NP complete problems

Some figures, text, and pseudocode from:

- Introduction to Algorithms, by Cormen, Leiserson, Rivest and Stein
- Algorithms, by Dasgupta, Papadimitriou, and Vazirani

Module objectives

- Some problems are too hard to solve in polynomial time
  - Example of such problems, and what makes them hard
- Class NP\P
  - NP: problems with solutions verifiable in poly time
  - P: problems not solvable in poly time
- NP-complete, fundamental class in Computer Science
  - reduction form on problem to another
- Approximation Algorithms:
  - since these problems are too hard, will settle for non-optimal solution
  - but close to the optimal
  - if we can find such solution reasonably fast

Module objectives

- WARNING: This presentation trades rigor for intuition and easiness
- The CLRS book ch 35 is rigorous, but considerably harder to read
  - hopefully easier after going through these slides
- For an introduction to complexity theory that is rigorous and somewhat more accessible, see
  - Michael Sipser : Introduction to Theory of Computation

2SAT problem

- 2-clause (aVb)
  - true (satisfied) if either a or b true, false (unsatisfied) if both false
  - a, b are binary true/false literals
  - $\underline{a}$ = not (a) = negation (a). ¬T=F ; ¬F=T
  - can have several clauses, e.g. (a∨b), (¬a∨c), (¬c∨d), (¬a∨¬b)
  - truth table for logical OR: (T∨T)=T; (T∨F)=T; (F∨T)=T; (F∨F)=F
- 2-SAT problem: given a set of clauses, find an assignment T/F for literals in order to satisfy all clauses

2 SAT solution

- Example: satisfy the following clauses:
  - $(a \vee b) \wedge (\neg a \vee c) \wedge (\neg d \vee b) \wedge (d \vee \neg c) \wedge (\neg c \vee f) \wedge (\neg f \vee \neg g) \wedge (g \vee \neg d)$
- try a=TRUE
  - $a=T \Rightarrow \neg a=F \Rightarrow c=T \Rightarrow d=f=T \Rightarrow \neg g=T \Rightarrow g=F \Rightarrow \neg d=T$ contradiction
- try a=FALSE
  - $a=F \Rightarrow b=T$, it works; eliminate first three clauses and a,b; now we have $(d \vee \neg c) \wedge (\neg c \vee f) \wedge (\neg f \vee \neg g) \wedge (g \vee \neg d)$
- try c=FALSE
  - it works, eliminate first two clauses and c, remaining $(\neg f \vee \neg g) \wedge (g \vee \neg d)$
- try g=TRUE
  - $g=T \Rightarrow \neg g=F \Rightarrow \neg f=T$; done.
- assignment : TRUE(b, g) ; FALSE(a, c, f), EITHER (d)

2SAT algorithm

- pick one literal not assigned yet, say "a", from a clause still to be satisfied
  - see if THINGS_WORK_OUT(a) *//try assign a=TRUE*
  - if NOT, see if THINGS_WORK_OUT($\neg$a)*// try assign a=FALSE*
- if still NOT, return "NOT POSSIBLE"
- if YES (either way), keep the assignments made, and delete all clauses that are satisfied by assignments
- repeat from the beginning until there are no clauses left, or until "NOT POSSIBLE" shows up

How to try an assignment for 2SAT

THINGS_WORK_OUT (a)

- queue Q={a}
- while x=dequeue(Q)
  - for each clause that contain ¬x like (y∨¬x) or (¬x∨y):
    - if y=FALSE (or ¬y=TRUE) already assigned, return "NOT POSSIBLE"
    - assign y=TRUE (or ¬y=FALSE), enqueue(y,Q)
- return the list of TRUE/FALSE assignments made.

2SAT algorithm

- running time: more than linear in number of clauses, if we are unlucky
  - easy to implement
  - n = number of literals, c=number of clauses.
  - definitely polynomial, less than O(nc)
  - 2SAT can be solved in linear time using graph path search

- 2SAT-MAX: if an instance to 2-SAT is not satisfiable, satisfy as many clauses as possible
  - this problem is much harder, "NP-hard"

3SAT

- CLRS book calls it "3-CNF satisfiability"
- same as 2SAT, but clauses contain 3 literals
  - example $(a \vee b \vee \neg c), (\neg b \vee c \vee \neg a), (d \vee c \vee b), (\neg d \vee e \vee c), (\neg e \vee b \vee d)$
- try to solve/satisfy this problem with an intelligent/ fast algorithm - can't find such a solution
  - exercise: why THINGS_WORK_OUT procedure is not applicable on 3SAT?
- this problem can be solved only by essentially trying [almost] all possibilities
  - even if done efficiently, still an exponential time/trials
- why is 3SAT problem so hard?

complexity = try all combinations

- why is 3SAT hard?
  - no one knows for sure, but widely believe to be true (no proof yet)
  - the answer seems to be that on problems that solution come from an exponential space
  - not enough space structure to search efficiently (polynomial time)
- proving either
  - that no polynomial solution exists for 3SAT
  - or finding a polynomial solution for 3SAT
- ... would make you rich and very famous

class NP = polynomial verification

- 2SAT, 3SAT very different for finding a solution
- but 2SAT, 3SAT same for verifying a solution : if someone proposes a solution, it can be verified immediately
  - proposed solution = all literals assigned T/F
  - just check every clause to be TRUE
- NP = problems for which possible solutions can be verified quickly (polynomial)
- P = problems for which solutions can be found quickly
  - obviously P$\subseteq$NP, since finding a solution is harder than verifying one
  - 2SAT, 3SAT$\in$NP
  - 2SAT$\in$P, 3SAT$\notin$P

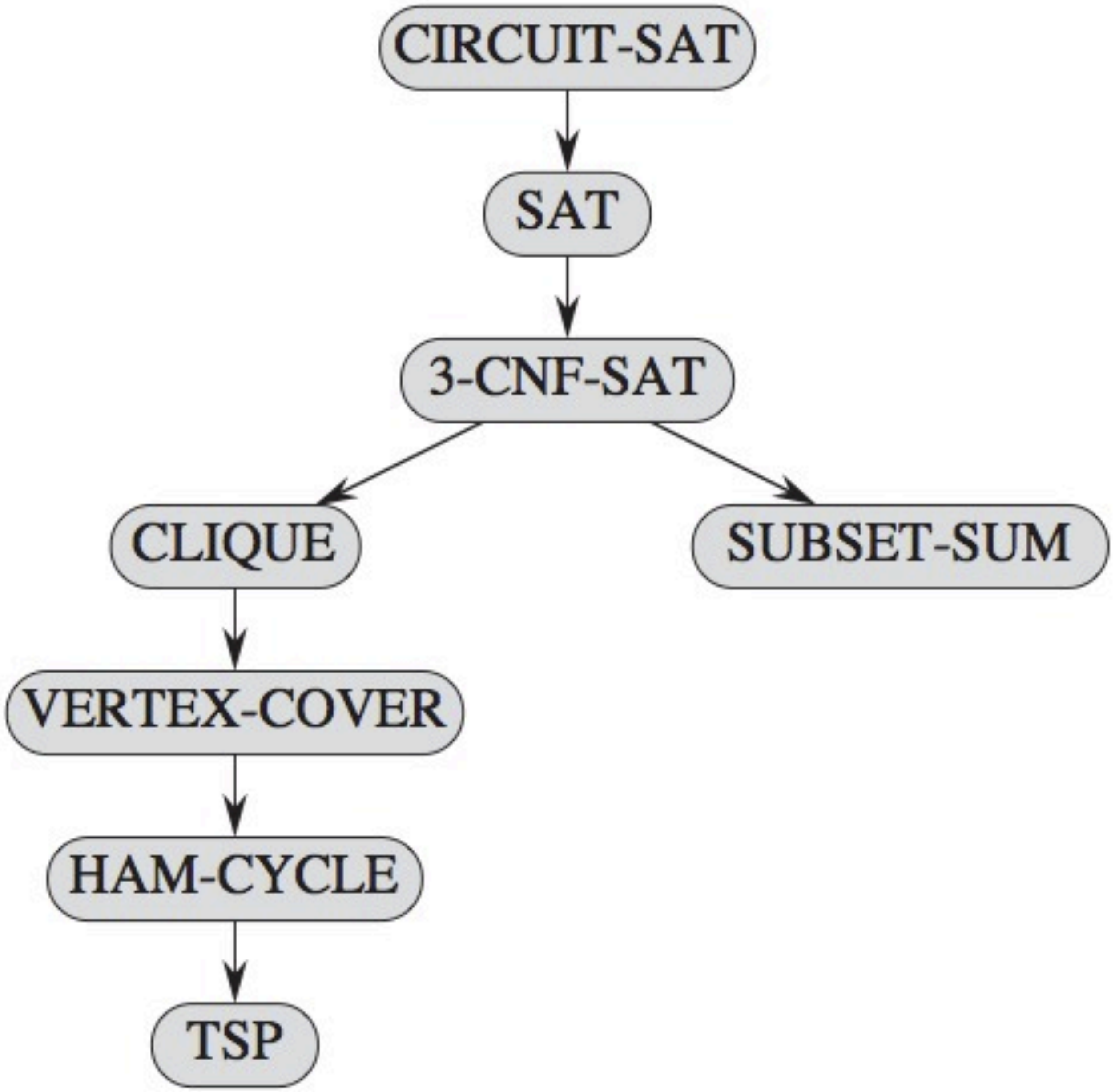
problems in NP\P

- NP\P problems : solutions are quickly verifiable, but hard to find
  - like 3SAT
  - also CIRCUIT-SAT,
  - CLIQUE
  - VERTEX-COVER
  - HAMILTONIAN-CYCLE
  - TSP
  - SUBSET-SUM
  - many many others, generally problems asking "find the subset that maximizes .... "

