

Progressions / series

arithmetic

$$1+2+3+\dots+n = \sum_{k=1}^n k = \frac{n(n+1)}{2}$$

quadratic

$$1^2+2^2+3^2+\dots+n^2 = \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

geometric

base $x > 0$ fixed

$$x^0 + x^1 + x^2 + \dots + x^n$$

if $x < 1$

$$x^0 + x^1 + x^2 + \dots + x^n \approx \frac{1}{1-x}$$

= finite

$$\frac{x^{n+1} - 1}{x - 1}$$

if $x \neq 1$

$x=1$ $1^0 + 1^1 + 1^2 + \dots + 1^n = n+1 = \Theta(n)$ if $x=1$

harmonic

$$H_n = \frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{n} \approx \ln(n) + \text{constant} = \Theta(\ln(n))$$

$$T(n) = 4T\left(\frac{n}{2}\right) + n$$

$$T(n) = 4t\left(\frac{n}{2}\right) + \Theta(n)$$

Divide & conquer

→ pb(n)
RT

→ solve
4 subproblems
of size $\frac{n}{2}$

→ non-rec load
= before solving subpb — split
= after solving subpb (combine) — decisions

ITERATIONS
= algebra or tree

$$k=1 \quad T(n) = n + 4T\left(\frac{n}{2}\right)$$

$$k=2 = n + 4\left[\frac{n}{2} + 4T\left(\frac{n}{4}\right)\right] = n + 2n + 4^2 T\left(\frac{n}{4}\right)$$

$$k=3 = n + 2n + 4^2\left[\frac{n}{4} + 4T\left(\frac{n}{8}\right)\right] = n + 2n + 4n + 4^3 T\left(\frac{n}{2^3}\right)$$

$$k=4 = n + 2n + 4n + 4^3\left[\frac{n}{2^3} + 4T\left(\frac{n}{2^4}\right)\right] = n + 2n + 4n + 8n + 4^4 T\left(\frac{n}{2^4}\right)$$

General k

$$= n + 2n + 4n + \dots + 2^{k-1}n + 4^k T\left(\frac{n}{2^k}\right)$$

$$= n(2^0 + 2^1 + 2^2 + \dots + 2^{k-1}) + 4^k T\left(\frac{n}{2^k}\right)$$

$$= n \frac{2^k - 1}{2 - 1} + 4^k T\left(\frac{n}{2^k}\right)$$

Last k we want $T\left(\frac{n}{2^k}\right) = \text{base} \approx T(1)$

$$\frac{n}{2^k} \approx 1 \Leftrightarrow k \approx \log(n)$$

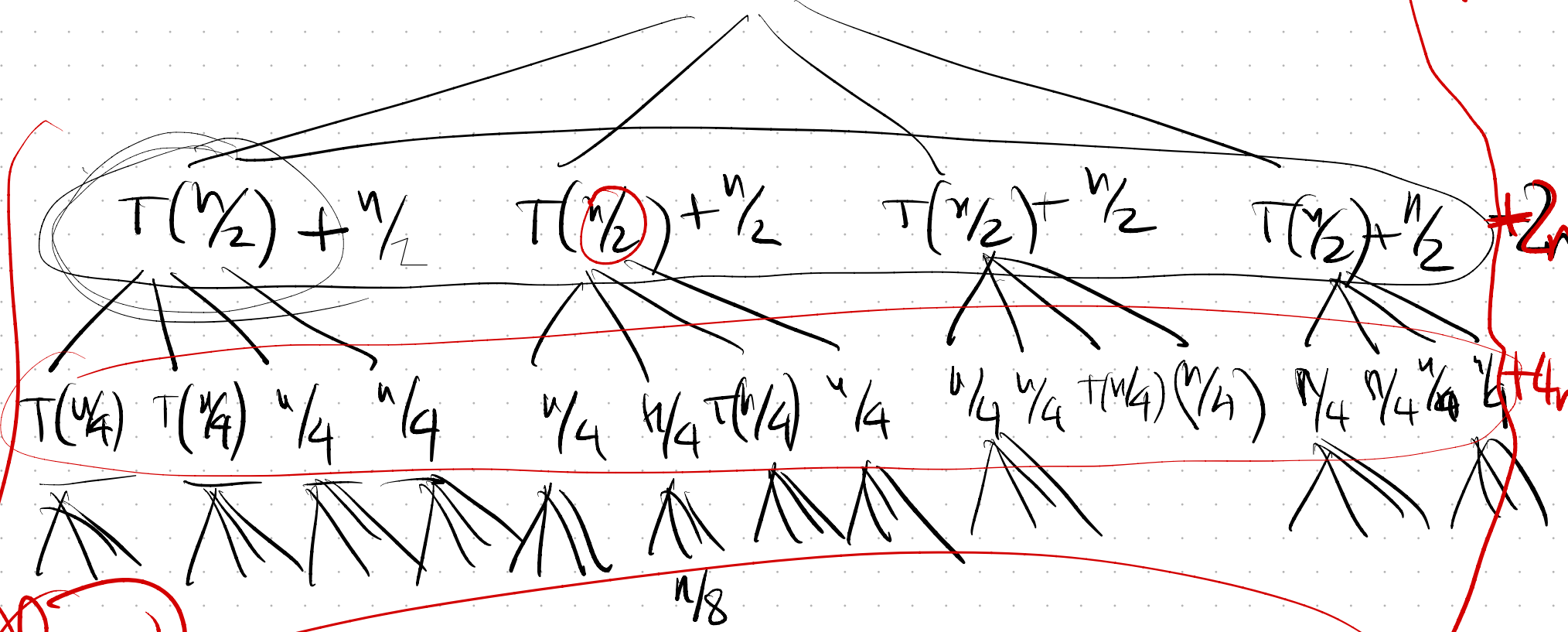
$$= n \cdot (2^{\log(n)} - 1) + 4^{\log(n)} T(1)$$

