X1 P1
(x-hour)
50 minutes

9/26/95

# CS 25 - Algorithms

**Today - Math Primer** (parts of chap 2, 3, 5)

- logs, exponents, common functions
- factorials, Stirling's approx.
- 4 types of series
- ~~summations~~
- induction
- ~~trees~~

## ① Logs, exponents, common functions

**Exponents:**

$$a^n = \overbrace{a \cdot a \cdot a \cdots a}^{n}$$

$$a^0 = 1$$

$$a^{-1} = 1/a$$

$$(a^m)^n = a^{mn} = (a^n)^m$$

$$a^m \cdot a^n = a^{m+n} \neq a^{mn} \quad \leftarrow \text{common mistake}$$

$$a^{b^c} = a^{(b^c)} \neq (a^b)^c$$

e.g. $3^{3^3} = 3^{27} \approx 7.63 \times 10^{12}$

$\neq 27^3 = 19683$

armageddon

**Logarithms:**

$$a = b^{\log_b a}$$

$\log_b a$ = "number you have to raise $b$ to in order to get $a$"

$$\lg n = \log_2 n$$

$$\ln n = \log_e n \qquad e \approx 2.71828$$

$$\log^k n = (\log n)^k \neq \log n^k$$

$$\log\log n = \log(\log n)$$

$$\log^{(i)} n = \overbrace{\log(\log(\cdots \log}^{i}(n)\cdots))$$

$$\lg^{*} n = \min i \text{ s.t. } \lg^{(i)} \leq 1$$

$$\log(ab) = \log a + \log b \neq (\log a)(\log b)$$

$$\log a^b = b \log a$$

$$\log_b a = \frac{\log_c a}{\log_c b} \qquad \text{e.g. } \log_2 100 = \frac{\log_{10} 100}{\log_{10} 2} = \frac{\ln 100}{\ln 2}$$

$$\log_b a = \frac{1}{\log_a b}$$

$$a^{\log_b c} = c^{\log_b a} \qquad \text{e.g. } 4^{\log_2 n} = n^{\log_2 4} = \underline{n^2}$$

**Functions:**

(1) $f(n)$ is "polylogarithmic" if $f(n) = O(\log^{k_1} n)$ for some constant $k_1$ & $k_2$. $f(n) = \Omega(\log^{k_2} n)$ ✓

(2) $f(n)$ is "polynomial" if $f(n) = O(n^{k_1})$ for some constant $k_1$ & $k_2$. $f(n) = \Omega(n^{k_2})$

(3) $f(n)$ is "exponential" if $f(n) = O(k_1^n)$ for some constant $k_1$ & $k_2$. $f(n) = \Omega(k_2^n)$

$\forall$ const. $k_1, k_2, k_3$, $\quad \log^{k_1}(n) = o(n^{k_2}) = o(k_3^n)$

"little-o"

## ② Factorials, Stirling's approx.

$$n! = \begin{cases} 1 & \text{if } n=0 \\ n\cdot(n-1)! & \text{if } n>0 \end{cases}$$

i.e. $n! = 1\cdot2\cdot3\cdots n$

show later?

$$\begin{aligned} n! &= 1\cdot2\cdots n \\ &\geq \lceil n/2\rceil \cdots n \\ &\geq (\lceil n/2\rceil)^{n-\lceil n/2\rceil+1} = (\lceil n/2\rceil)^{\lfloor n/2\rfloor+1} \\ &\geq (\lceil n/2\rceil)^{n/2} \\ &\geq (n/2)^{n/2} \end{aligned}$$

Weak bounds: $(n/2)^{n/2} \leq n! \leq n^n$ "superexponential"

Stirling's approx: $n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\Theta(1/n)\right)$

So,
$$n! = o(n^n)$$
$$n! = \omega(2^n)$$
$$\log(n!) = \Theta(n\log n)$$

Useful bounds: $\sqrt{2\pi n}\,(n/e)^n \leq n! \leq \sqrt{2\pi n}\left(\frac{n}{e}\right)^{n+\frac{1}{12n}}$