NP-reduction

- problem A reduces to problem B if
  - any input x for pb A $\xrightarrow{\text{map}}$ input y for pb B
  - solution/answer for (y,B) $\xrightarrow{\text{map}}$ solution/answer for (x,A)
  - "map" has to be done in polynomial time
  - A $\xrightarrow{\text{poly-map}}$ B or A $\leq_p$ B ($\leq_p$ stands for "polynomial-easier-than")
- think "B harder than A", since solving B means also solving to A via reduction
- 3SAT reduces to CLIQUE
  - 3SAT $\leq_p$ CLIQUE
- CLIQUE reduces to VERTEX-COVER
  - CLIQUE $\leq_p$ VERTEX-COVER

reductions


CIRCUIT-SAT
SAT
3-CNF-SAT
CLIQUE
SUBSET-SUM
VERTEX-COVER
HAM-CYCLE
TSP

CLIQUE problem

- a clique in undirected graph G=(V,E) is a set of vertices $S \subset V$ in which all edges exist: $\forall u,v \in S\ (u,v) \in E$
  - a clique of size n must have all (n choose 2) edges
- Task: find the maximal set S that is a clique

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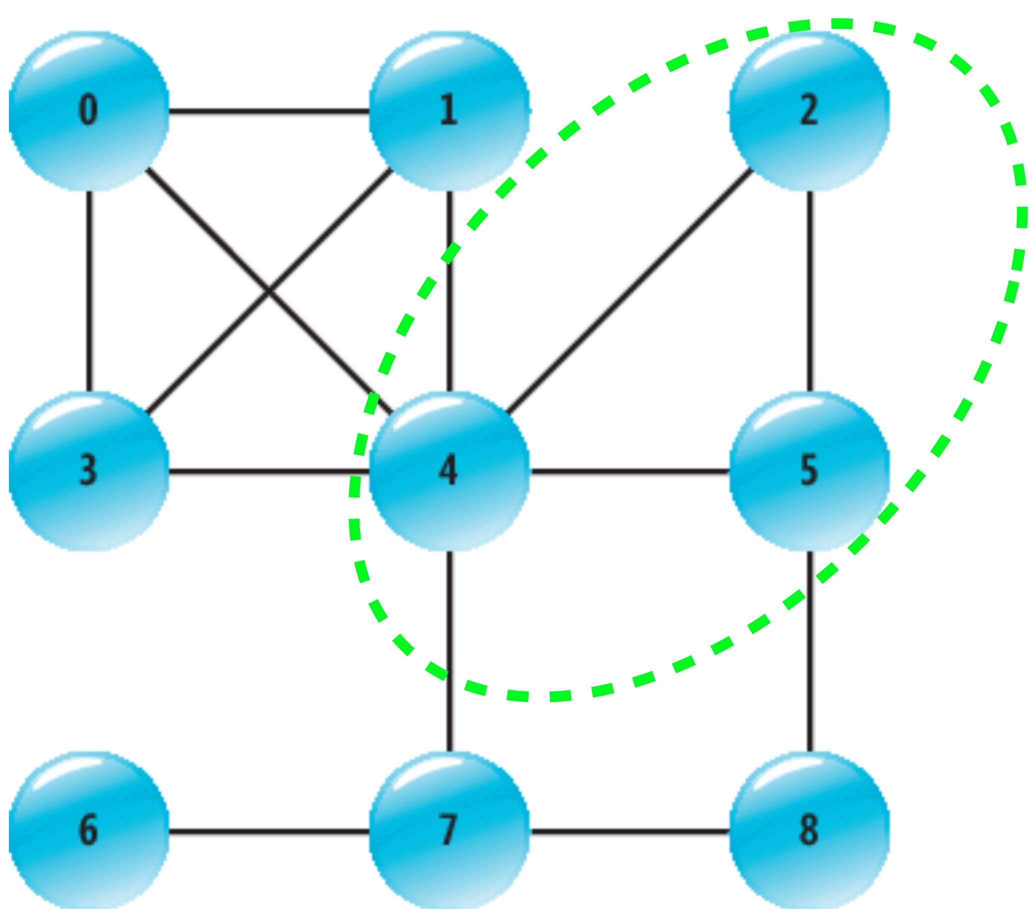

- in the picture, two cliques are shown of size 3 and 4

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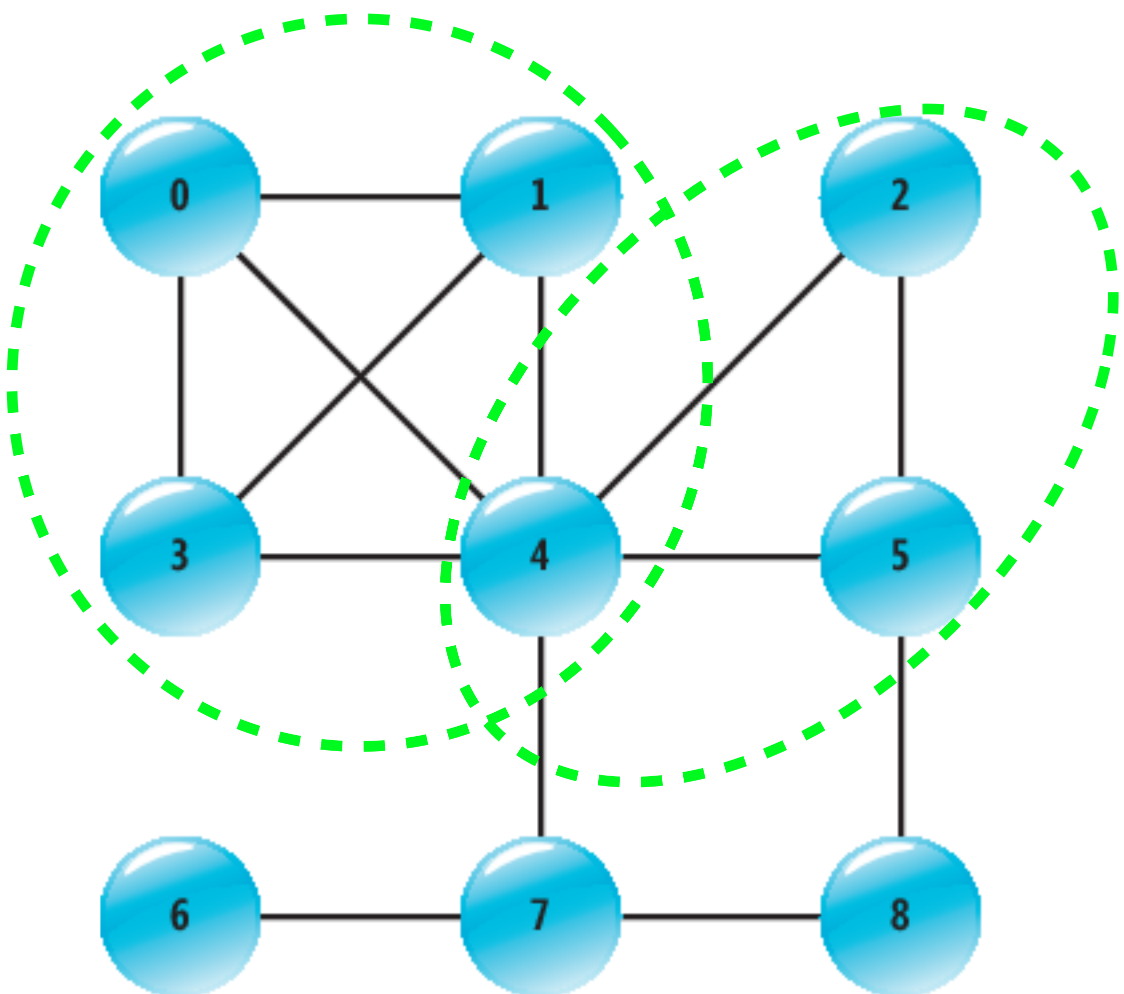