$$= n \cdot (n - 1) + 2^{2 \log n} T(1)$$

$$\Theta(n^2) + \Theta(n^2)$$

$$\Theta(n^2)$$

ITERATION TREE $T(n)$



$\log(n)$ layers

rec component

$8n$ non-rec

Total calls?

$4^{\log(n)}$ leaves in tree

$n + 2n + 4n + \dots + n$
 $\dots \Theta(n^2)$

$$T(n) = n^2 + T\left(\frac{n}{2}\right) + T\left(\frac{n}{4}\right)$$

$$= n^2 + \left[\left(\frac{n}{2}\right)^2 + T\left(\frac{n}{4}\right) + T\left(\frac{n}{8}\right) \right] + \left[\left(\frac{n}{4}\right)^2 + T\left(\frac{n}{8}\right) + T\left(\frac{n}{16}\right) \right]$$

$$= n^2 + \frac{5}{16}n^2 + T\left(\frac{n}{4}\right) + 2T\left(\frac{n}{8}\right) + T\left(\frac{n}{16}\right)$$

$$= n^2 + \frac{5}{16}n^2 + \left[\left(\frac{n}{4}\right)^2 + T\left(\frac{n}{8}\right) + T\left(\frac{n}{16}\right) \right] + 2 \left[\left(\frac{n}{8}\right)^2 + T\left(\frac{n}{16}\right) + T\left(\frac{n}{32}\right) \right] +$$

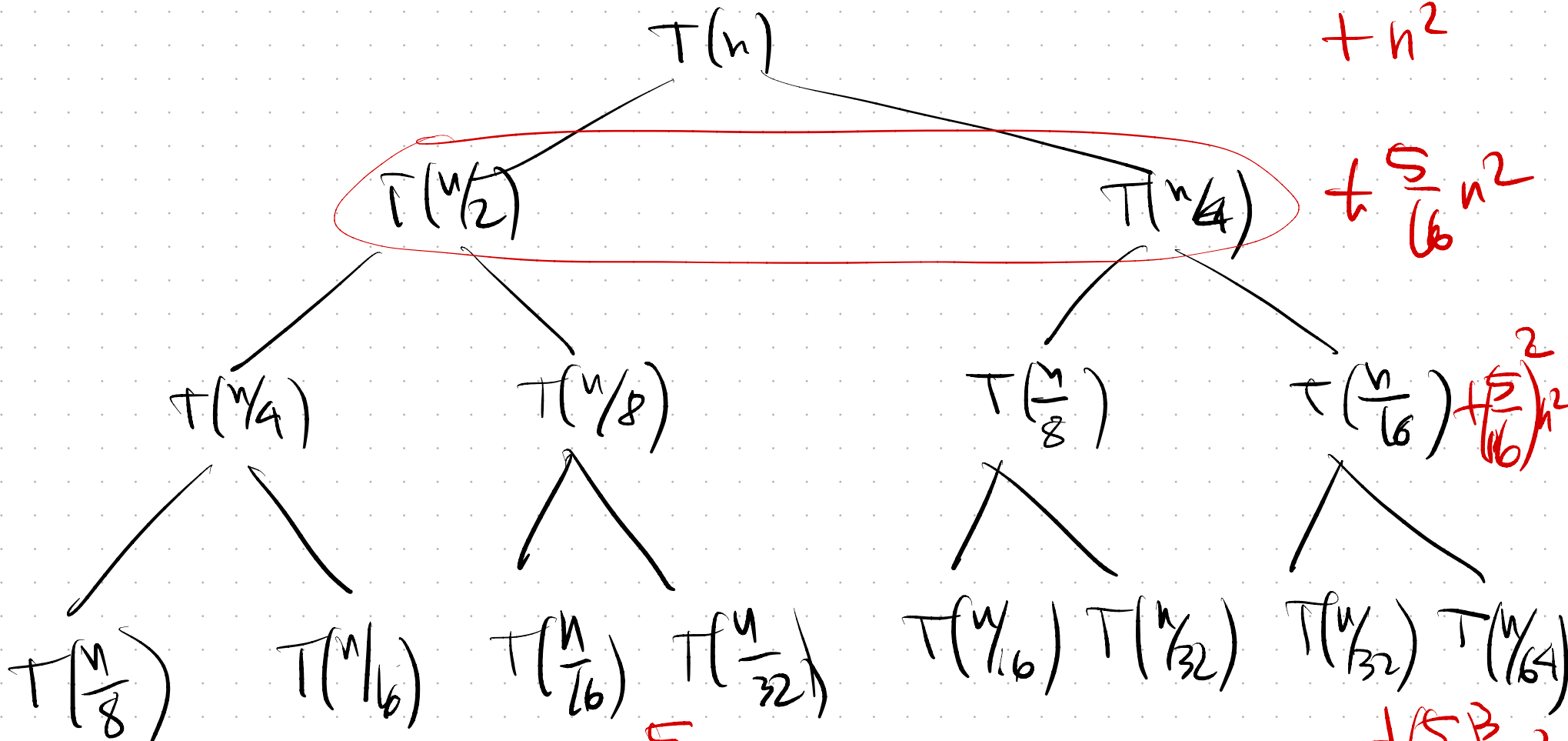
$$+ \left[\left(\frac{n}{16}\right)^2 + T\left(\frac{n}{32}\right) + T\left(\frac{n}{64}\right) \right]$$

