to work on or think about your final project proposals.

Back to CS166!

Analyzing Insertions

- Each ***enqueue*** into a binomial heap takes time O(log n), since we have to meld the new node into the rest of the trees.
- However, it turns out that the amortized cost of an insertion is lower in the case where we do a series of n insertions.

Adding One

- Suppose we want to execute n++ on the binary representation of n.
- Do the following:
  - Find the longest span of 1's at the right side of n.
  - Flip those 1's to 0's.
  - Set the preceding bit to 1.
- Runtime: $\Theta(b)$, where b is the number of bits flipped.

An Amortized Analysis

- ***Claim:*** Starting at zero, the amortized cost of adding one to the total is O(1).
- ***Idea:*** Use as a potential function the number of 1's in the number.

$\boxed{\Phi = 0}$ 0 0 0 0 0

An Amortized Analysis

- ***Claim:*** Starting at zero, the amortized cost of adding one to the total is O(1).
- ***Idea:*** Use as a potential function the number of 1's in the number.

$\Phi = 1$ 0 0 0 0 1

Actual cost: 1
$\Delta\Phi$: +1

Amortized cost: **2**

An Amortized Analysis

- ***Claim:*** Starting at zero, the amortized cost of adding one to the total is O(1).
- ***Idea:*** Use as a potential function the number of 1's in the number.

$\Phi = 1$ 0 0 0 1 0

Actual cost: 2
$\Delta\Phi$: 0

Amortized cost: **2**

An Amortized Analysis

- ***Claim:*** Starting at zero, the amortized cost of adding one to the total is O(1).
- ***Idea:*** Use as a potential function the number of 1's in the number.

$\Phi = 2$

0 0 0 1 1

Actual cost: 1
$\Delta\Phi$: 1

Amortized cost: **2**

An Amortized Analysis

- ***Claim:*** Starting at zero, the amortized cost of adding one to the total is O(1).
- ***Idea:*** Use as a potential function the number of 1's in the number.

$\Phi = 1$ 0 0 1 0 0

Actual cost: 3
$\Delta\Phi$: -1

Amortized cost: **2**

Properties of Binomial Heaps

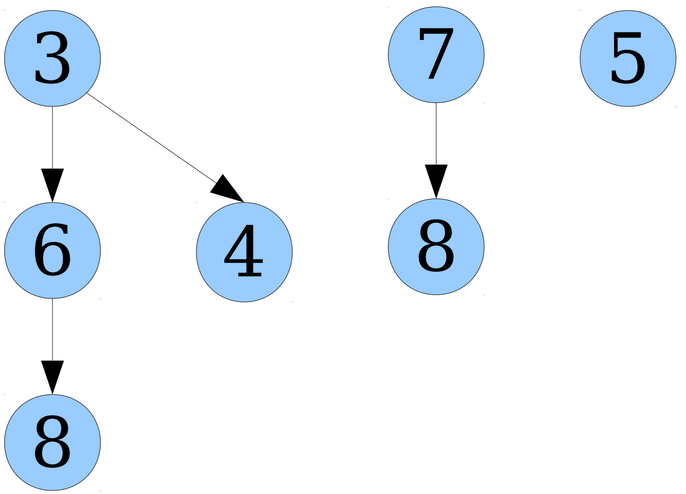
- Starting with an empty binomial heap, the amortized cost of each insertion into the heap is O(1), assuming there are no deletions.
- ***Rationale:*** Binomial heap operations are isomorphic to integer arithmetic.
- Since the amortized cost of incrementing a binary counter starting at zero is O(1), the amortized cost of enqueuing into an initially empty binomial heap is O(1).

Binomial vs Binary Heaps

- Interesting comparison:
  - The cost of inserting n elements into a binary heap, one after the other, is $\Theta(n \log n)$ in the worst-case.
  - If n is known in advance, a binary heap can be constructed out of n elements in time $\Theta(n)$.
  - The cost of inserting n elements into a binomial heap, one after the other, is $\Theta(n)$, even if n is not known in advance!

A Catch

- This amortized time bound does not hold if ***enqueue*** and ***extract-min*** are intermixed.
- ***Intuition:*** Can force expensive insertions to happen repeatedly.



Question: Can we make insertions amortized O(1), regardless of whether we do deletions?

Where's the Cost?