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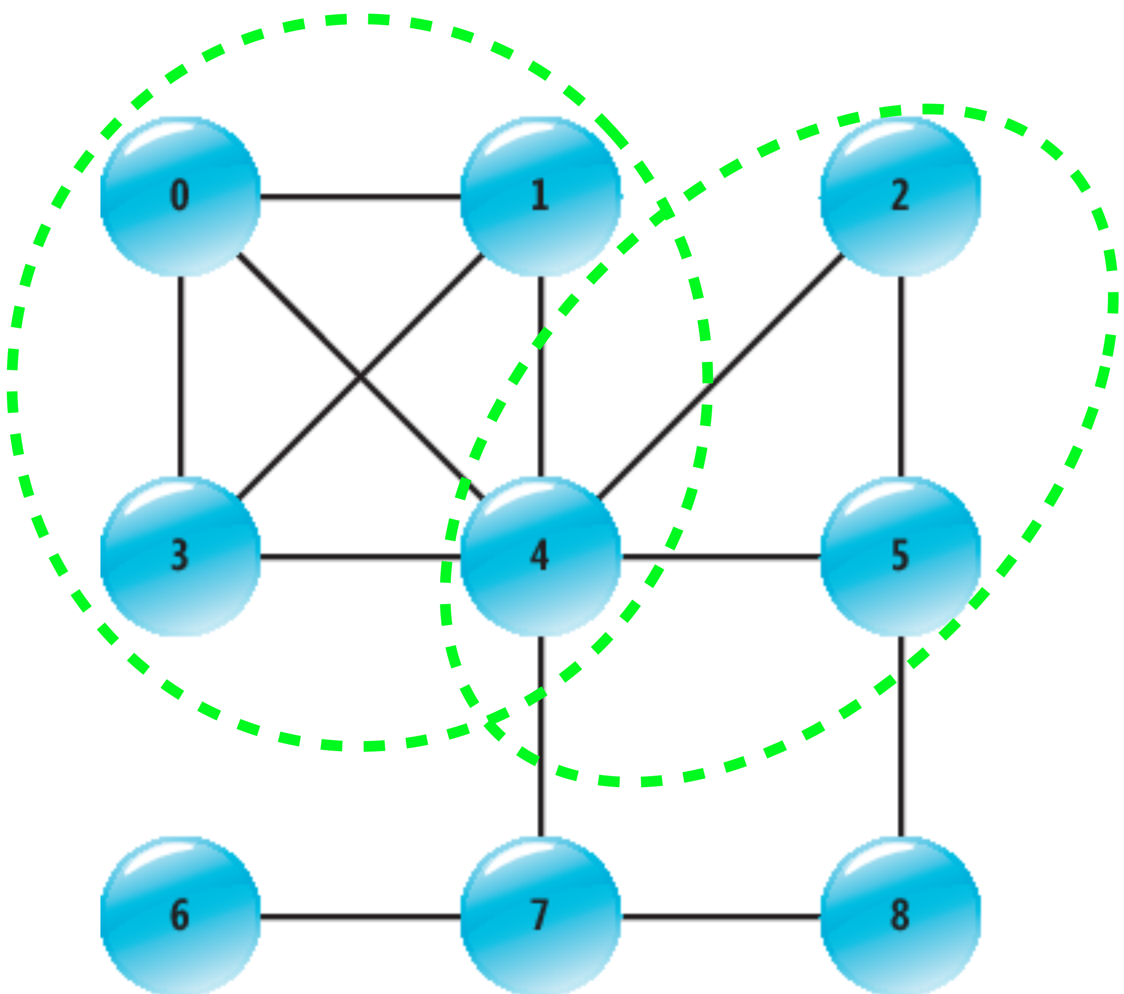
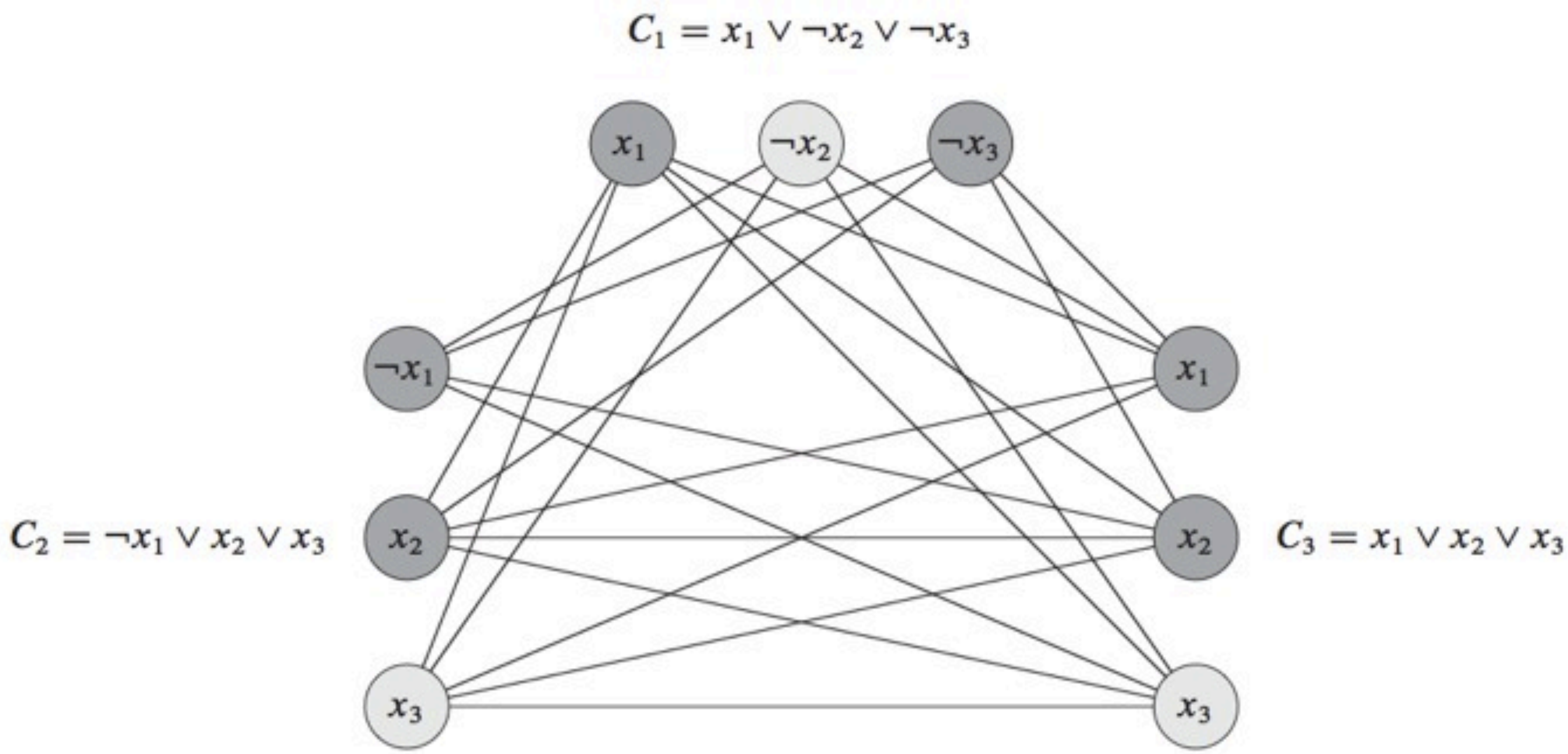


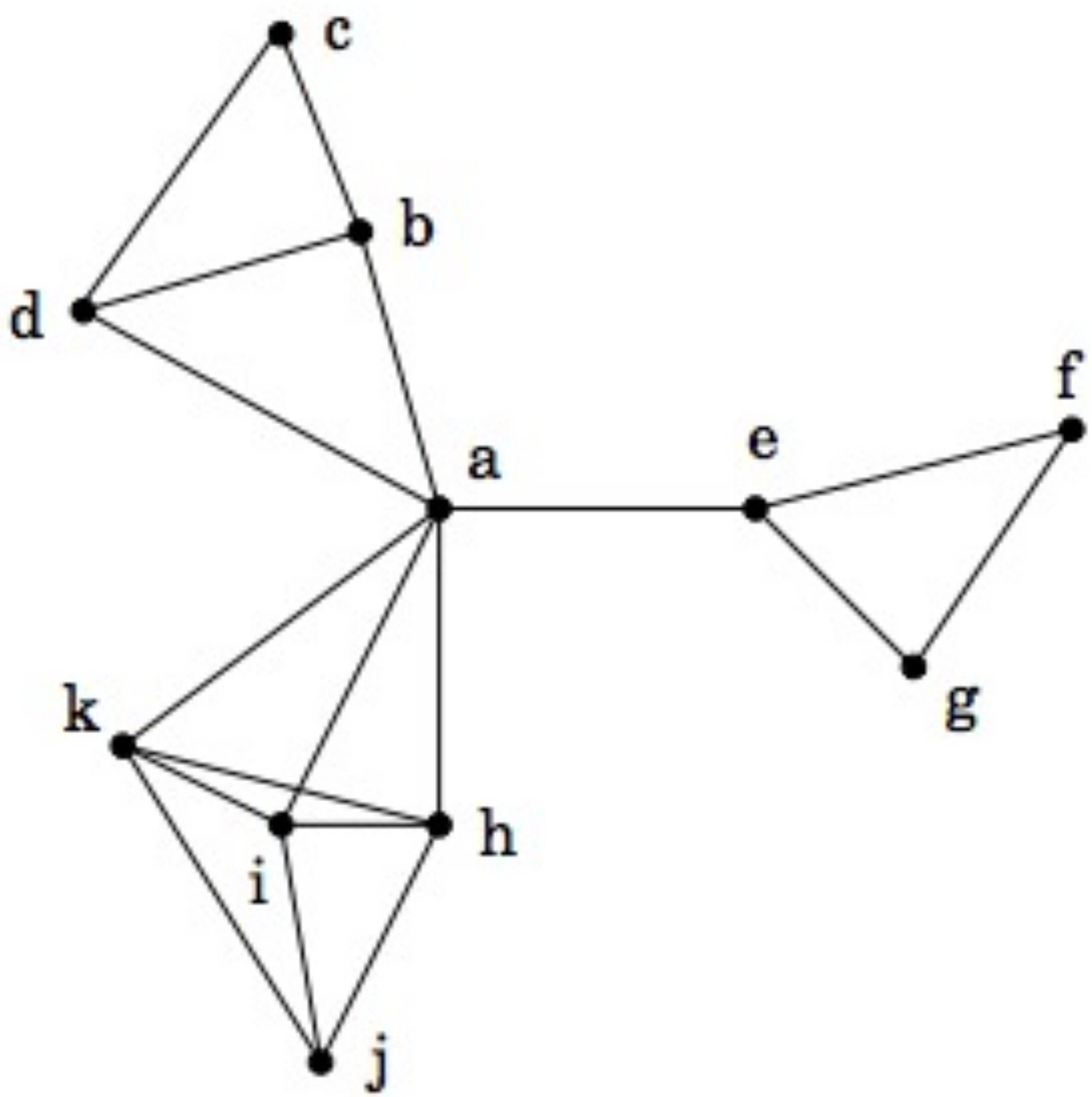
- in the picture, two cliques are shown of size 3 and 4
- the maximal clique is of size 4, as no clique of size 5 exists
- CLIQUE is hard to solve: we dont know any efficient algorithm to search for cliques.

