$$T(n) = 2T(n/2) + 1$$

$k=2$ $$= 2[2T(n/4)+1] + 1 = 4T(n/4) + 1+2$$

$k=3$ $$= 4[2T(n/8)+1] + 1+2 = 8T(n/8) + 1+2+4$$

$k$ $$2^k T(n/2^k) + \boxed{1+2+ \dots + 2^{k-1}}$$

geometric series base $r=2$

last $k$ $$n/2^k \simeq 1 \iff k \simeq \log_2 n \simeq \log n$$

$$n T(1) + 2^k - 1$$

$$n T(1) + n - 1 = \Theta(n)$$

**Geom Series**

**base $x$**

- $1+x+x^2+x^3+\dots+x^{n-1} = \dfrac{x^n-1}{x-1}$ if $x \neq 1$

- $n = 1+x+x^2+\dots+x^{n-1}$ if $x=1$

**proof**

$$S = 1 + \cancel{x} + \cancel{x^2} \dots + \cancel{x^{n-1}}$$

$$S \cdot x = \cancel{x} + \cancel{x^2} + x^3 + \dots + \cancel{x^{n-1}} + x^n$$

$$S(x-1) = Sx - S = x^n - 1$$

$$S = \frac{x^n-1}{x-1}$$

ind step

$$1+x+x^2+\cdots+x^{n-1}=\frac{x^n-1}{x-1} \Rightarrow 1+x+x^2+\cdots+x^n=\frac{x^{n+1}-1}{x-1}$$