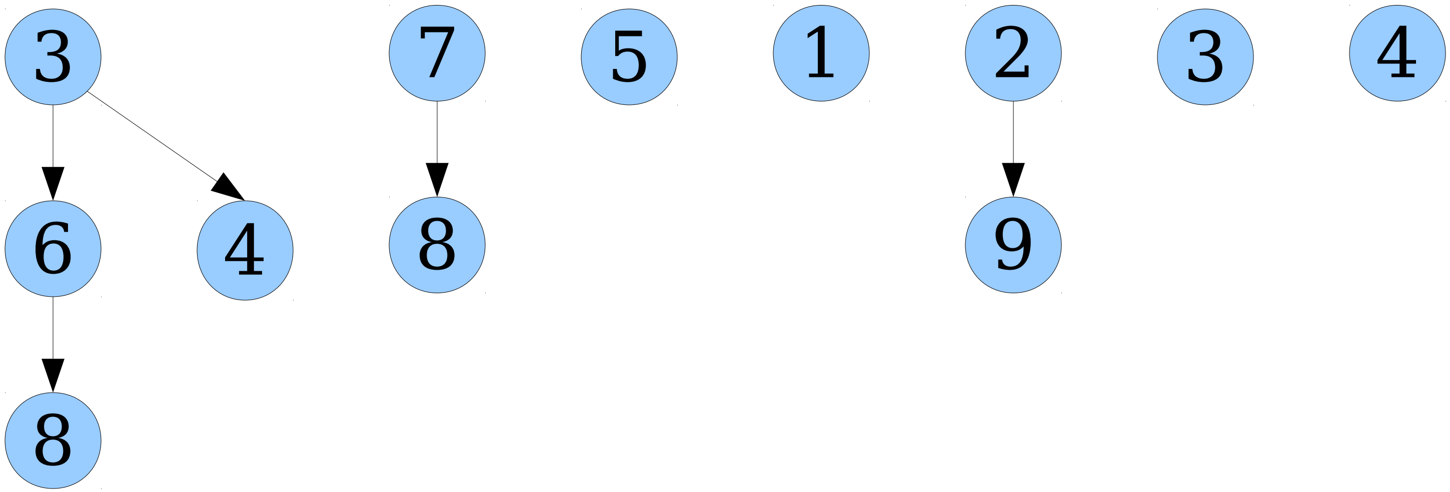
- Why does ***enqueue*** take time O(log n)?
- ***Answer***: May have to combine together O(log n) different binomial trees together into a single tree.
- ***New Question***: What happens if we don't combine trees together?
- That is, what if we just add a new singleton tree to the list?

Lazy Melding

- More generally, consider the following lazy melding approach:

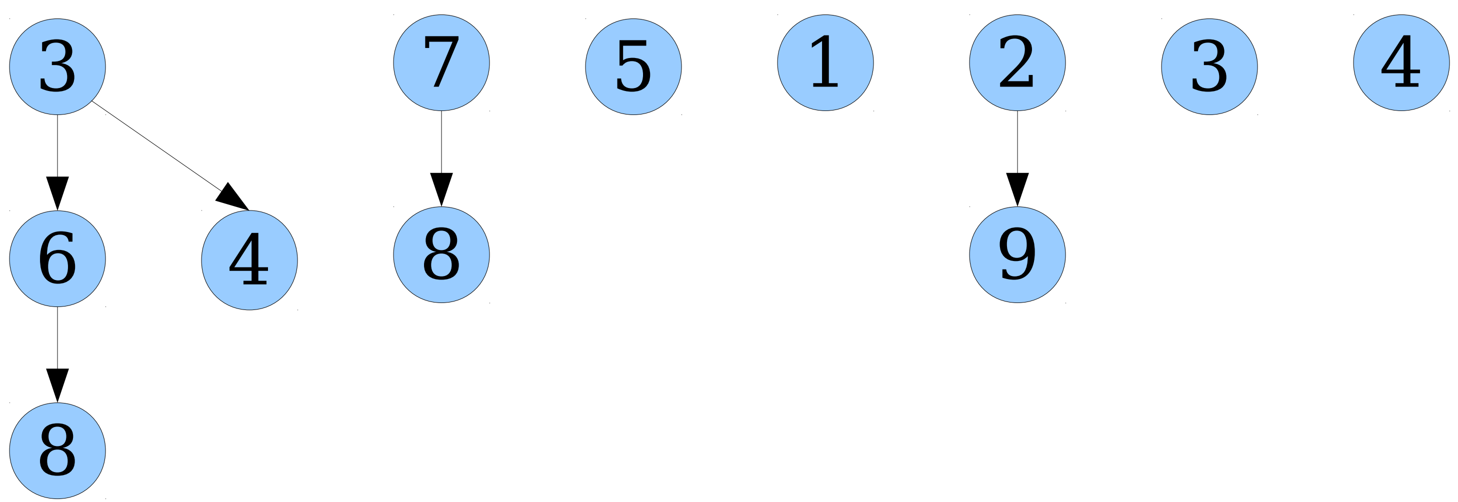
  To meld together two binomial heaps, just combine the two sets of trees together.