3SAT reduces to CLIQUE



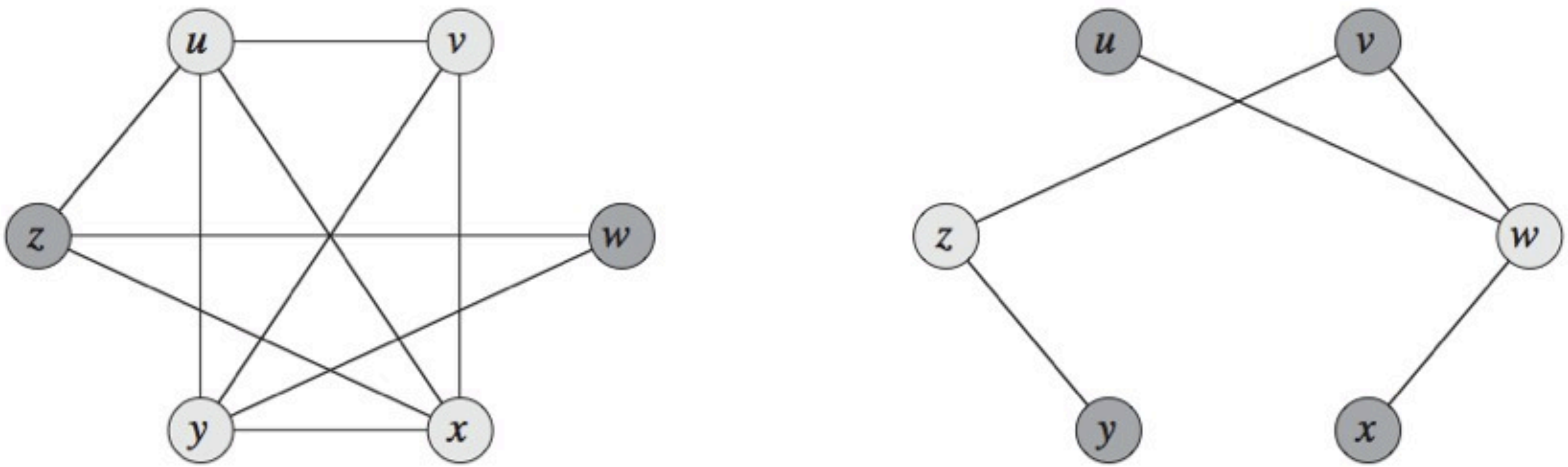
- idea: for the K clauses input to 3SAT, draw literals as vertices, and all edges between vertices except
  - across clauses only (no edges inside a clause)
  - not between x and ¬x
- reduction takes poly time
- a satisfiable assignment ⇒ a clique of size K
- a clique of size K ⇒ satisfiable assignment

VERTEX COVER



- Graph undirected G = (V,E)
- Task: find the minimum subset of vertices $T \subset V$, such that any edge $(u,v) \in E$ has at least on end u or v in T.
- NP-hard

CLIQUE reduces to VERTEX-COVER



(a) (b)

- idea: start with graph G=(V,E) input of the CLIQUE problem
- construct the complement graph G'=(V,E') by only considering the missing edges from E: E'= {all (u,v)}\E
  - poly time reduction
- clique of size K in G⇒ vertex cover of size |V|-k in G'
- vertex cover of size k in G' ⇒ clique of size |V|-K in G

SUBSET-SUM problem

- Given a set of positive integers S={a1,a2,..,an} and an integer size t
- Task: find a subset of numbers from S that sum to t
  - there might be no such subset
  - there might be multiple subsets
- Close related to discrete Knapsack (module 7)

3SAT reduction to SUBSET-SUM

		x_1	x_2	x_3	C_1	C_2	C_3	C_4
v_1	=	1	0	0	1	0	0	1
v'_1	=	1	0	0	0	1	1	0
v_2	=	0	1	0	0	0	0	1
v'_2	=	0	1	0	1	1	1	0
v_3	=	0	0	1	0	0	1	1
v'_3	=	0	0	1	1	1	0	0
s_1	=	0	0	0	1	0	0	0
s'_1	=	0	0	0	2	0	0	0
s_2	=	0	0	0	0	1	0	0
s'_2	=	0	0	0	0	2	0	0
s_3	=	0	0	0	0	0	1	0
s'_3	=	0	0	0	0	0	2	0
s_4	=	0	0	0	0	0	0	1
s'_4	=	0	0	0	0	0	0	2
t	=	1	1	1	4	4	4	4

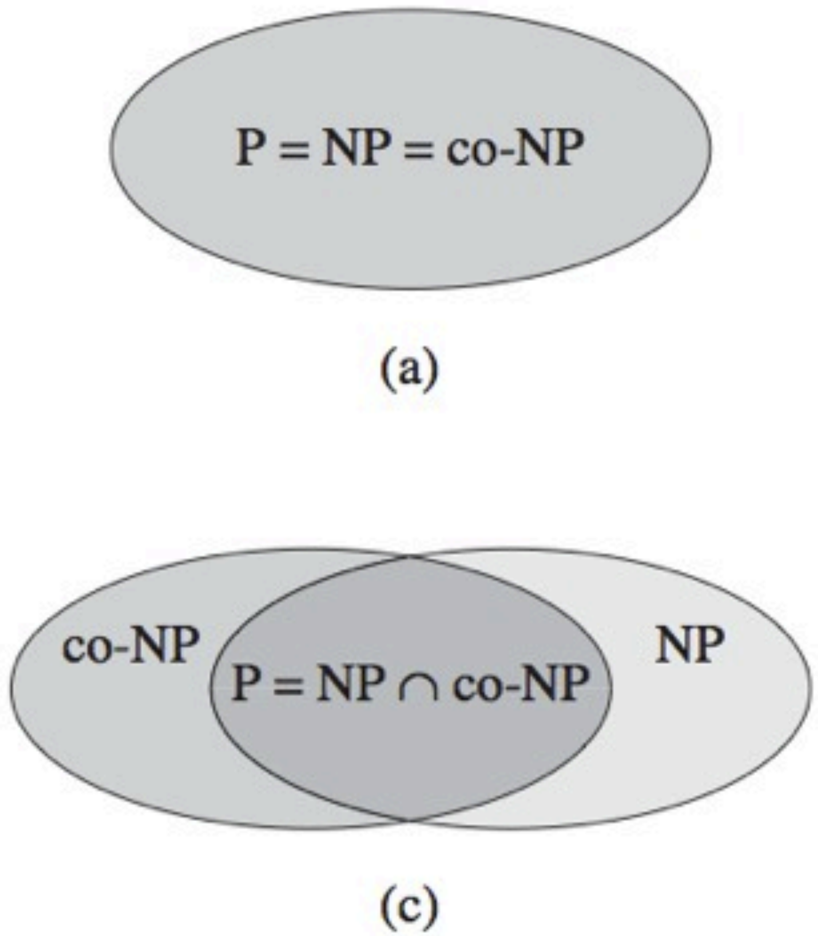
- poly-time reduction
- SUBSET-SUM is NP complete
- CLRS book 34.5.5

Figure 34.19 The reduction of 3-CNF-SAT to SUBSET-SUM. The formula in 3-CNF is $\phi = C_1 \wedge C_2 \wedge C_3 \wedge C_4$, where $C_1 = (x_1 \vee \neg x_2 \vee \neg x_3)$, $C_2 = (\neg x_1 \vee \neg x_2 \vee \neg x_3)$, $C_3 = (\neg x_1 \vee \neg x_2 \vee x_3)$, and $C_4 = (x_1 \vee x_2 \vee x_3)$. A satisfying assignment of ϕ is $\langle x_1 = 0, x_2 = 0, x_3 = 1 \rangle$. The set S produced by the reduction consists of the base-10 numbers shown; reading from top to bottom, $S = \{1001001, 1000110, 100001, 101110, 10011, 11100, 1000, 2000, 100, 200, 10, 20, 1, 2\}$. The target t is 1114444. The subset $S' \subseteq S$ is lightly shaded, and it contains v'_1, v'_2, and v_3, corresponding to the satisfying assignment. It also contains slack variables s_1, s'_1, s'_2, s_3, s_4, and s'_4 to achieve the target value of 4 in the digits labeled by C_1 through C_4.

NP complete problems

- problem A is NP-complete if
  - A is in NP (poly-time to verify proposed solution)
  - any problem in NP reduces to A
- second condition says: if one solves pb A, it solves via polynomial reductions all other problems in NP
- CIRCUIT SAT is NP-complete (see book)
  - and so the other problems discussed here, because they reduce to it
- NP-complete contains as of 2013 thousands well known "apparently hard" problems
  - unlikely one (same as "all") of them can be solved in poly time. . .
  - that would mean P=NP, which many believe not true.