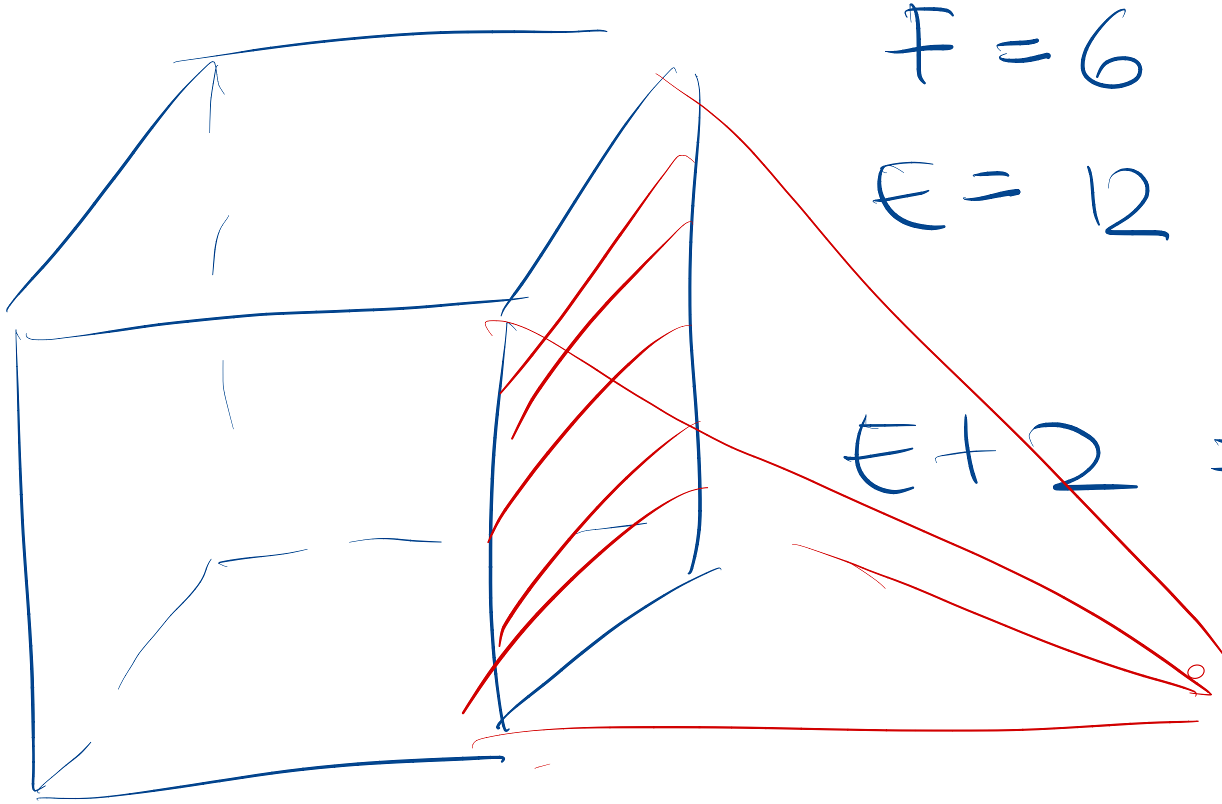
proof

$$1+x+x^2+\cdots+x^n = 1+x+x^2+\cdots+x^{n-1}+x^n$$

$$=\frac{x^n-1}{x-1}+x^n$$

$$=\frac{\cancel{x^n}-1+x^{n+1}-\cancel{x^n}}{x-1}$$

$$=\frac{x^{n+1}-1}{x-1} \checkmark$$



$F = 6 \quad V = 8$

$E = 12$

$E + 2 = F + V$

$$T(n) = 4T(n/2) + n$$

$$= 4\left[4T(n/4) + n/2\right] + n = 4^2 T(n/4) + n + 2n$$

$k=3$ $$= 4^2\left[4T(n/8) + n/4\right] + n + 2n = 4^3 T(n/2^3) + n + 2n + 4n$$

$$\boxed{k \qquad 4^k T(n/2^k) + n(1 + 2 + \dots + 2^{k-1})}$$

pattern to prove

last k $n/2^k \simeq 1 \iff k \simeq \log_2 n$

$$4^{\log_2 n} T(1) + n(2^k - 1)$$

$$2^{2 \cdot \log_2 n} T(1) + n \cdot (n-1)$$

$$n^2 T(1) + n^2 - n = \Theta(n^2)$$

$T(n) = 4T(n/2) + n$

Substitution
- guess
- proof

guess $\Theta(n^2) \Leftrightarrow c_1 n^2 \le T(n) \le c_2 n^2$

ind proof lower bound

$$c_1 (n/2)^2 \le T(n/2) \Rightarrow c_1 n^2 \le T(n) \quad \left( c_1 n^2 \le T(n) \Rightarrow c_1 (2n)^2 \le T(2n) \right)$$

proof

$$T(n) = 4T(n/2) + n \ge 4\left[c_1 (n/2)^2\right] + n$$

$$= 4c_1 n^2/4 + n$$

$$= c_1 n^2 + n \ge c_1 n^2$$

say e.g. $c_1 = 1$

ind proof Upper Bound

$$T\left(\frac{n}{2}\right) \le c_2 \left(\frac{n}{2}\right)^2 \Longrightarrow T(n) \le c_2 n^2$$

Proof

$$T(n) = 4T(n/2) + n \le 4\left[c_2 (n/2)^2\right] + n$$

$$4 c_2 n^2/4 + n = c_2 n^2 + n$$

want

$$\le c_2 n^2$$

not true

it only means

its not strong enough

$$T(n) \le Cn^2$$

Ind Proof Upper Bound (Stronger Claim)

