

Solving a Recurrence via Induction

Consider the recurrence

$$T(n) = 4T(n/2) + \frac{n^2}{\lg n},$$

where n is a power of two and $T(1) = \Theta(1)$. We prove by induction, separately for the upper and lower bounds, that

$$T(n) = \Theta(n^2 \log \log n).$$

We begin the induction at $n = 4$ so that the logarithms below are positive. Small values of n can be absorbed into the constants. Let

$$L = \lg n \quad \text{and} \quad g(n) = n^2 \ln(\lg n) = n^2 \ln L.$$

The function $g(n)$ is $\Theta(n^2 \log \log n)$; the bases of the logarithms affect only constant factors and lower-order terms.

We will use the following inequalities. For $L \geq 2$,

$$\frac{1}{L} \leq \ln L - \ln(L-1) = -\ln\left(1 - \frac{1}{L}\right) \leq \frac{1}{L-1}. \quad (1)$$

They follow from $x \leq -\ln(1-x) \leq x/(1-x)$ for $0 < x < 1$, with $x = 1/L$.

Notation convention. An inequality with a question mark above its relation symbol, such as $A \overset{?}{\leq} B$ or $A \overset{?}{\geq} B$, is a desired inequality that has not yet been established. We retain the question mark through every logically equivalent reformulation of that desired inequality. An unmarked relation states a known fact that has already been justified.

Upper bound

Claim. *There is a constant $C > 0$ such that*

$$T(n) \leq Cn^2 \ln(\lg n)$$

for every power of two $n \geq 4$.

Proof. Choose

$$C \geq \max\left\{1, \frac{T(4)}{4^2 \ln(\lg 4)}\right\}.$$

This choice establishes the base case $n = 4$.

For the inductive step, let $n \geq 8$ and assume

$$T(n/2) \leq C \left(\frac{n}{2}\right)^2 \ln\left(\lg \frac{n}{2}\right).$$

Since $\lg(n/2) = L - 1$, the recurrence and the inductive hypothesis give

$$\begin{aligned} T(n) &= 4T(n/2) + \frac{n^2}{L} \\ &\leq 4C \left(\frac{n}{2}\right)^2 \ln(L-1) + \frac{n^2}{L} \\ &= Cn^2 \ln(L-1) + \frac{n^2}{L}. \end{aligned}$$

To complete the inductive step, we want the last expression to be at most the target $Cn^2 \ln L$. Working algebraically on this desired inequality gives the following equivalent statements:

$$\begin{aligned} Cn^2 \ln(L-1) + \frac{n^2}{L} &\stackrel{?}{\leq} Cn^2 \ln L \\ \iff C \ln(L-1) + \frac{1}{L} &\stackrel{?}{\leq} C \ln L \\ \iff \frac{1}{L} &\stackrel{?}{\leq} C(\ln L - \ln(L-1)) \\ \iff \frac{1}{L} &\stackrel{?}{\leq} C \ln \left(\frac{L}{L-1} \right) \\ \iff C &\stackrel{?}{\geq} \frac{1}{L \ln \left(\frac{L}{L-1} \right)}. \end{aligned}$$

By the left-hand inequality in (1),

$$L \ln \left(\frac{L}{L-1} \right) \geq 1, \quad \text{so} \quad \frac{1}{L \ln \left(\frac{L}{L-1} \right)} \leq 1.$$

Thus our choice $C \geq 1$ makes the wanted inequality true, and hence

$$T(n) \leq Cn^2 \ln L = Cn^2 \ln(\lg n).$$

This completes the induction and proves $T(n) = O(n^2 \log \log n)$. □

Lower bound

Claim. *There is a constant $c > 0$ such that*

$$T(n) \geq cn^2 \ln(\lg n)$$

for every power of two $n \geq 4$.

Proof. Choose

$$0 < c \leq \min \left\{ \frac{1}{2}, \frac{T(4)}{4^2 \ln(\lg 4)} \right\}.$$

This establishes the base case $n = 4$.

For the inductive step, let $n \geq 8$ and assume

$$T(n/2) \geq c \left(\frac{n}{2}\right)^2 \ln \left(\lg \frac{n}{2} \right).$$

The recurrence and the inductive hypothesis imply

$$\begin{aligned} T(n) &= 4T(n/2) + \frac{n^2}{L} \\ &\geq cn^2 \ln(L-1) + \frac{n^2}{L}. \end{aligned}$$

To complete the inductive step, we want the last expression to be at least the target $cn^2 \ln L$. Working algebraically on this desired inequality gives the following equivalent statements:

$$\begin{aligned} &cn^2 \ln(L-1) + \frac{n^2}{L} \stackrel{?}{\geq} cn^2 \ln L \\ \iff &c \ln(L-1) + \frac{1}{L} \stackrel{?}{\geq} c \ln L \\ \iff &\frac{1}{L} \stackrel{?}{\geq} c(\ln L - \ln(L-1)) \\ \iff &\frac{1}{L} \stackrel{?}{\geq} c \ln \left(\frac{L}{L-1} \right) \\ \iff &c \stackrel{?}{\leq} \frac{1}{L \ln \left(\frac{L}{L-1} \right)}. \end{aligned}$$

By the right-hand inequality in (1), and because $L \geq 2$,

$$L \ln \left(\frac{L}{L-1} \right) \leq \frac{L}{L-1} \leq 2, \quad \text{so} \quad \frac{1}{L \ln \left(\frac{L}{L-1} \right)} \geq \frac{1}{2}.$$

Thus our choice $c \leq 1/2$ makes the wanted inequality true, and hence

$$T(n) \geq cn^2 \ln L = cn^2 \ln(\lg n).$$

This completes the induction and proves $T(n) = \Omega(n^2 \log \log n)$. □

Combining the independently proved upper and lower bounds gives

$$\boxed{T(n) = \Theta(n^2 \log \log n)}.$$