P vs NP problem



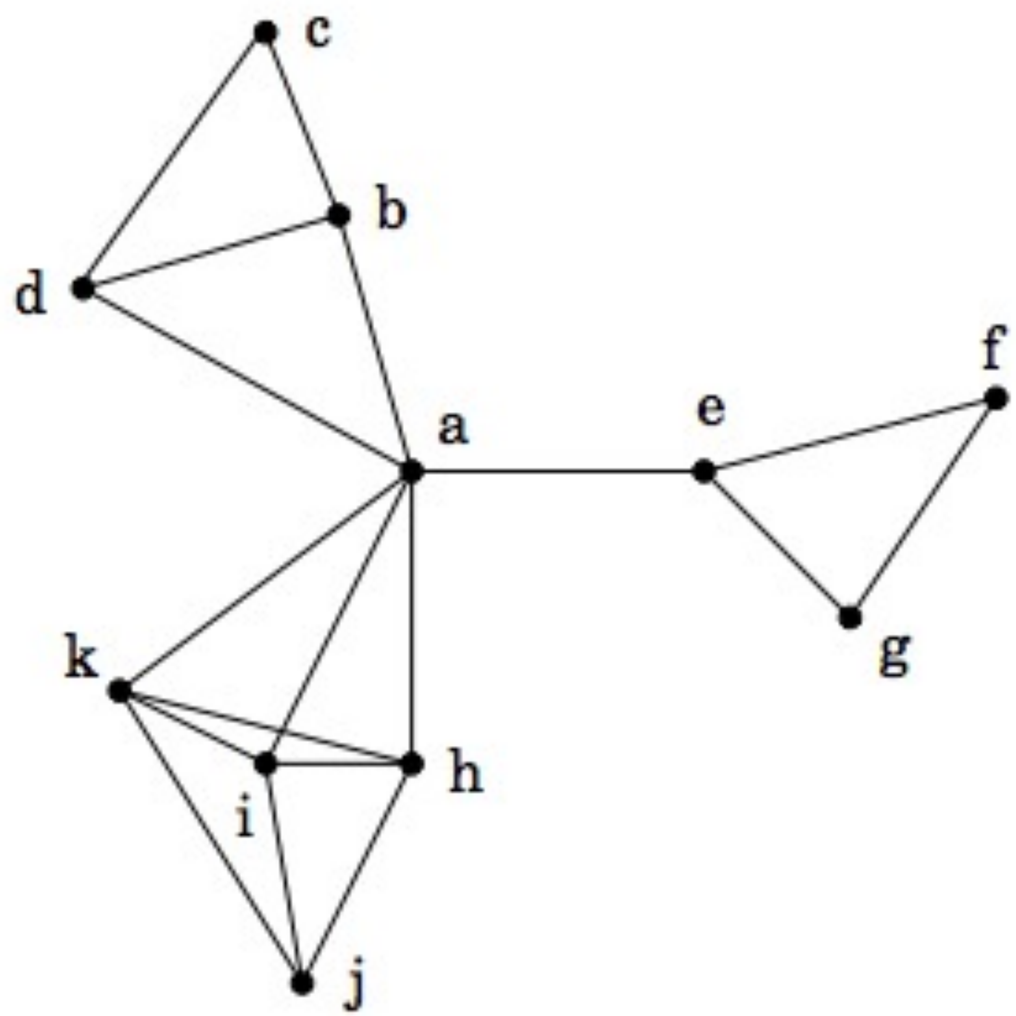
- see book for co-NP class definition
- four possibilities, no one knows which one is true
- most believe (d) to be true
- prove P=NP: find a poly time solver for an NP-complete pb, for ex 3SAT
- prove P≠NP: prove that an NP-complete pb cant have poly-time solver

Approximation Algorithms

Some problems too hard

- ... to solve exactly
- so we settle for a non-optimal solution
- use an efficient algorithm, sometime Greedy
- solution wont be optimal, but how much non-optimal?
  - objective(SOL) VS objective(OPTSOL)

Vertex Cover approx algorithm



- choose an edge (u,v)
  - add u,v to VCover
  - delete all edges with ends in u or v
- repeat until no edges left
- for the example in the picture:

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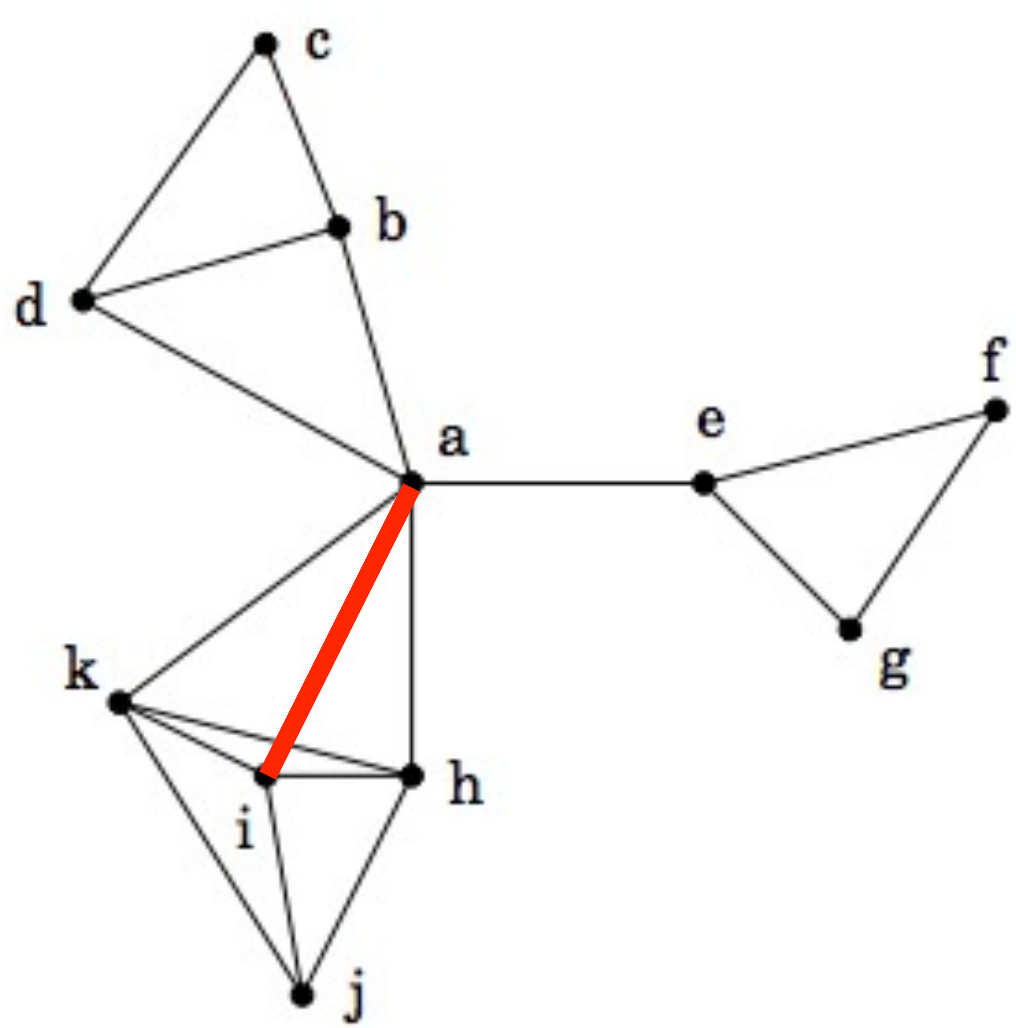

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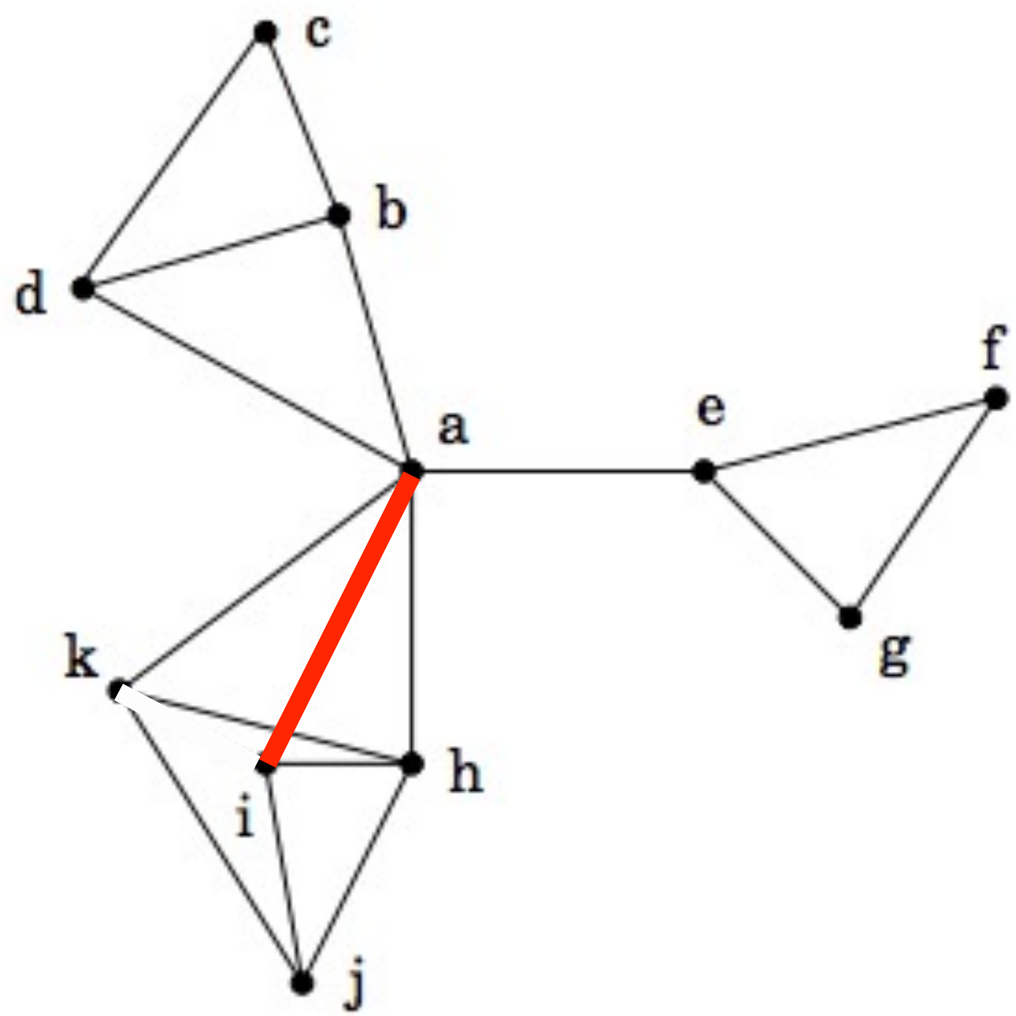