$$= n^2 + \frac{5}{16}n^2 + \frac{25}{256}n^2 + T\left(\frac{n}{8}\right) + 3T\left(\frac{n}{16}\right) + 3T\left(\frac{n}{32}\right) + T\left(\frac{n}{64}\right)$$

||

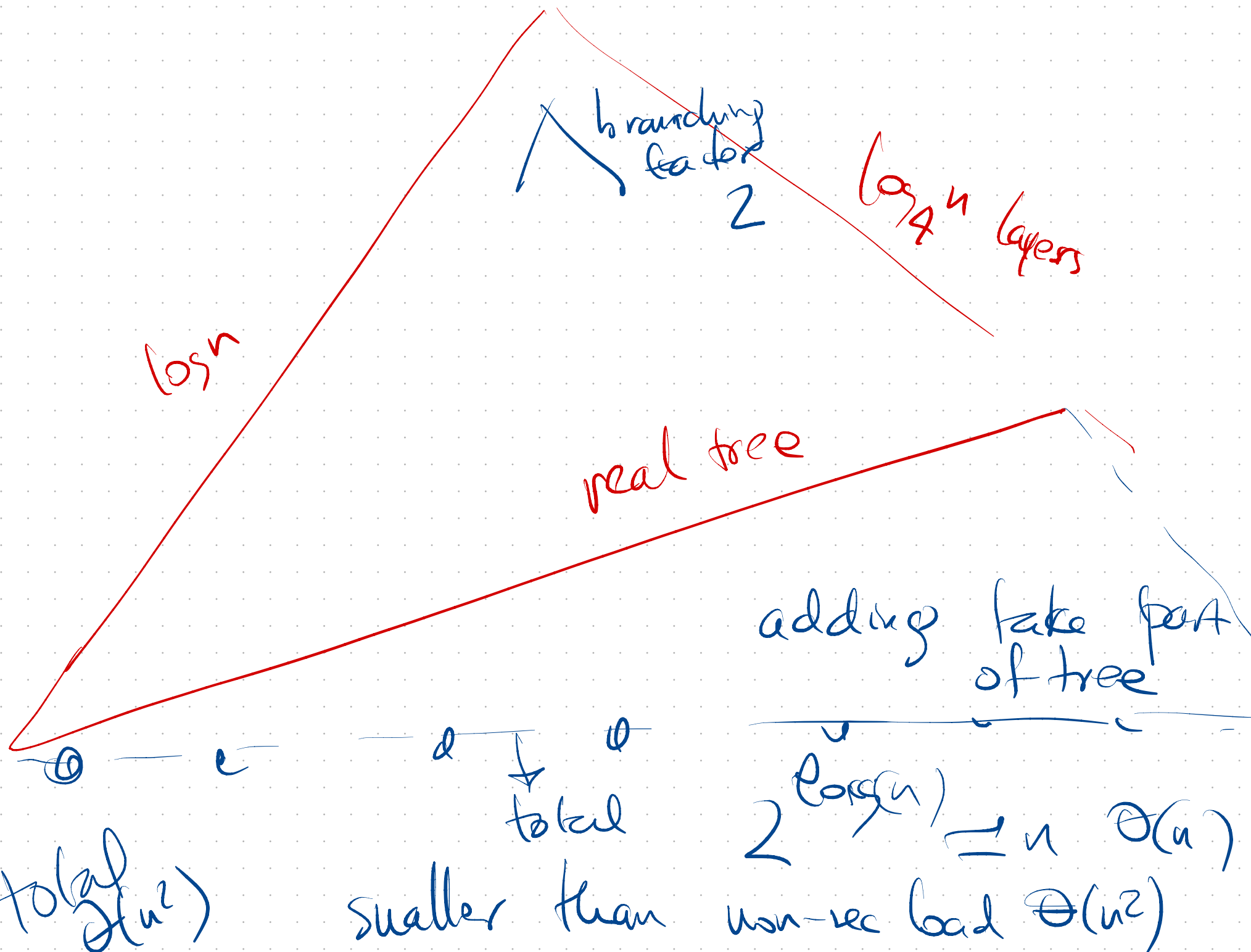
$$+ \left(\frac{5}{16}\right)^3 n^2$$

~ Pascal Δ
??