③ Series

sequence : $a_1, a_2, a_3, \dots a_n$

series : $a_1 + a_2 + \dots a_n = \sum_{i=1}^{n} a_i$

infinite series : $\sum_{i=1}^{\infty} a_i = \lim_{n \to \infty} \sum_{i=1}^{n} a_i$

linearity : $\sum_{i=1}^{n} (c_1 a_i + c_2 b_i) = c_1 \sum_{i=1}^{n} a_i + c_2 \sum_{i=1}^{n} b_i$

$$\Longrightarrow \sum_{i=1}^{n} \Theta(f(i)) = \Theta\left(\sum_{i=1}^{n} f(i)\right)$$

4 types of series

(a) arithmetic $\sum_{i=1}^{n} i = 1+2+3+ \dots n = \frac{n(n+1)}{2}$

trick: $\begin{array}{r} 1+2+\dots n \\ n+n-1+\dots 1 \\ \hline (n+1)\ (n+1) \dots (n+1) \end{array} = n(n+1) \div 2$ ✓

e.g. $\sum_{i=1}^{n} (a \cdot i + b) = a \sum_{i=1}^{n} i + \sum_{i=1}^{n} b$

$= \frac{a\,n(n+1)}{2} + bn \quad \dots$

$\sum_{k=s}^{t} k = s + (s+1) + (s+2) + \dots + t$

$\begin{array}{r} t + (t-1) + (t-2) + \dots \quad s \\ \hline s+t \quad (s+t) \dots \end{array} = \frac{(t-s+1)\cdot(s+t)}{2}$

(b) geometric $\sum_{i=0}^{n} x^i = 1 + x + x^2 + \dots x^n$

$\boxed{\text{for } x \neq 1}$, $\sum_{i=0}^{n} x^i = \frac{x^{n+1} - 1}{x - 1}$

trick:

$$\begin{aligned} xS &= \quad\; x + x^2 + \dots x^n + x^{n+1} \\ - \quad S &= 1 + x + x^2 + \dots x^n \\ \hline xS - S &= x^{n+1} - 1 \\ S &= \frac{x^{n+1}-1}{x-1} \end{aligned}$$

$S = \sum_{i=0}^{n} x^i$

$\boxed{\text{for } |x| < 1}$ $\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$

idea: $\sum_{i=0}^{\infty} x^i = \lim_{n \to \infty} \sum_{i=0}^{n} x^i = \lim_{n \to \infty} \frac{x^{n+1}-1}{x-1} = \frac{-1}{x-1} = \frac{1}{1-x}$

(3) (c) telescoping $\sum_{i=1}^{n}(a_i - a_{i-1}) = a_n - a_{n-1}$

$$+ a_{n-1} - a_{n-2}$$

$$+ a_{n-2} - a_{n-3}$$

$$+ a_1 - a_0$$

$$= a_n - a_0$$

e.g. $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots \frac{1}{(n-1)n} = \sum_{i=1}^{n-1} \frac{1}{i(i+1)}$

$$= \sum_{i=1}^{n-1} \left(\frac{1}{i} - \frac{1}{i+1}\right)$$

$$= 1 - \frac{1}{n} \; !$$

(d) harmonic $\sum_{i=1}^{n} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n}$

$$= \ln n + O(1)$$

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Bounding Series:

upper bound - bound by largest term

lower bound - split sum & bound

$n!$

$\sum_{i=1}^{n} \sqrt{i}$

$$\sum_{i=1}^{n} \sqrt{i} \le \sum_{i=1}^{n} \sqrt{n} = n\sqrt{n}$$

$$\sum_{i=1}^{n} \sqrt{i} \ge \sum_{i=\lceil n/2 \rceil}^{n} \sqrt{i} \ge \sum_{i=\lceil n/2 \rceil}^{n} \sqrt{\lceil n/2 \rceil} = (n - \lceil n/2 \rceil + 1)\sqrt{\lceil n/2 \rceil} = (\lfloor n/2 \rfloor + 1)\sqrt{\lceil n/2 \rceil} \ge n/2 \sqrt{n/2}$$

$$\Rightarrow \sum_{i=1}^{n} \sqrt{i} = \Theta(n\sqrt{n})$$

→ Do factorial

etc.