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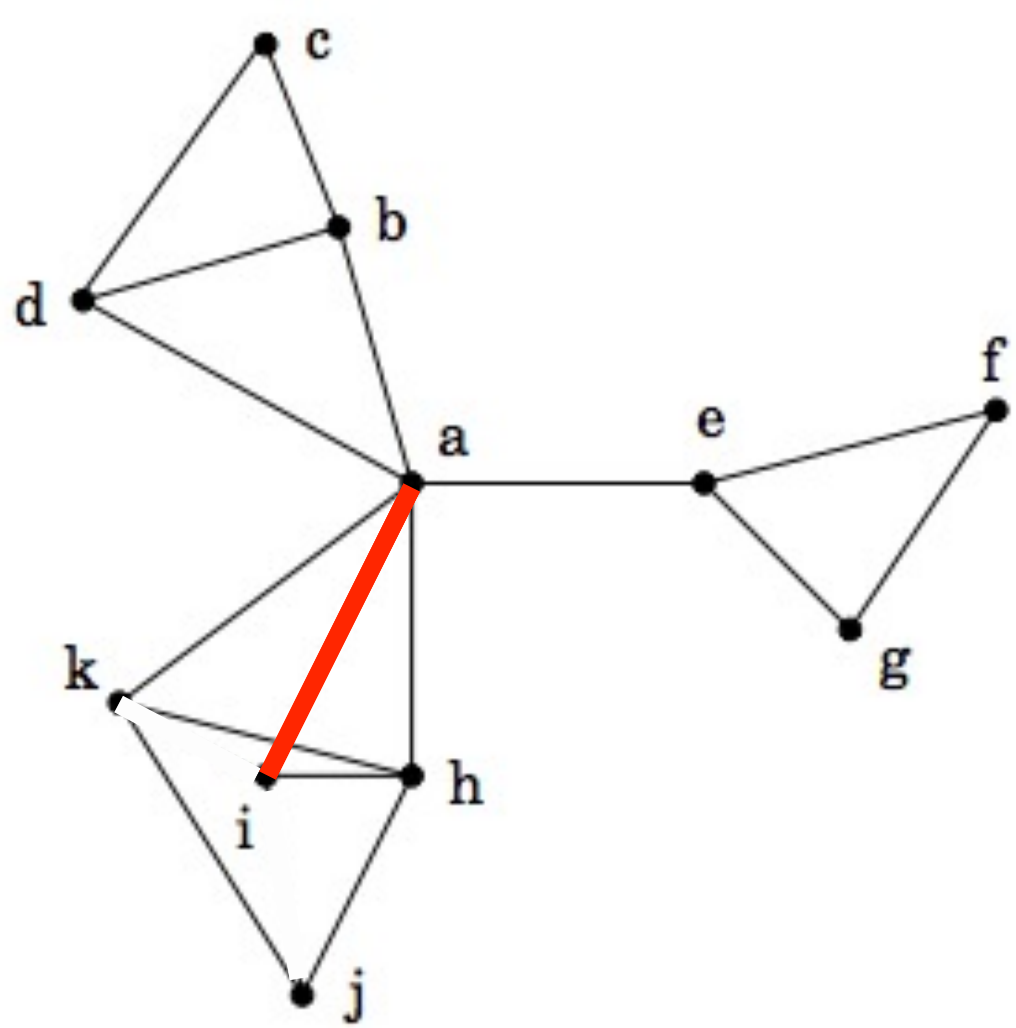

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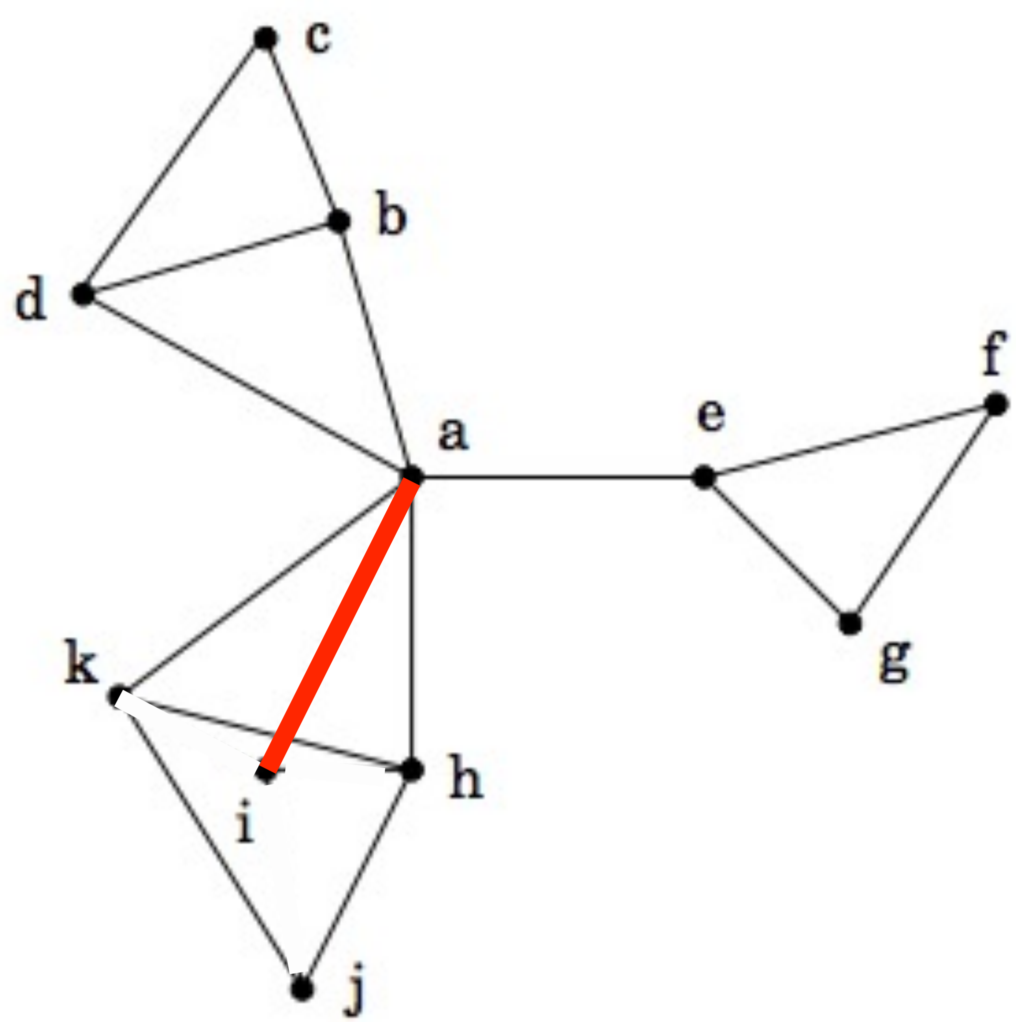

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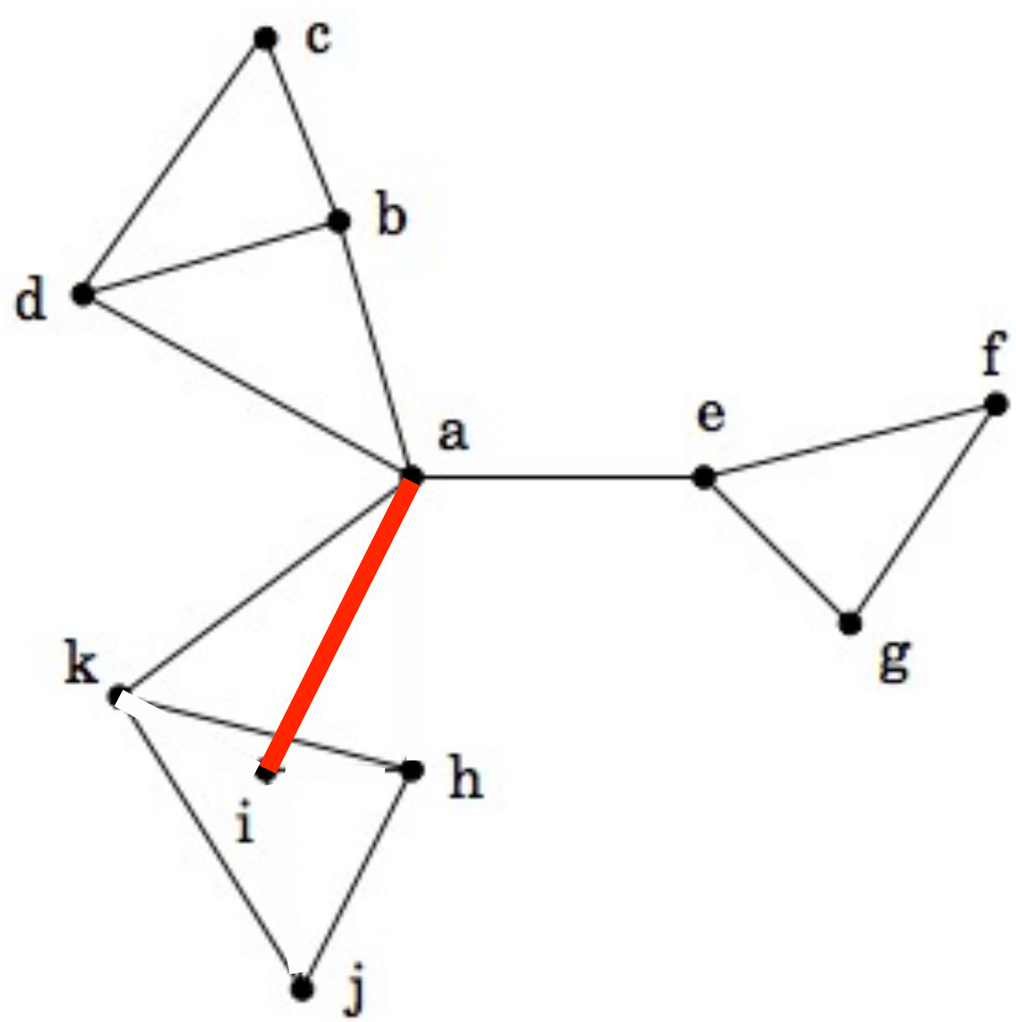