- If we assume the trees are stored in doubly-linked lists, this can be done in time O(1).



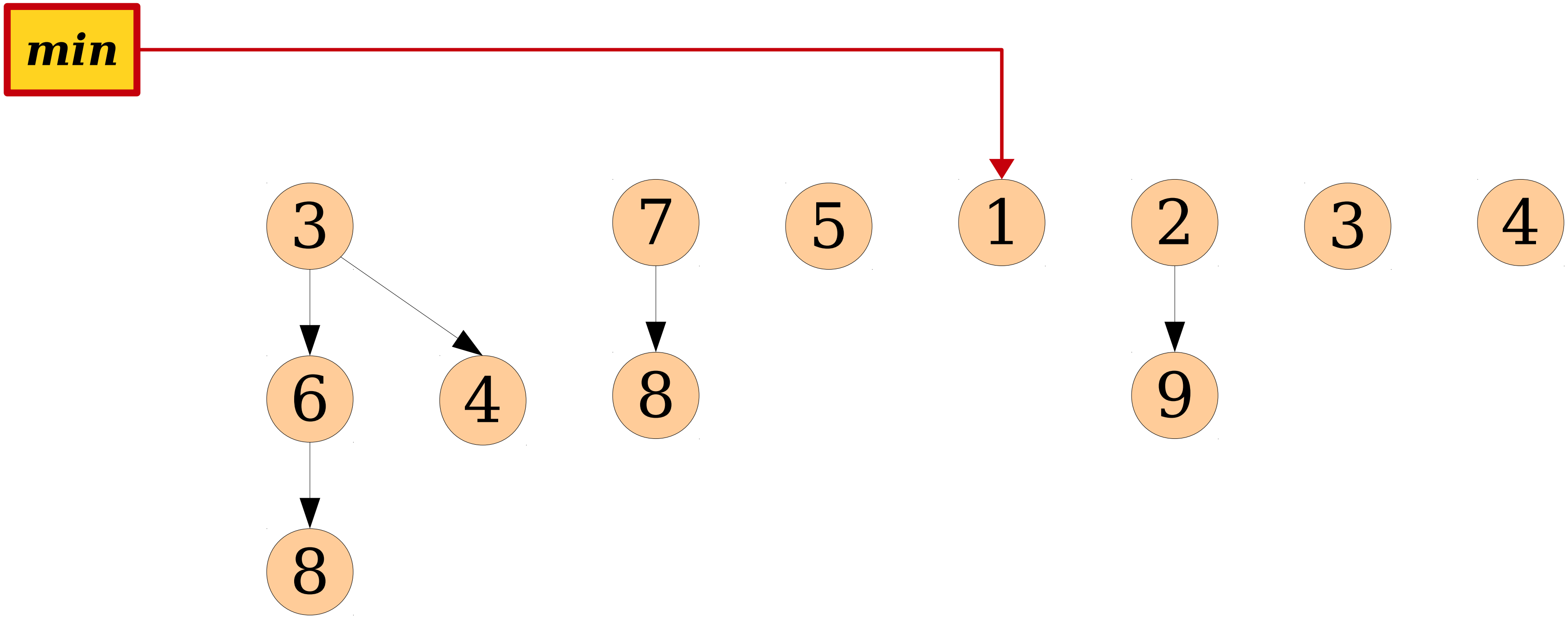
The Catch: Part One

- When we use eager melding, the number of trees is O(log n).
- Therefore, ***find-min*** runs in time O(log n).
- ***Problem:*** ***find-min*** no longer runs in time O(log n) because there can be Θ(n) trees.