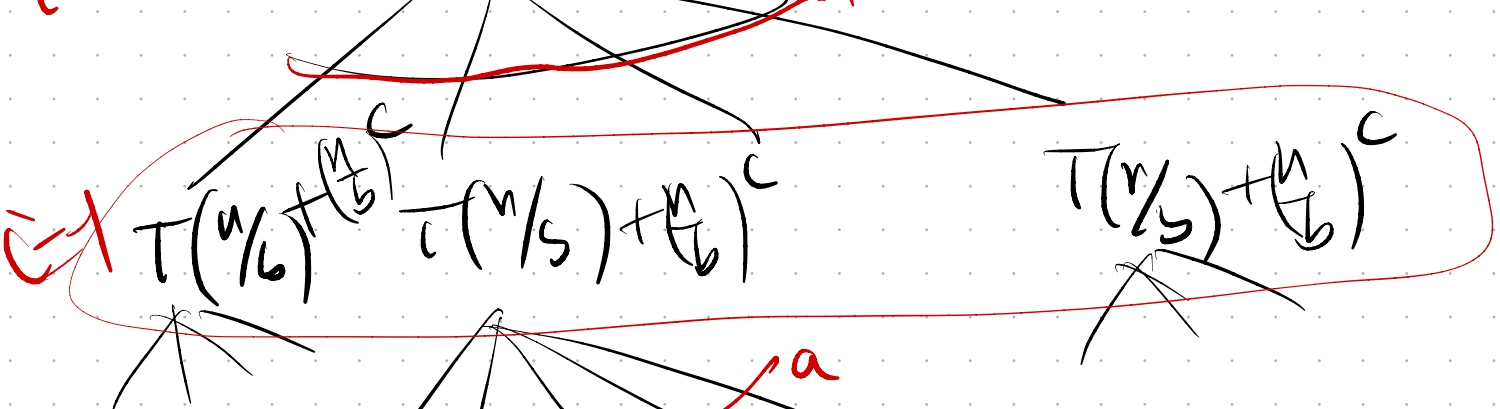
$+n^2$
 $+ \frac{5}{16} n^2$
 $+ \left(\frac{5}{16}\right)^2 n^2$
 $+ \left(\frac{5}{16}\right)^3 n^2$

non HC $n^2 \left(a^0 + a^1 + \dots + a^{\text{last layer}} \right) = \Theta(n^2)$
 $a = \frac{5}{16}$
 constant $< \frac{1}{1-a}$

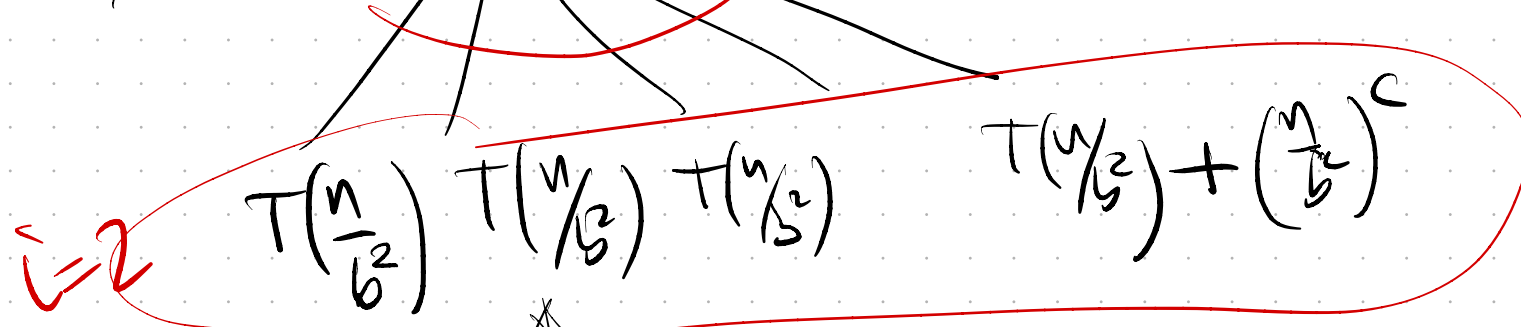


Simple Master Theorem $T(n) = aT(n/b) + \Theta(n^c)$
 a branches, $\frac{n}{b}$ subproblems each n^c non rec load

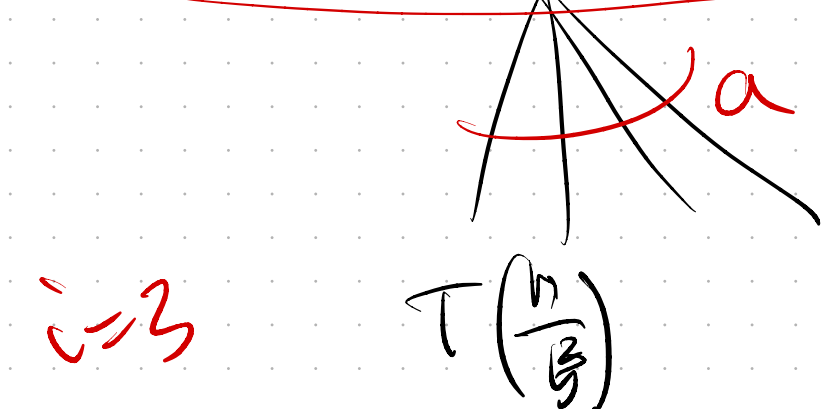
$i=0$ $T(n) + n^c \left(\frac{a}{b^c}\right)^0$



