More generally (see text):

$$T(n) = a\,T(n/b) + f(n) \longleftarrow \text{not necessarily a simple polynomial}$$

e.g. $$T(n) = 4\,T(n/2) + \Theta(n^2 \log n)$$

<u>Case 1</u>: $\cancel{\dfrac{f(n)}{n^{\log_b a}} = O(n^{-\varepsilon})}$ for some $\varepsilon > 0$ $\quad \boxed{f(n) = O\left(n^{\log_b a - \varepsilon}\right)}$

$$T(n) = \Theta\left(n^{\log_b a}\right)$$

<u>Case 2</u>: $\cancel{\dfrac{f(n)}{n^{\log_b a}} = \Theta(\log^k n)}$ for some $k \geq 0$ $\quad \boxed{f(n) = \Theta\left(n^{\log_b a} \log^k n\right)}$

$$T(n) = \Theta\left(n^{\log_b a} \log^{k+1} n\right)$$

<u>Case 3</u>: $\cancel{\dfrac{f(n)}{n^{\log_b a}} = \Omega(n^{\varepsilon})}$ for some $\varepsilon > 0$ $\quad \boxed{f(n) = \Omega\left(n^{\log_b a + \varepsilon}\right)}$

(<u>regularity condition</u>) Then, as long as $a\,f(n/b) \leq \alpha\, f(n)$ for some const. $\alpha < 1$,

$$T(n) = \Theta(f(n))$$

Examples

(1) $T(n) = 4T(n/2) + \Theta(n^2 \log n)$

$$\frac{f(n)}{n^{\log_b a}} = \frac{n^2 \log n}{n^{\log_2 4}} = \log n$$

Case 2 : $T(n) = \Theta(n^2 \log^2 n)$

(2) $T(n) = 4T(n/2) + \Theta(n^3)$

$$\frac{f(n)}{n^{\log_b a}} = \frac{n^3}{n^{\log_2 4}} = \frac{n^3}{n^2} = n$$

Case 3 : Check regularity condition

$4(n/2)^3 \leq \alpha n^3$ for some $\alpha < 1$?

$4 \frac{n^3}{8} \leq \alpha n^3$

$\alpha \geq 1/2$ ✓

$T(n) = \Theta(n^3)$

(3) $T(n) = 4T(n/2) + \frac{n^2}{\lg n}$

$$\frac{f(n)}{n^{\log_b a}} = \frac{n^2/\lg n}{n^{\log_2 4}} = \frac{1}{\lg n}$$

Case 1 — no $\quad \frac{1}{\lg n} = O(n^{-\varepsilon}) \iff \lg n = \Omega(n^{\varepsilon})$ X

Case 2 — no $\quad \frac{1}{\lg n} = \Theta(\log^k n)$ for $k \geq 0$ X

Case 3 — no $\quad \frac{1}{\lg n} = \Omega(n^{\varepsilon})$ X

No case applies ... can show $T(n) = \Theta(n^2 \log\log n)$ by careful iteration.