$$T(n/2) \le c(n/2)^2 - d\,n/2 \implies T(n) \le cn^2 - dn$$

proof $T(n) = 4T(n/2) + n \le 4\left[c(n/2)^2 - d\,n/2\right] + n$

$$= 4cn^2/4 - 4d\,n/2 + n$$

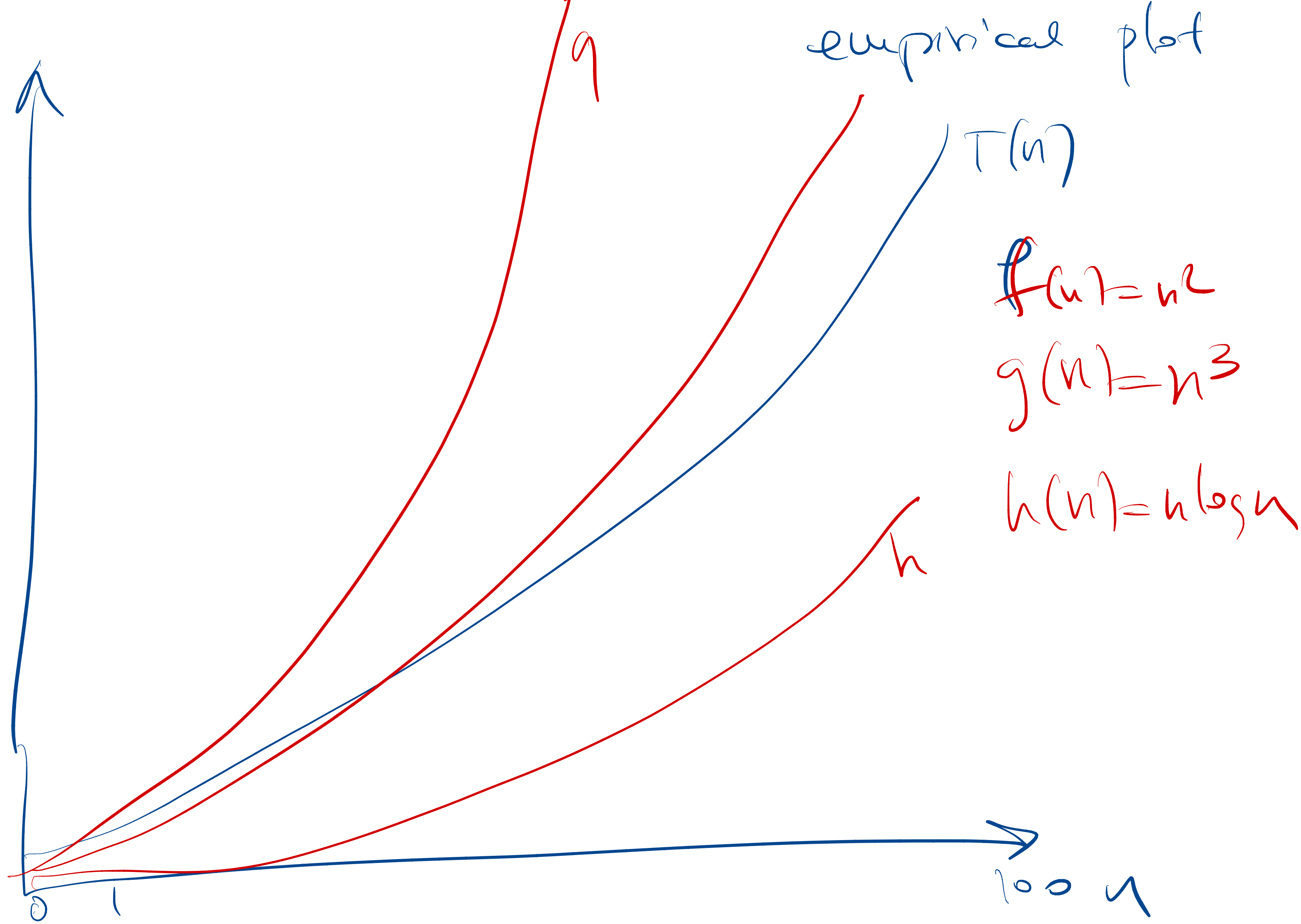
$$= cn^2 - n(2d-1)$$

want $\overset{?}{\le} cn^2 - dn$

$c = 1$

$$\iff d\cancel{n} \overset{?}{\le} (2d-1)\cancel{n}$$

$$\iff d \overset{?}{\le} 2d-1 \quad \text{say choose } d=2$$



empirical plot
g
T(n)
f(n) = n^2
g(n) = n^3
h(n) = n log n
h
0
1
100
n

$$T(n) = T(n/2) + T(n/4) + n^2$$

$$= \left[T(n/4) + T(n/8) + (n/2)^2\right] + \left[T(n/8) + T(n/16) + \left(\frac{n}{4}\right)^2\right] + n^2$$

$$= T(n/4) + 2T(n/8) + T(n/16) + n^2\left(1 + \frac{1}{4} + \frac{1}{16}\right)$$

$$= \left[T(n/8) + T(n/16) + (n/4)^2\right] + 2\left[T(n/16) + T(n/32) + \left(\frac{n}{8}\right)^2\right] +$$

$$+ \left[T(n/32) + T(n/64) + \left(\frac{n}{16}\right)^2\right] + n^2\left(1 + \frac{5}{16}\right)$$

$$= T(n/8) + 3T(n/16) + 3T(n/32) + T(n/64) + n^2\left(1 + \frac{5}{16} + \frac{1}{16} + \frac{2}{64} + \right.$$

$$= T(n/8) + 3T(n/16) + 3T(n/32) + T(n/64) + n^2\left(1 + \frac{5}{16} + \left(\frac{5}{16}\right)^2\right)$$