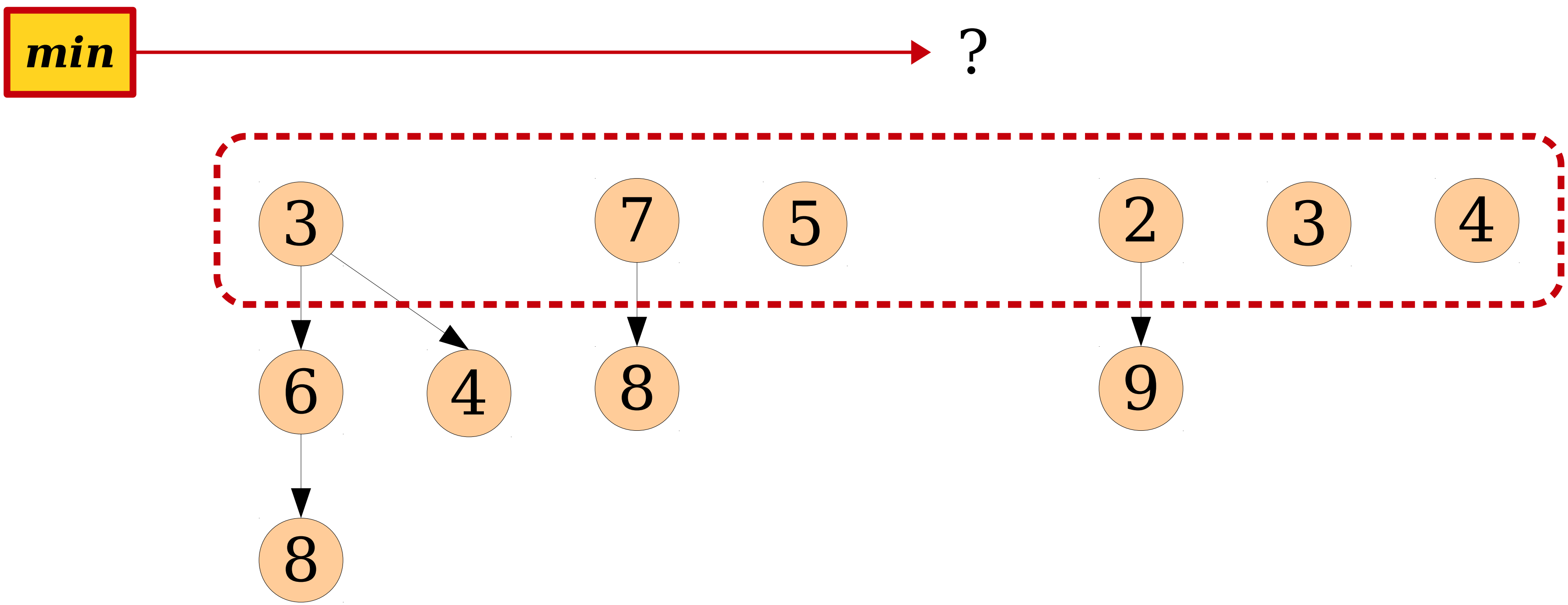
A Solution

- Have the binomial heap store a pointer to the minimum element.
- Can be updated in time O(1) after doing a meld by comparing the minima of the two heaps.



The Catch: Part Two

- Even with a pointer to the minimum, deletions might now run in time $\Theta(n)$.
- ***Rationale:*** Need to update the pointer to the minimum.