$a \cdot \left(\frac{n}{b}\right)^c$



a^2 branches $a^2 \cdot \left(\frac{n}{b^2}\right)^c$



layers $\log_b(n)$

$a^3 \cdot \left(\frac{n}{b^3}\right)^c$
 $\left(\frac{a}{b^c}\right)^3 \cdot n^c$

$\frac{a}{b^c} = \text{base}$

Total

$$n^c \sum_{i=0}^{\log_b n - 1} \left(\frac{a}{b^c}\right)^i$$

non-rec
geom series

$$+ a \log_b n$$

$(n^{\log_b a})$

rec
calls

Case 1 - exp

base $\frac{a}{b^c} > 1$

$C < \log_b a$

Case 2 - linear

base $\frac{a}{b^c} = 1$

$C = \log_b a$

Case 3 - const

base $\frac{a}{b^c} < 1$

$C > \log_b a$

Case 1 $c < \log_b a \Leftrightarrow \frac{a}{bc} > 1$

$$T(n) = n^c \sum_{i=0}^{\log_b n - 1} \left(\frac{a}{bc}\right)^i + \Theta(n^{\log_b a})$$

$$= n^c \frac{\left(\frac{a}{bc}\right)^{\log_b n} - 1}{\left(\frac{a}{bc}\right) - 1} + \Theta(n^{\log_b a})$$

$$\geq \Theta\left(n^c \frac{a^{\log_b n}}{(bc)^{\log_b n}}\right) + \Theta(n^{\log_b a})$$

$$\geq \Theta\left(\frac{n^c}{n^{\log_b(bc)}}\right)$$

$$x^0 + x^1 + \dots + x^{r-1} = \frac{x^r - 1}{x - 1}$$

$$x = \frac{a}{bc}$$

$$r = \log_b n - 1$$

???

$$\frac{a^{\log_b x} \log_b a}{a} = x$$

$$= \Theta(n^{\log_5 a}) \quad \checkmark$$

Case 2 $c = \log_5 a$ $\frac{a}{b^c} = 1$

$$T(n) = n^c \sum_{i=0}^{\log_5 n} 1^i + \Theta(n^{\log_5 a}) =$$

$$= n^c \cdot \log_5 n + \Theta(n^{\log_5 a})$$

$$\Theta(n^c \cdot \log_5 n) + \Theta(n^c)$$

$$\Theta(n^c \log_5 n) = \Theta(n^{\log_5 a} \cdot \log_5 n)$$

case 3

$$c > \log_2 a$$

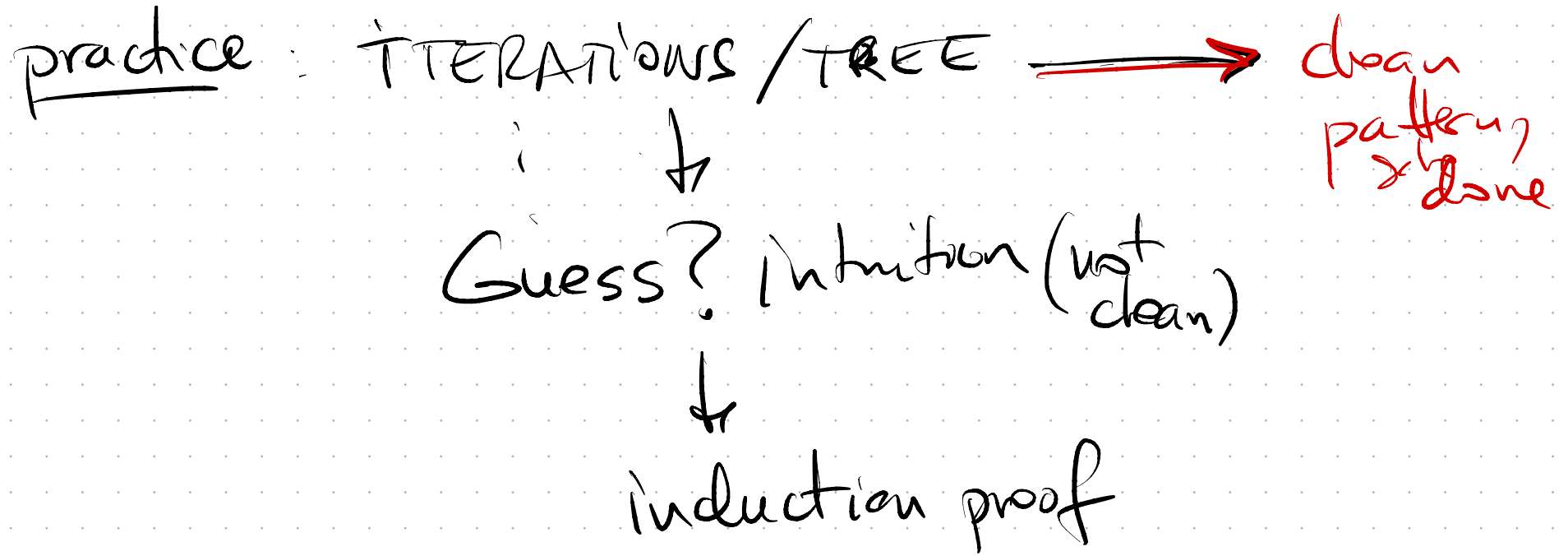
$$\frac{a}{5c} < 1$$

$$T(n) = n^c \sum_{i=0}^{\log_2 n - 1} \left(\frac{a}{5c}\right)^i + \Theta(n^{\log_2 a})$$