$1/256$

Pascal $\triangle$ coeff

$n/2^k$ arg

This PB short cut $T(\,)$ rec part is too small to matter

$$\Rightarrow \Theta\left[n^2\left(1+5/16+\left(\frac{5}{16}\right)^2+\cdots+\left(\frac{5}{16}\right)^{\text{lastle}}\right)\right]$$

base $= \frac{5}{16} < 1 \Rightarrow$ FINITE $\Theta(1)$

$$\Rightarrow \Theta(n^2)$$

---

Substitute/Guess $T(n) = T(n/2) + T(n/4) + n^2 = \Theta(n^2)$

proof ind Upper Bound

$$\left.\begin{aligned} T(n/2) &\le c(n/2)^2 \\ T(n/4) &\le c(n/4)^2 \end{aligned}\right\} \Longrightarrow T(n) \le cn^2$$

proof:

$$T(n) = T(n/2) + T(n/4) + n^2 \le c(n/2)^2 + c(n/4)^2 + n^2$$

$$= \cancel{n^2}\left(c/4 + c/16 + 1\right)$$

want

$$\stackrel{?}{\le} \cancel{n^2} \cdot c$$

$$\iff \frac{c}{4} + \frac{c}{16} + 1 \le c$$

---

$$T(n) = T(n/2) + T(n/4) + n$$

$n/2$ $n/4$

$$\Theta(n)$$

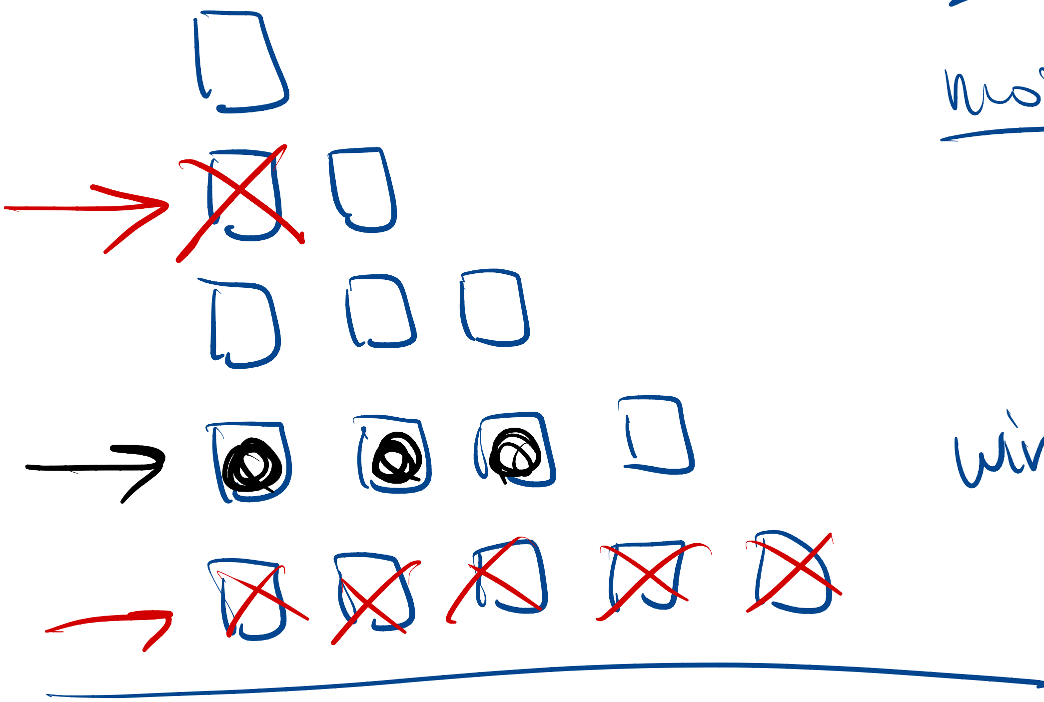
$$\frac{c}{2} + \frac{c}{4} \le c$$

---

$$T(n) = T(n/2) + T(n/4) + T(n/4) + n$$

$$c/2 + c/4 + c/4 \le c$$

# Square Game (Diversion, not lecture)

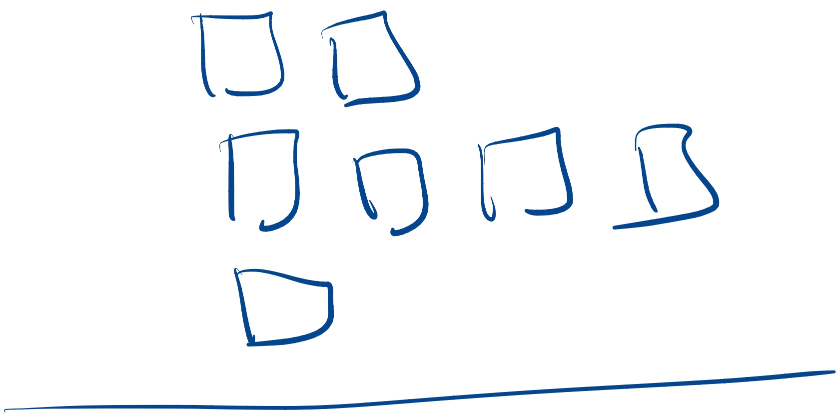


2 players, alternate turns

move: pick a row; remove any squares $\geq 1$ from that row

wins: who picks the last sq.

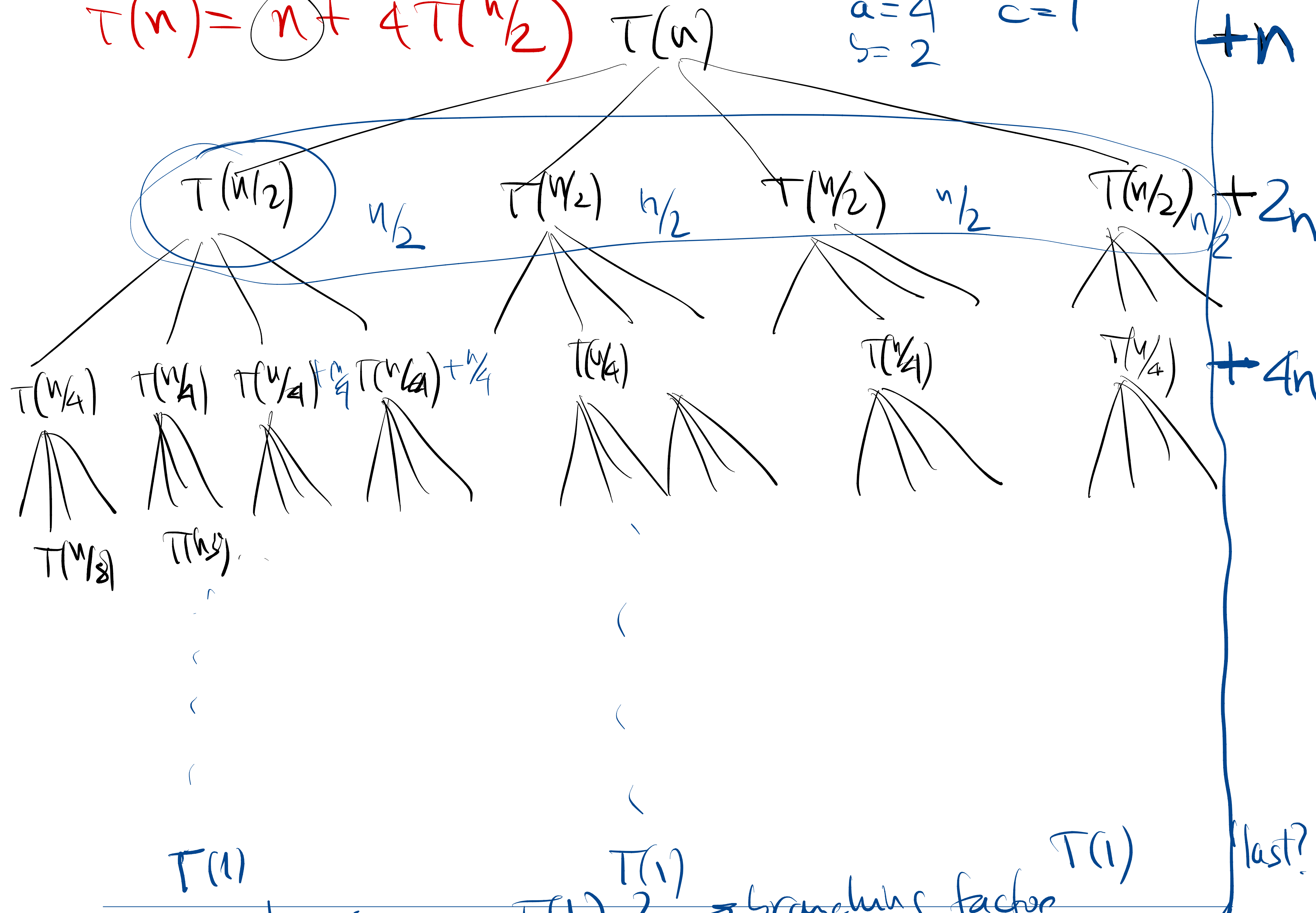
$$n^{\log_b a} = a^{\log_b n} \quad ??$$