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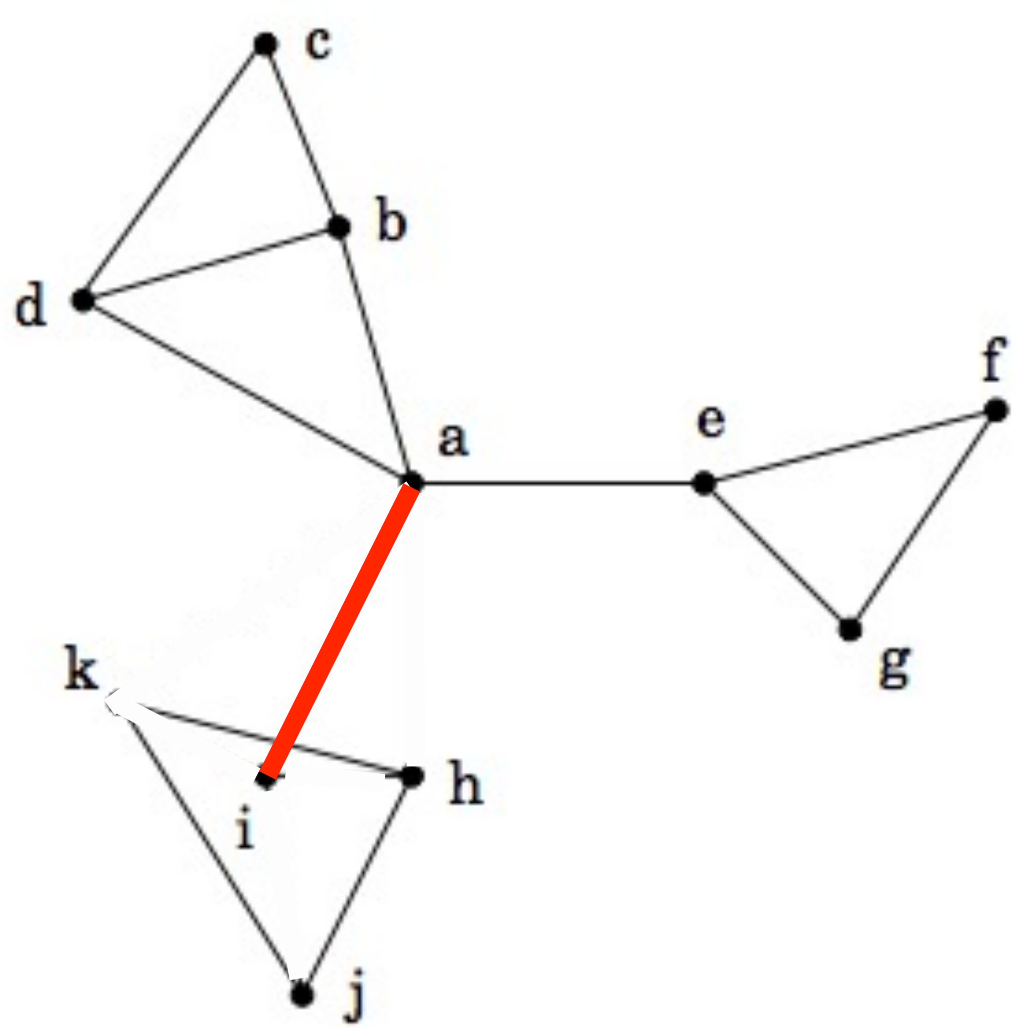

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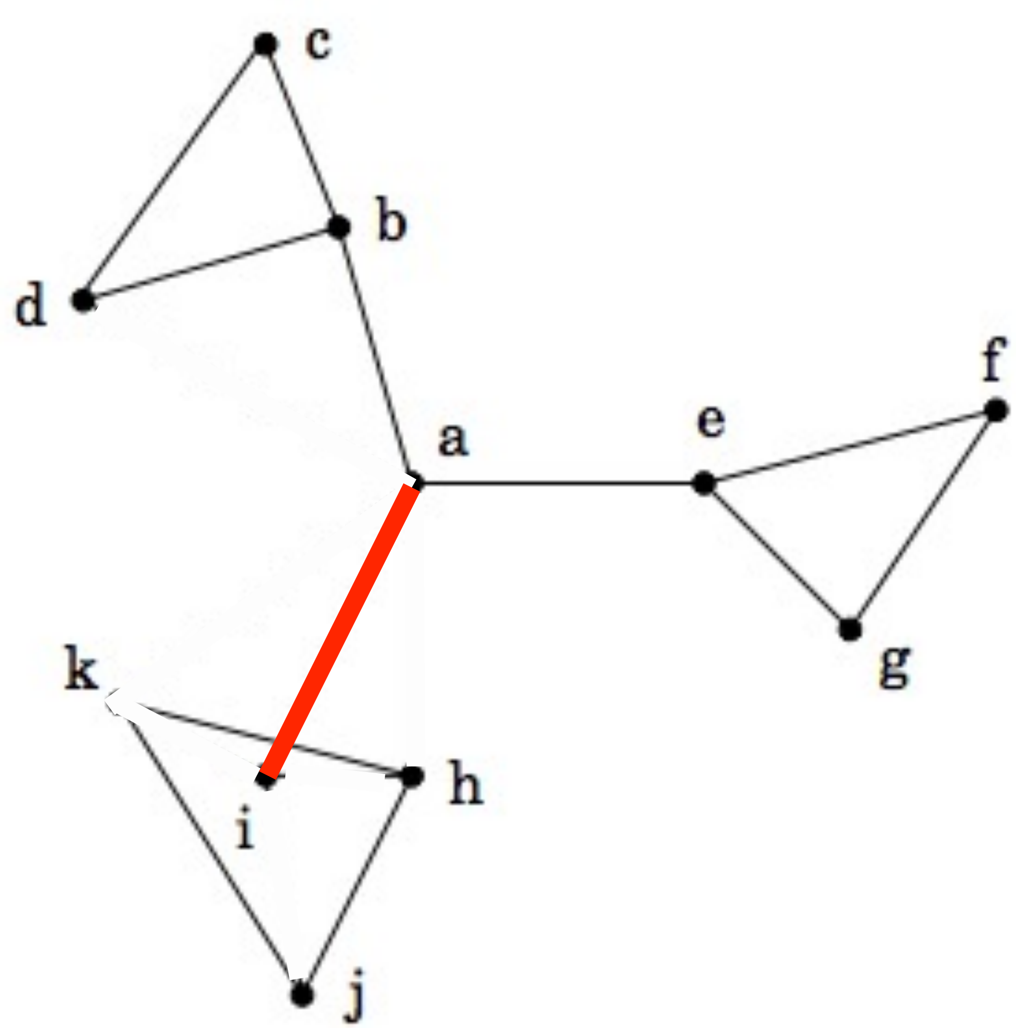

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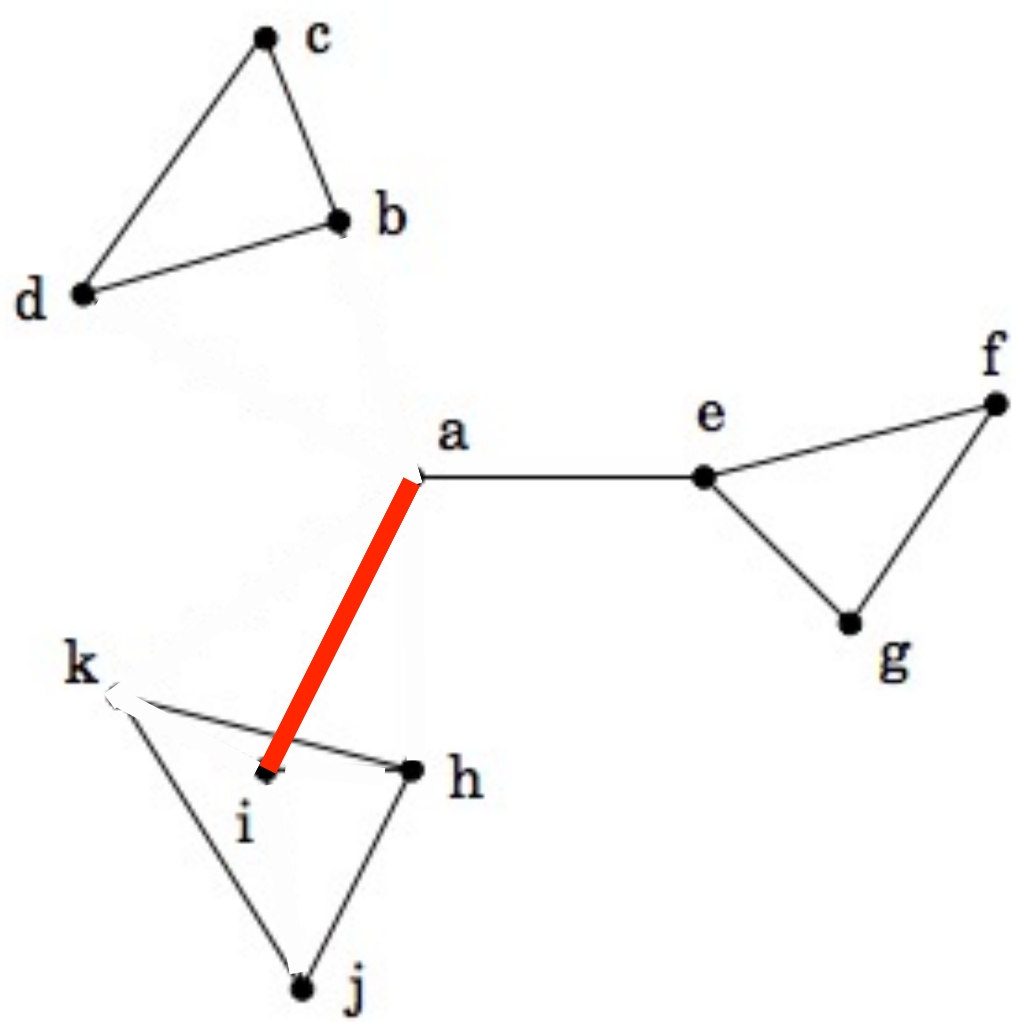