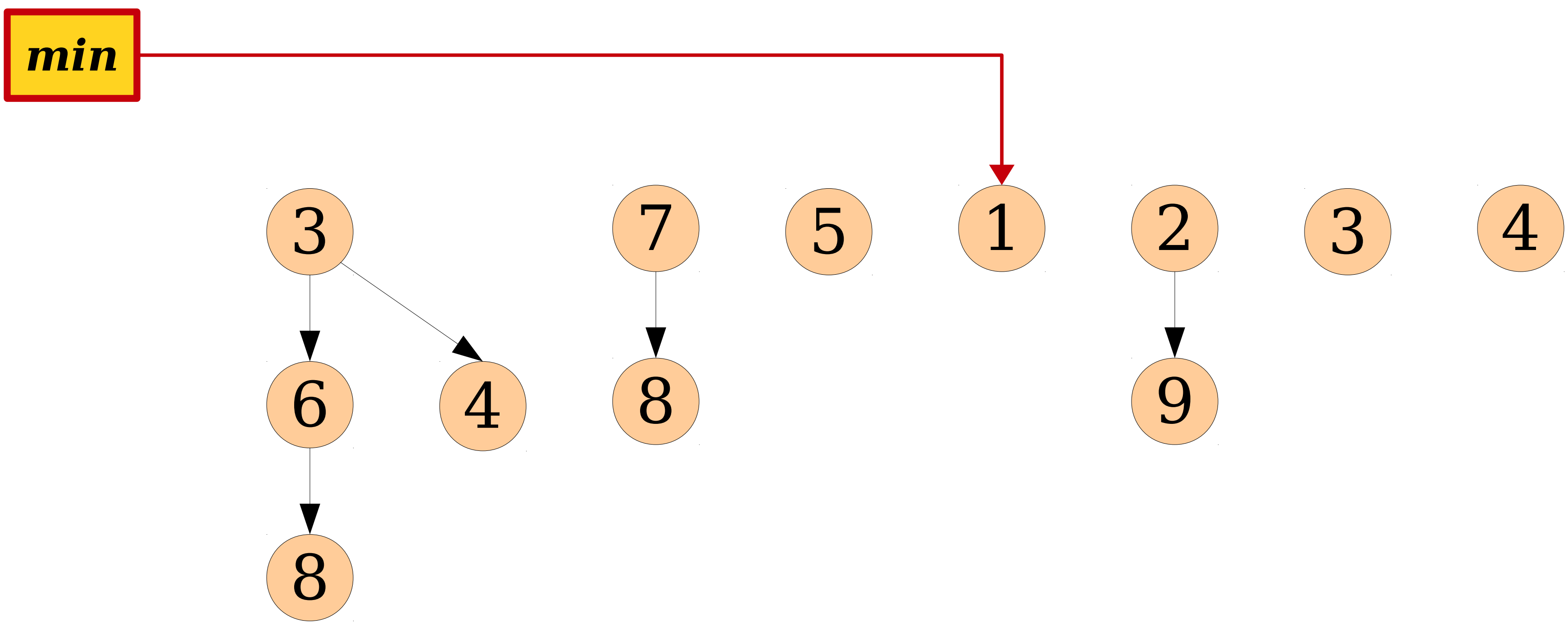


Resolving the Issue

- ***Idea:*** When doing an ***extract-min***, coalesce all of the trees so that there's at most one tree of each order.
- Intuitively:
  - The number of trees in a heap grows slowly (only during an insert or meld).
  - The number of trees in a heap drops rapidly after coalescing (down to O(log n)).
  - Can backcharge the work done during an ***extract-min*** to ***enqueue*** or ***meld***.

Coalescing Trees

- Our eager melding algorithm assumes that
  - there is either zero or one tree of each order, and that
  - the trees are stored in ascending order.
- ***Challenge:*** When coalescing trees in this case, neither of these properties necessarily hold.



Wonky Arithmetic

- Compute the number of bits necessary to hold the sum.
  - Only O(log n) bits are needed.
- Create an array of that size, initially empty.
- For each packet:
  - If there is no packet of that size, place the packet in the array at that spot.
  - If there is a packet of that size:
    - Fuse the two packets together.
    - Recursively add the new packet back into the array.

Now With Trees!

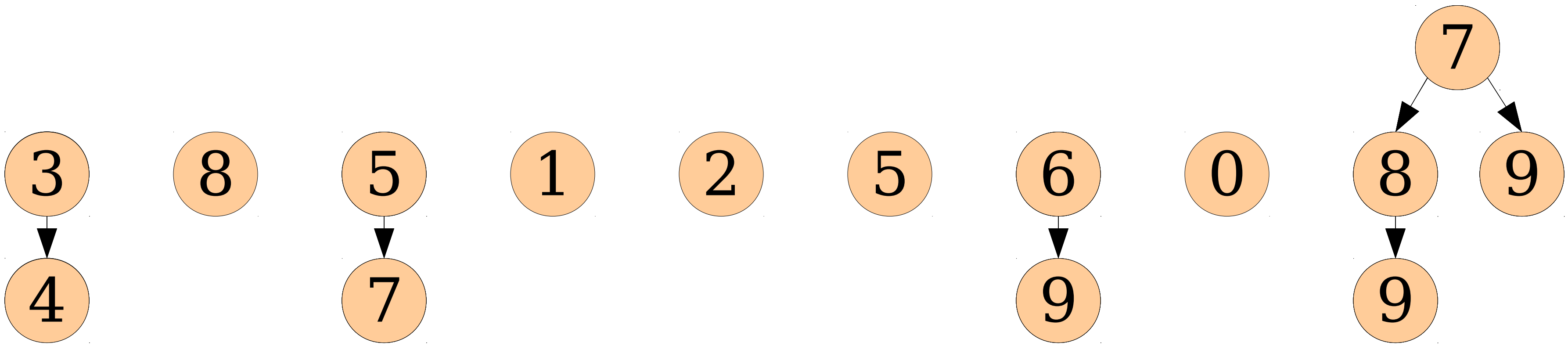
- Compute the number of *trees* necessary to hold the *nodes*.
  - Only O(log n) *trees* are needed.
- Create an array of that size, initially empty.
- For each *tree*:
  - If there is no *tree* of that size, place the *tree* in the array at that spot.
  - If there is a *tree* of that size:
    - Fuse the two *trees* together.
    - Recursively add the new *tree* back into the array.