**Weak Induction** P is a property of the integers. Prove true $\forall n$.

Base case: typically prove $P(0)$ or $P(1)$

(weak) Induction step: assume $P(n-1)$, show $P(n)$ (or assume $P(n)$, show $P(n+1)$, etc.)

( gave proof of $P(n) =$ "The sum of $1, 2, \ldots n$ is $\frac{n(n+1)}{2}$"

base $P(1) = \frac{1(1+1)}{2} = \frac{1 \cdot 2}{2} = 1$ ✓

induction: assume $P(n-1)$, prove $S(n)$

$$\underbrace{1 + 2 + 3 + \cdots (n-1)} + n$$

$$\frac{(n-1) \cdot n}{2} + n = \frac{n^2 - n}{2} + \frac{2n}{2}$$

$$= \frac{n^2 + n}{2}$$

$$= \frac{n(n+1)}{2}$$ ✓

**Strong Induction**

(strong) induction step: assume $P(i) \; \forall \, i \leq n$

show $P(n+1)$

Fibo
- recursively $\rightarrow \Omega(2^{n/2})$ time, linear space (stack depth)
- iteratively $\rightarrow \Theta(n)$ time, $\Theta(1)$ space (store just last two)
- repeated squaring of matrix $\rightarrow \Theta(\lg n)$ time, $\Theta(1)$ space (precalculate which matrices needed, keep a running product...)
- direct calculation $\rightarrow \dfrac{\phi^i - \hat{\phi}^i}{\sqrt{5}}$ where $\phi = \frac{1+\sqrt{5}}{2}$, $\hat{\phi} = \frac{1-\sqrt{5}}{2}$ $\Theta(1)$ time (?), $\Theta(1)$ space.

## Proof by weak induction: Generating the Fibonacci numbers

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | |
| 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | ... |

$$f_i = \begin{cases} 1 & \text{if } i = 1 \text{ or } 2 \\ f_{i-1} + f_{i-2} & i > 2 \end{cases}$$

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$M^2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$M^3 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$$

$$M^4 = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$$

$$10^{80} \approx 2^{266} \approx \phi^{383}$$

$$M^5 = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 8 & 5 \\ 5 & 3 \end{pmatrix}$$

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$$

$$M^6 = \begin{pmatrix} 8 & 5 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 13 & 8 \\ 8 & 5 \end{pmatrix}$$

$$10^{80} \text{ ops} \cdot \frac{\text{sec}}{10^9 \text{ ops}} \cdot \frac{\text{min}}{60 \text{ sec}} \frac{\text{hr}}{60 \text{ min}} \cdot \frac{\text{day}}{24 \text{ hr}} \frac{\text{yr}}{365 \text{ day}}$$

↑ Gigahertz

$$= 3.17 \times 10^{63} \text{ yrs.}$$

**Claim:** $M^i = \begin{pmatrix} f_{i+1} & f_i \\ f_i & f_{i-1} \end{pmatrix}$

**Pf** (weak induction)

- Case $i=1$ $\quad M^1 = M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ ✓

- assume for $i$, prove for $i+1$

$$M^{i+1} = M^i \cdot M = \begin{pmatrix} f_{i+1} & f_i \\ f_i & f_{i-1} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} f_{i+1}+f_i & f_{i+1} \\ f_i+f_{i-1} & f_i \end{pmatrix} = \begin{pmatrix} f_{i+2} & f_{i+1} \\ f_{i+1} & f_i \end{pmatrix} \checkmark$$