bound constant

$$\Theta(n^c) + \Theta(n^{\log_2 a})$$

$$\Theta(n^c) \quad \checkmark$$



$$T(n) = n + 4T(n/2)$$

iterations

not clean pattern

Guess $\Theta(n^2)$

$$c_1 n^2 \leq T(n) \leq c_2 n^2$$

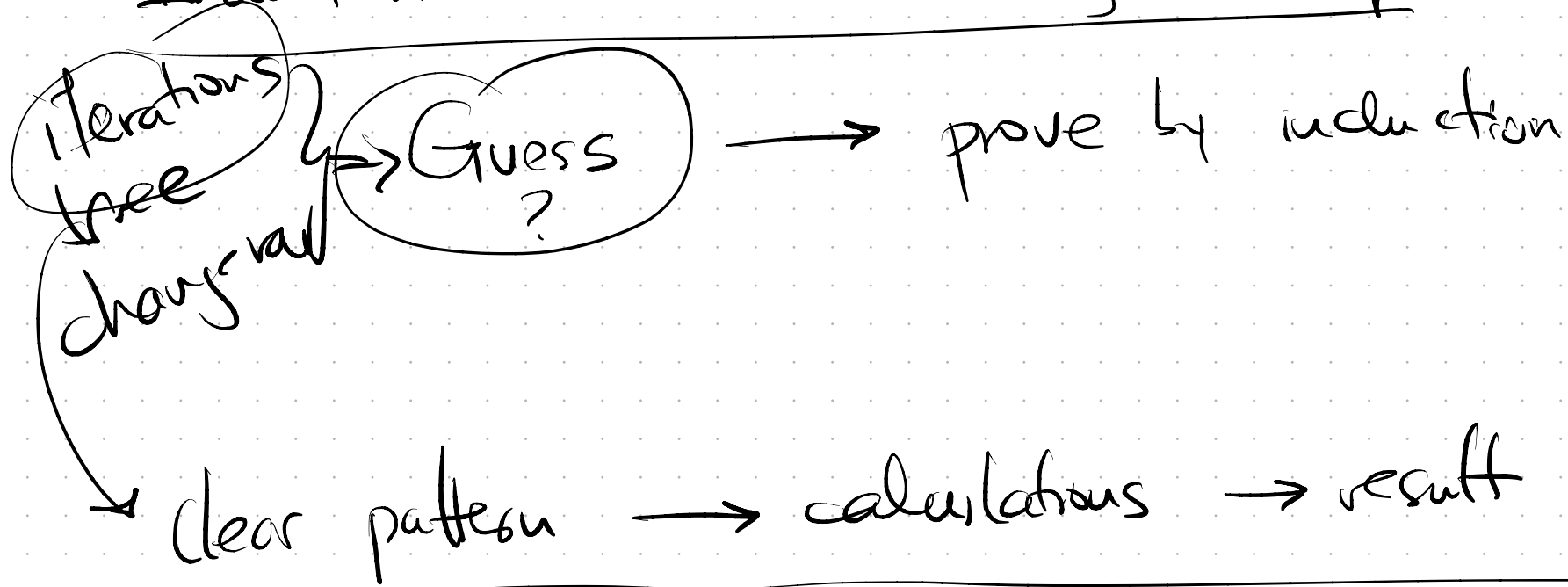
ind step

$$c_1 \left(\frac{n}{2}\right)^2 \leq T\left(\frac{n}{2}\right) \Rightarrow c_1 n \leq T(n)$$

$$T\left(\frac{n}{2}\right) \leq c_2 \left(\frac{n}{2}\right)^2 \Rightarrow$$

$$T(n) \leq c_2 n^2$$

Induction Method for solving rec eq

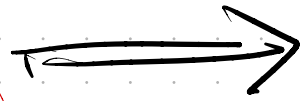


$$T(n) = 2T(n/2) + n \quad \text{guess} \quad \Theta(n \log n)$$

$$\underbrace{c_1 n \log n}_{LB} \leq T(n) \leq \underbrace{c_2 n \log n}_{UB} \quad \forall n \geq n_0$$