Coalescing Trees

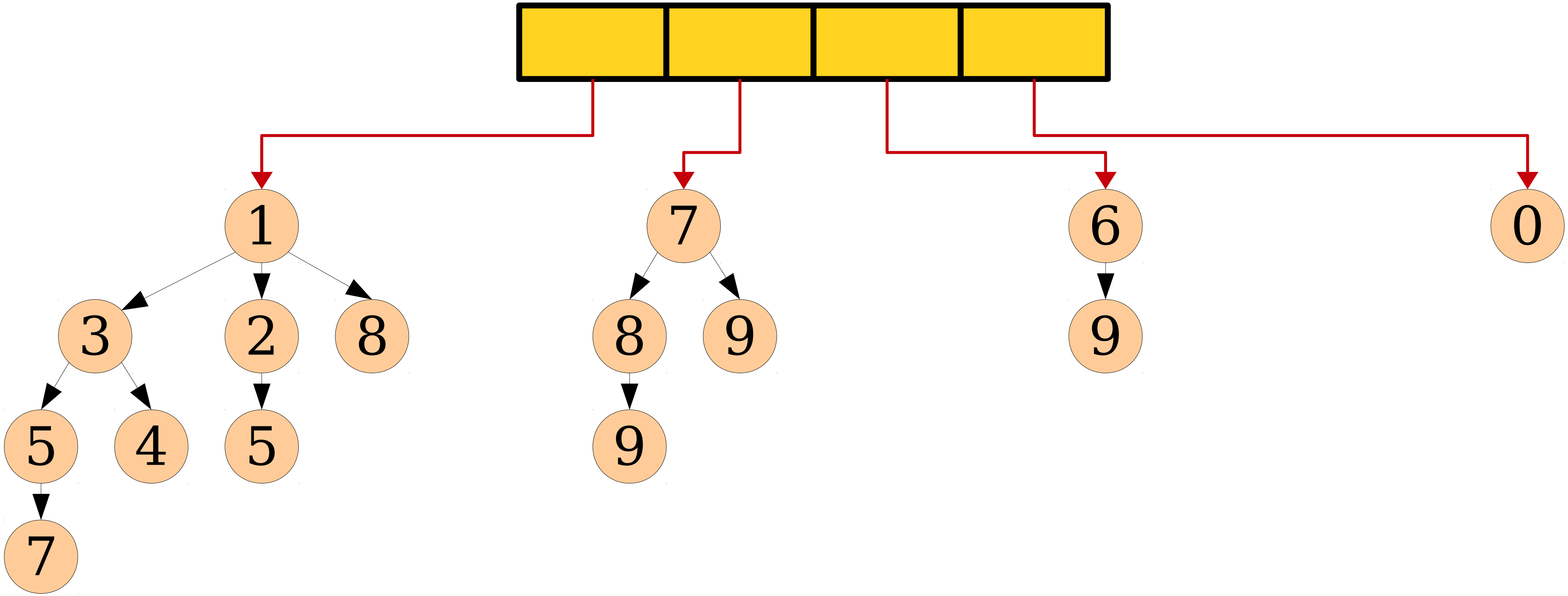
Total number of nodes: **15**

(Can compute in time $\Theta(T)$, where T is the number of trees, if each tree is tagged with its order)

Bits needed: **4**



Coalescing Trees



Analyzing Coalesce

- Suppose there are T trees.
- We spend $\Theta(T)$ work iterating across the main list of trees twice:
  - Pass one: Count up number of nodes (if each tree stores its order, this takes time $\Theta(T)$).
  - Pass two: Place each node into the array.
- Each merge takes time O(1).
- The number of merges is $O(T)$.
- Total work done: $\boldsymbol{\Theta(T)}$.
- In the worst case, this is $O(n)$.

The Story So Far

- A binomial heap with lazy melding has these worst-case time bounds:
  - ***enqueue***: O(1)
  - ***meld***: O(1)
  - ***find-min***: O(1)
  - ***extract-min***: O(n).
- These are *worst-case* time bounds. What about an *amortized* time bounds?

An Observation

- The expensive step here is ***extract-min***, which runs in time proportional to the number of trees.
- Each tree can be traced back to one of three sources:
  - An ***enqueue***.
  - A ***meld*** with another heap.
  - A tree exposed by an ***extract-min***.
- Let's use an amortized analysis to shift the blame for the ***extract-min*** performance to other operations.