$T(n) = n + 4T(n/2)$

$a = 4 \quad c = 1$

$b = 2$




how many T(1)? → branching factor
→ depth of tree

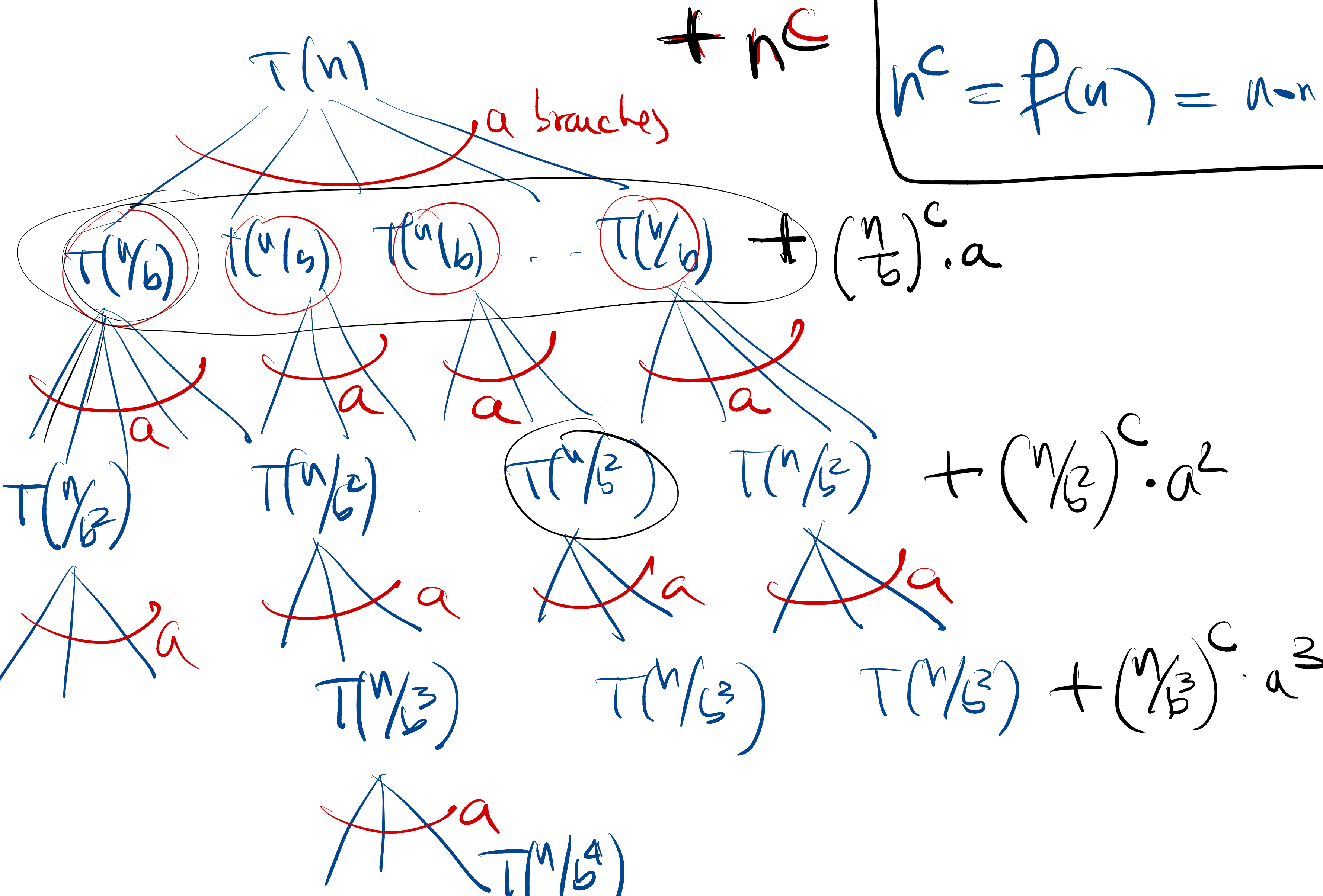
$T(n) = aT(n/b) + n^c$

simplified $\Theta(n^c)$

branching factor = a = # of rec calls.

$1/b = \frac{\text{size of subproblem}}{\text{size of problem}}$

$n^c = f(n)$ = non-rec load



Recursive load

$$T(n/b^k) \simeq T(1)$$

$$k = \log_b n$$

$\# T(1)$ $a^k = a^{\log_b n}$

$$\Theta\left(a^{\log_b n} \cdot T(1)\right)$$

$$\| $$

$n^{\log_b a}$

Non Rec load

$n^c + \left(\frac{n}{b}\right)^c \cdot a + \left(\frac{n}{b^2}\right)^c \cdot a^2$

$+ \left(\frac{n}{b^3}\right)^c \cdot a^3 + \dots$ last

Total $\quad n^c \displaystyle\sum_{i=0}^{\log_b n - 1} \left(\frac{a}{b^c}\right)^i + \Theta\left(n^{\log_b a}\right)$

non-rec

rec

geom series base $= \frac{a}{b^c}$

$x =$

geom series $\sum_{i=0}^{\log_b n - 1} x^i = \frac{x^{\log_b n} - 1}{x - 1}$ if $\boxed{x \neq 1}$

Case 1 : $\boxed{x > 1} \Leftrightarrow \frac{a}{b^c} > 1 \Leftrightarrow \underline{c < \log_b a} \Leftrightarrow \underline{b^c < a}$

Total $n^c \frac{\left(a/b^c\right)^{\log_b n} \cancel{-1}}{\cancel{(a/b^c) - 1}} + \Theta(n^{\log_b a})$