LB ind step $n/2 \rightarrow n$:

$$T(n/2) \geq c \frac{n}{2} \log \frac{n}{2}$$



$$T(n) \geq c n \log n$$

proof

ind hyp

ind hyp

ind cond

$$T(n) = \underline{2T(n/2)} + n \geq 2 c \frac{n}{2} \log \left(\frac{n}{2}\right) + n \stackrel{\text{want}}{\geq} c n \log n$$

$$c n (\log n - \log 2) + n \stackrel{\text{want}}{\geq} c n \log n$$

~~$$c n \log n - c_1 n + n \stackrel{\text{want}}{\geq} c n \log n$$~~

$$n \stackrel{\text{want}}{\geq} c_1 n$$

$$1 \stackrel{\text{want}}{\geq} c_1$$

✓ $c_1 = 0.5$

UB ind step $n/2 \rightarrow n$ $T(n/2) \leq c_2 n/2 \log_2(n/2) \Rightarrow$

$T(n) \leq c_2 n \log_2 n$

proof

$T(n) = 2T(n/2) + n \stackrel{\text{ind hyp}}{\leq} 2c_2 n/2 \log_2(n/2) + n \stackrel{\text{want}}{\leq} c_2 n \log_2 n$

$c_2 \cdot n \cdot (\log_2 n - 1) + n \stackrel{\text{want}}{\leq} c_2 n \log_2 n$

~~$c_2 n \log_2 n - c_2 n + n \stackrel{\text{want}}{\leq} c_2 n \log_2 n$~~

$n(1 - c_2) \stackrel{\text{want}}{\leq} 0 \quad \checkmark \quad c_2 \geq 1$

$$T(n) = 4T(n/2) + n \quad \Theta(n^2)$$

$$c_1 n^2 \leq T(n) \leq c_2 n^2$$

LB ind. step: $n/2 \rightarrow n$

$$\boxed{c_1 \left(\frac{n}{2}\right)^2 \leq T(n/2) \Rightarrow c_1 n^2 \leq T(n)}$$

ind hyp cond

proof

$$T(n) = 4T(n/2) + n \geq 4c_1 \left(\frac{n}{2}\right)^2 + n \stackrel{\text{want}}{\geq} c_1 n^2$$

$$\cancel{4c_1 n^2} + n \stackrel{\text{want}}{\geq} c_1 n^2$$

$$c_1 n^2 + n \stackrel{\text{want}}{\geq} c_1 n^2 \quad \checkmark$$

$$n \geq 0$$

UB ind step $T(n/2) \leq C_2(n/2)^2 \Rightarrow T(n) \leq C_2 n^2$

Proof

$$T(n) = 4T(n/2) + n \leq 4C_2 \frac{n^2}{4} + n \stackrel{\text{want}}{\leq} C_2 n^2$$

broken

$$C_2 n^2 + n \stackrel{\text{want}}{\leq} C_2 n^2$$

impossible

Proof 2

$$T(n) = 4T(n/2) + n \leq 4C_2 \frac{n^2}{2} + n$$

$$= C_2 n^2 + n = O(n^2) \leq \underline{\text{constant}} \cdot n^2$$

diff constant

proof 3 Stronger induction claim $T(n) \leq c_2 n^2 - dn$

ind step

$$T\left(\frac{n}{2}\right) \leq c_2 \left(\frac{n}{2}\right)^2 - d \frac{n}{2} \implies T(n) \leq c_2 n^2 - dn \quad \text{for } n \geq n_0$$

$$T(n) = 4T\left(\frac{n}{2}\right) + n \stackrel{\text{ind hyp}}{\leq} 4\left(c_2 \left(\frac{n}{2}\right)^2 - d \frac{n}{2}\right) + n \stackrel{\text{want}}{\leq} c_2 n^2 - dn$$

$$4c_2 \frac{n^2}{4} - 4d \frac{n}{2} + n \stackrel{\text{want}}{\leq} c_2 n^2 - dn$$

$$\cancel{c_2 n^2} - 2dn + n \stackrel{\text{want}}{\leq} \cancel{c_2 n^2} - dn$$

$$n \stackrel{\text{want}}{\leq} dn \quad \checkmark$$

$$d \geq 1 \quad d=2 \text{ ex}$$

$$c_2 > \frac{T(n_0+1)}{(n_0+1)^2}$$

~~$c_2 > \frac{T(n_0+1)}{(n_0+1)^2}$~~

$$T(n) \leq c_2 n^2 - dn \leq c_2 n^2$$

$$T(n) = n^2 + T(n/2) + T(n/4) \quad \Theta(n^2)$$

$$c_1 n^2 \leq T(n) \leq c_2 n^2$$

obvious

DB ind step

$$\left. \begin{array}{l} T(n/2) \leq c_2 (n/2)^2 \\ \text{and} \\ T(n/4) \leq c_2 (n/4)^2 \end{array} \right\} \Rightarrow T(n) \leq c_2 n^2$$

proof: $T(n) = n^2 + T(n/2) + T(n/4) \leq n^2 + c_2 (n/2)^2 + c_2 (n/4)^2$

want

$$\leq c_2 n^2$$

$$\cancel{n^2} + c_2 \frac{\cancel{n^2}}{4} + c_2 \frac{\cancel{n^2}}{16}$$

$$16 + 4c_2 + c_2$$

want

$$\leq \cancel{c_2 n^2}$$

$$\leq 16c_2$$

$$16 + 5c_2 \leq 16c_2 \quad \checkmark$$

$$T(n) = \frac{n^2}{\log n} + 4T\left(\frac{n}{2}\right)$$

$k=1$

Iterations

$$= \frac{n^2}{\log n} + 4 \left[\frac{(n/2)^2}{\log(n/2)} + 4T\left(\frac{n}{4}\right) \right]$$