The Potential Method

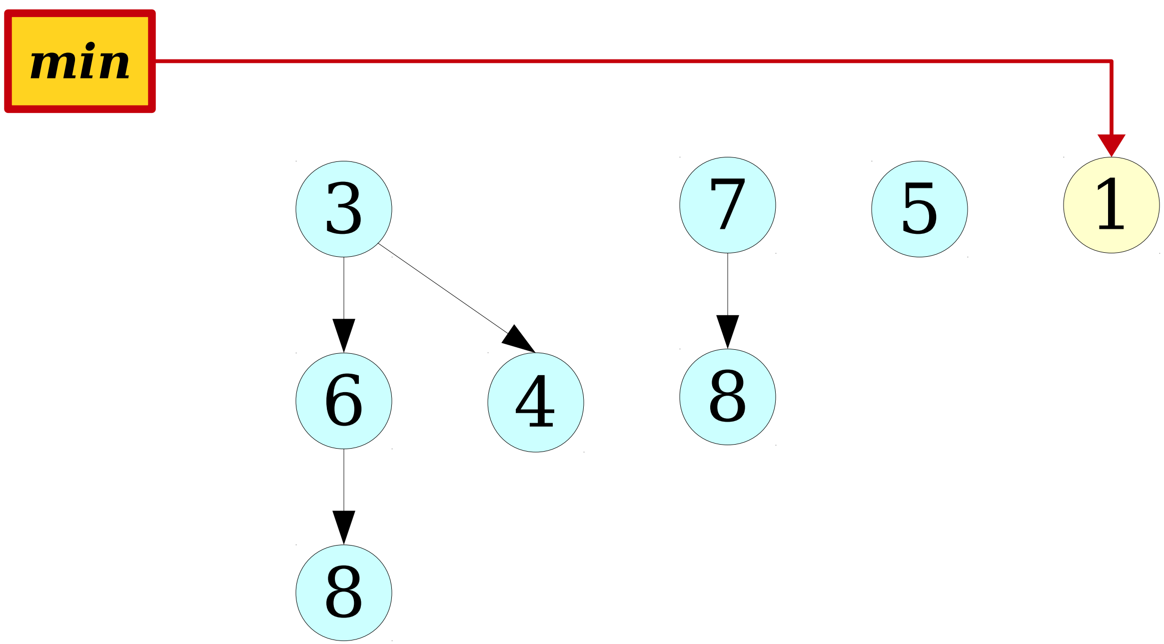
- We will use the potential method in this analysis.
- When analyzing insertions with eager merges, we set $\Phi(D)$ to be the number of trees in D.
- Let's see what happens if we use this Φ here.

Analyzing an Insertion

- To ***enqueue*** a key, we add a new binomial tree to the forest and possibly update the *min* pointer.

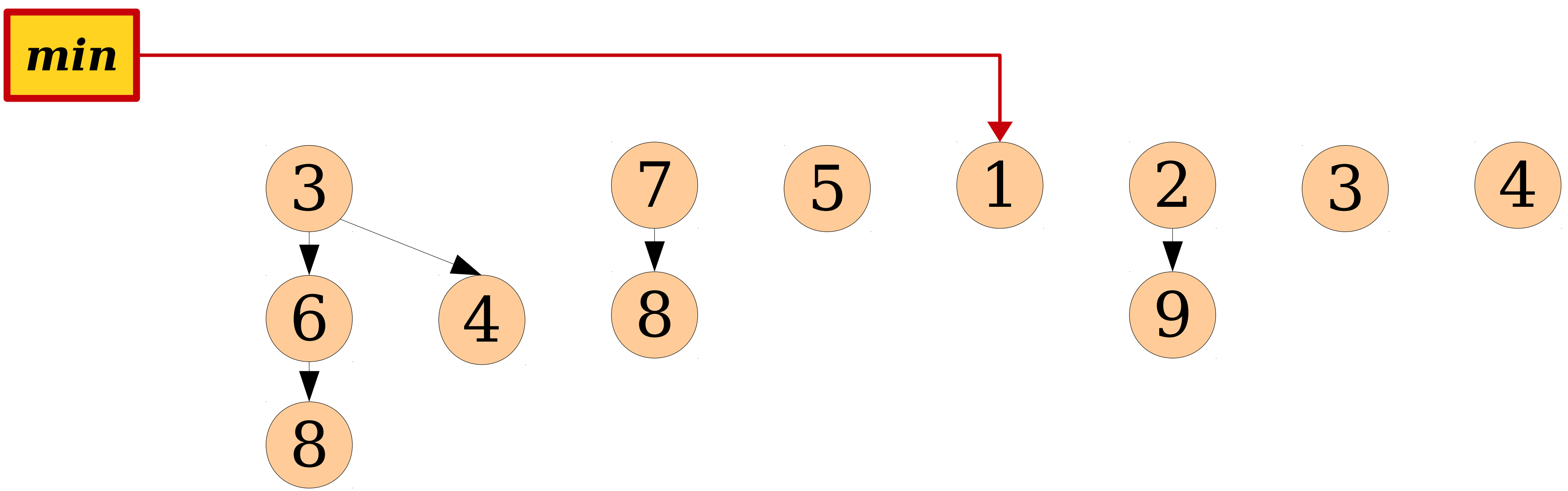
  Actual time: O(1). ΔΦ: +1

  Amortized cost: **O(1)**.