$$\Theta\left(n^c \cdot \frac{a^{\log_b n}}{(b^c)^{\log_b n}}\right) + \Theta\left(n^{\log_b a}\right)$$

$$\Theta\left(\cancel{n^c} \frac{n^{\log_b a}}{(b^{\log_b n})^c \cancel{= n^c}}\right) + \Theta\left(n^{\log_b a}\right)$$

$$\Theta\left(n^{\log_b a}\right)$$

Case 2 $\boxed{c = \log_b a} \iff b^c = a \iff x = 1$

Total

$$n^c \sum_{i=0}^{\log_b n - 1} (1)^i + \Theta\left(n^{\log_b a}\right) =$$

$$= n^c \cdot \log_b n + \Theta\left(n^{\log_b a}\right)$$

$$\Theta\left(n^{\log_b a} \cdot \log_b n\right) + \Theta\left(n^{\log_b a}\right)$$

bigger

$$\Theta\left(n^{\log_b a} \cdot \log n\right)$$

MergeSort $a = b = 2$
$c = 1$

R.T $\Theta(n \log n)$

Case 3 $\quad x < 1 \iff \frac{a}{b^c} < 1 \iff \boxed{c > \log_b a}$

Total $\quad n^c \underbrace{\sum_{i=0}^{\log_b n - 1} \left(\frac{a}{b^c}\right)^i}_{\Theta(1)} + \Theta\left(n^{\log_b a}\right)$

$$= \underbrace{\Theta(n^c)}_{\text{bigger}} + \Theta\left(n^{\log_b a}\right)$$

$$\Theta(n^c)$$

Book-Master Th $T(n) = aT(n/b) + f(n)$

$f = n^2/\log n$

---

$T(n) = 4T(n/2) + \Theta(n^3)$ $a=4$ $b=2$ $c=3$

case $\frac{a}{b^c} < 1$ case 3

$\Theta(n^c)$

binary search

$T(n) = T(n/2) + 1 \Rightarrow a=1$ $b=2$ $c=0$

$\frac{a}{b^c} = 1$ case 2

$\Theta(n^{\log_b a} \cdot \log n) = \Theta(\log n)$