$$= \frac{n^2}{\log n} + \frac{n^2}{\log(n/2)} + 4^2 T\left(\frac{n}{4}\right) \quad k=2$$

$$= \frac{n^2}{\log n} + \frac{n^2}{\log(n/2)} + \cancel{4^2} \left[\frac{\cancel{(n/4)^2}}{\log(n/4)} + 4T\left(\frac{n}{8}\right) \right]$$

$$= \frac{n^2}{\log n} + \frac{n^2}{\log(n/2)} + \frac{n^2}{\log(n/4)} + 4^3 T\left(\frac{n}{2^3}\right) \quad k=3$$

General k

$$\frac{n^2}{\log n} + \frac{n^2}{\log(n/2)} + \frac{n^2}{\log(n/4)} + \dots + \frac{n^2}{\log(n/2^{k-1})} + 4^k T\left(\frac{n}{2^k}\right)$$

Last k want $T\left(\frac{n}{2^k}\right) \approx T(1) \Leftrightarrow k \approx \log n$

$$\sum_{j=0}^{k-1} \frac{n^2}{\log\left(\frac{n}{2^j}\right)} + 4^k T\left(\frac{n}{2^k}\right)$$

$k = \log n$

$$\sum_{j=0}^{\log n - 1} \frac{n^2}{\log(n) - \log(2^j)} + 4^{\log n} \underline{T(1)}$$

$$\sum_{j=0}^{\log n - 1} \frac{1}{\log(n) - j} + \frac{n^2 T(1)}{\Theta(n^2)}$$

$$\sum_{j=0}^{\log n - 1} \frac{1}{\log(n) - j} = \sum_{l=1}^{\log n} \frac{1}{l}$$

$$\frac{1}{\log n} + \frac{1}{\log n - 1} + \frac{1}{2} + \frac{1}{\log n - (\log n - 1)} = \frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{\log n}$$

$$\left\{ H_n = \left[1 + \frac{1}{2} + \dots + \frac{1}{n} \right] \approx \ln(n) + \text{const} \right. = \Theta(\log(\log n))$$

$$n^2 \cdot \Theta(\log(\log n)) + \Theta(n^2)$$

$$\Theta(n^2 \log \log n)$$

Computer level logs $f=4$

$$f(n) = \log_2(\log_2(\log_2(\log_2(n)))) \approx 10$$

$$\log_2(\log_2(\log_2(n))) \approx 2^{10}$$

$$\log_2(\log_2(n)) \approx 2^{2^{10}}$$

$$n = 2^{2^{2^{2^{10}}}}$$

$$\log_a x = \log_b x \cdot \log_a b$$

$$\Theta(\log_a n) = \Theta(\log_b n) \text{ const} \quad \checkmark$$

$$\Theta(2^{\log_a n}) \neq \Theta(2^{\log_b n})$$

$$T(n) = \frac{n^2}{\log n} + 4T\left(\frac{n}{2}\right)$$

MT? no
class

$$f(n) = \frac{n^2}{\log n}$$

$$f = \frac{n^2}{\log n}$$

~~n^c~~

$$a=4 \quad b=2 \quad \log_b a = 2$$

$$f \text{ ? } n^{\log_b a} = n^2$$

$$\frac{f}{n^2} = \frac{1}{\log(n)} \rightarrow 0$$

$$f(n) \text{ want } \leq c \cdot n^{2-\epsilon}$$

$$\frac{n^2}{\log n} \text{ want } \leq c n^{2-\epsilon}$$

$n^\epsilon \text{ want } \leq c \cdot \log(n)$
not true

MT book $T(n) = at\left(\frac{n}{b}\right) + f(n)$

$$f(n) \ll n^{\log_b a}$$

$$f(n) = O(n^{\log_b a - \epsilon}) \Rightarrow \Theta(n^{\log_b a})$$

$$f(n) = \Theta(n^{\log_b a}) \Rightarrow \Theta(n^{\log_b a} \cdot \log n)$$

$$f(n) \gg n^{\log_b a}$$

$$f(n) = \Omega(n^{\log_b a + \epsilon}) \Rightarrow \Theta(f(n))$$

$$af(n/b) < \text{const} \cdot f(n)$$

$$T(n) = \frac{n}{\log n} + 4T\left(\frac{n}{2}\right)$$

$$f(n) = \frac{n}{\log n} \text{ vs } n^2$$

$$\frac{n}{\log n} < n < n^{2-\epsilon} \quad \checkmark$$

$$T(n) = \frac{n^3}{\log n} + 4T\left(\frac{n}{2}\right) \quad f(n) = \frac{n^3}{\log n} \text{ vs } n^2$$

$$\frac{n^3}{\log n} \stackrel{\text{want}}{>} c \cdot n^{2+\epsilon} \quad \text{yes}$$

$$\frac{n^3}{\log n} > n^{2.5} > n^{2+\epsilon} \quad \checkmark$$

$$f\left(\frac{n}{b}\right) < \text{const} \cdot f(n) \quad \text{control growth}$$