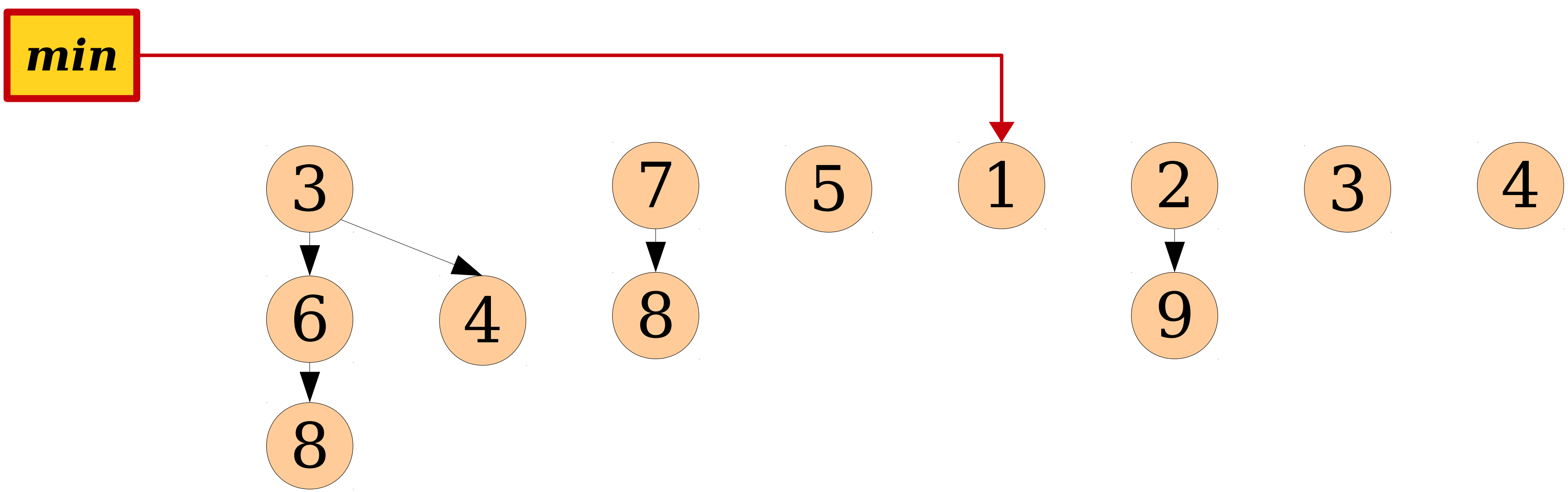
Analyzing a Meld

- Suppose that we ***meld*** two lazy binomial heaps B_1 and B_2. Actual cost: O(1).
- Let Φ_{B_1} and Φ_{B_2} be the initial potentials of B_1 and B_2.
- The new heap B has potential $\Phi_{B_1} + \Phi_{B_2}$ and B_1 and B_2 have potential 0.
- $\Delta\Phi$ is zero.
- Amortized cost: **O(1)**.



Analyzing a Find-Min

- Each ***find-min*** does O(1) work and does not add or remove trees.
- Amortized cost: **O(1)**.



Analyzing Extract-Min

- Suppose we perform an ***extract-min*** on a binomial heap with T trees in it.
- Initially, we expose the children of the minimum element. This increases the number of trees to $T + O(\log n)$.
- The runtime for coalescing these trees is $O(T + \log n)$.
- When we're done merging, there will be $O(\log n)$ trees remaining, so $\Delta\Phi = -T + O(\log n)$.
- Amortized cost is

$$\begin{aligned} &\Theta(T + \log n) + O(1) \cdot (-T + O(\log n)) \\ = \; &\Theta(T) - O(1) \cdot T + O(1) \cdot O(\log n) \\ = \; &\mathbf{O(\log n)}. \end{aligned}$$

The Overall Analysis

- The *amortized* costs of the operations on a lazy binomial heap are as follows:
  - ***enqueue***: O(1)
  - ***meld***: O(1)
  - ***find-min***: O(1)
  - ***extract-min***: O(log n)
- Any series of e ***enqueues*** mixed with d ***extract-min***s will take time $O(e + d \log e)$.

Why This Matters

- Lazy binomial heaps are a powerful building block used in many other data structures.
- We'll see one of them, the *Fibonacci heap,* when we come back on Thursday.
- You'll see another (supporting ***add-to-all***) on the problem set.

Next Time

- ***The Need for decrease-key***
  - A powerful and versatile operation on priority queues.
- ***Fibonacci Heaps***
  - A variation on lazy binomial heaps with efficient decrease-key.
- ***Implementing Fibonacci Heaps***
  - … is harder than it looks!