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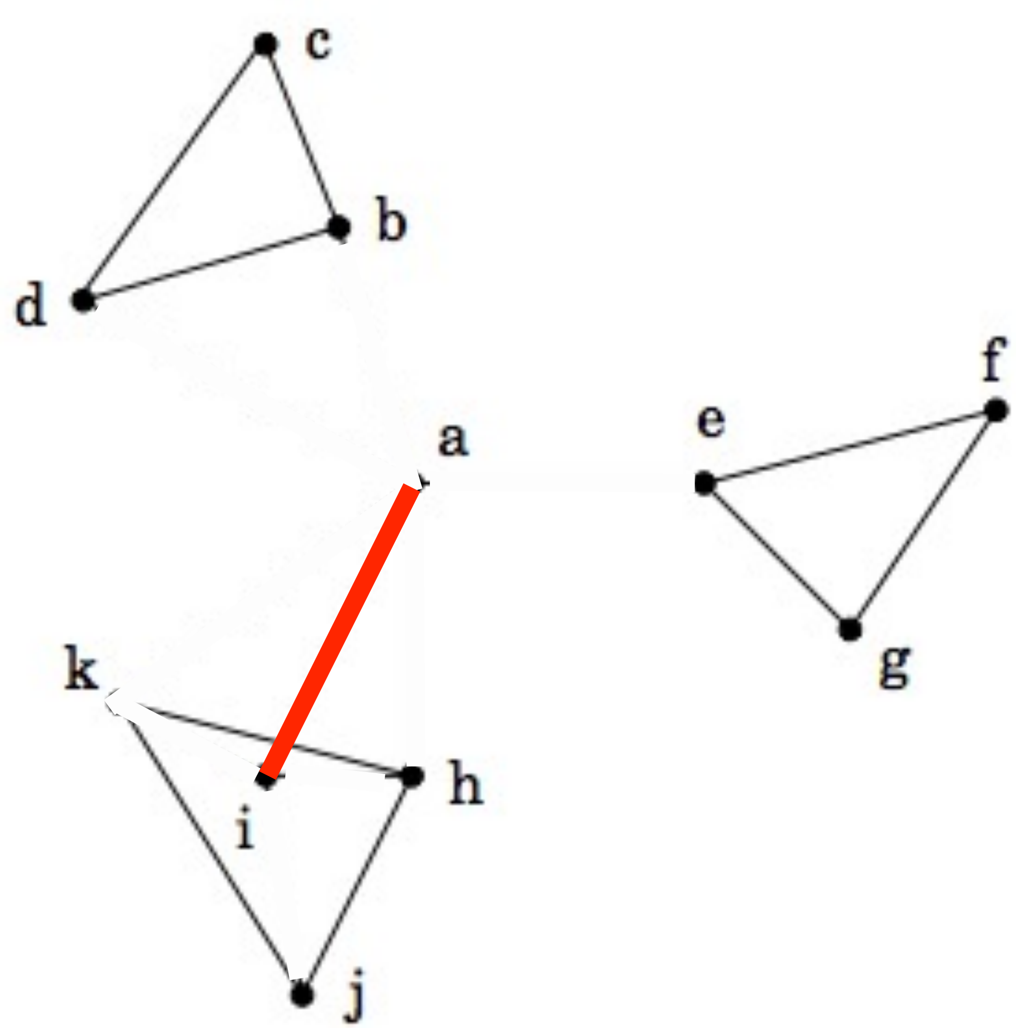

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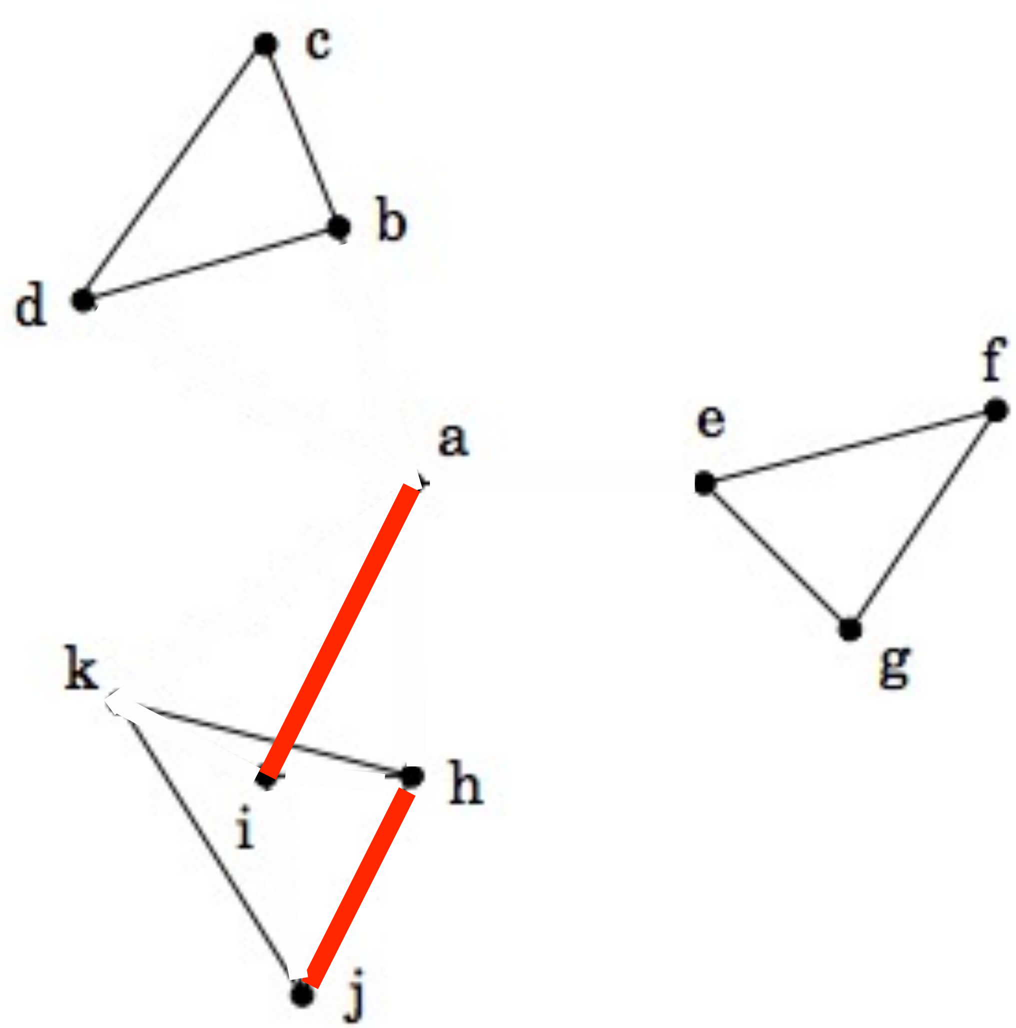

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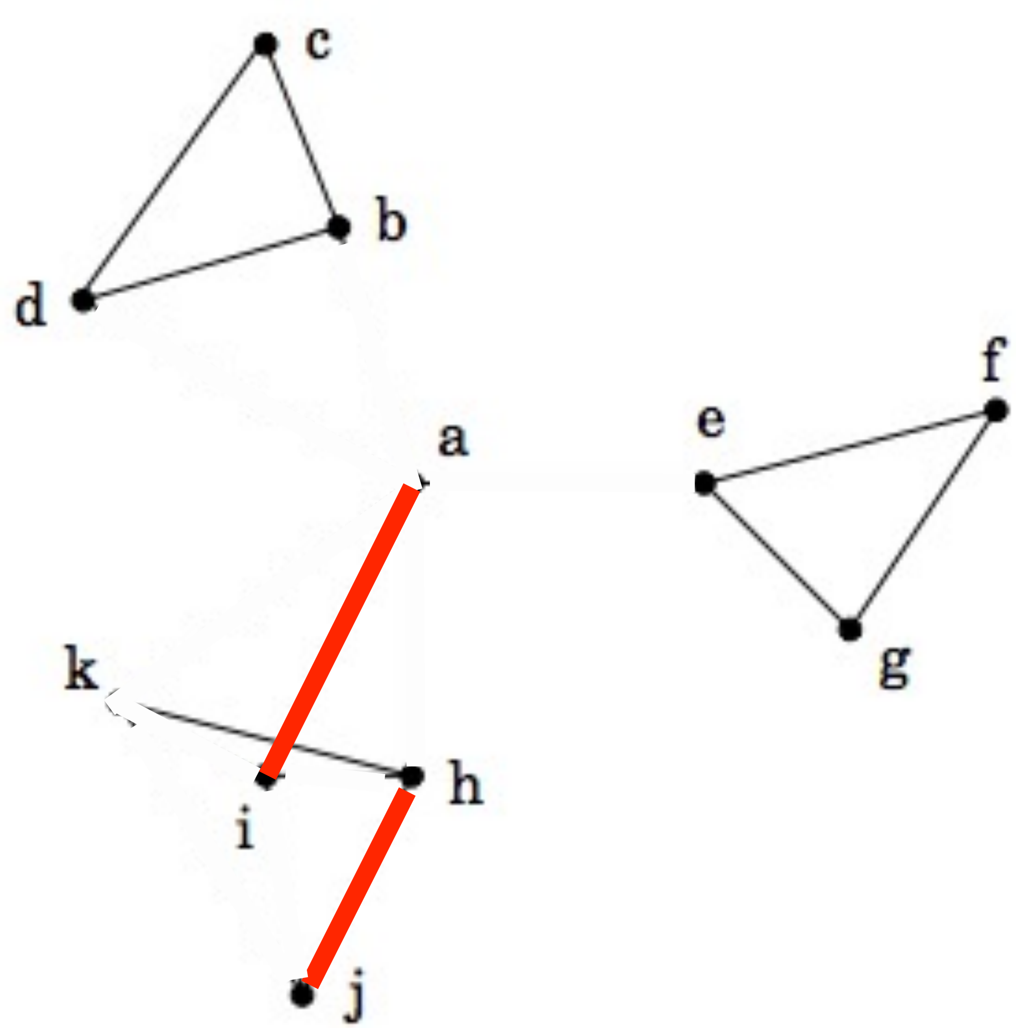

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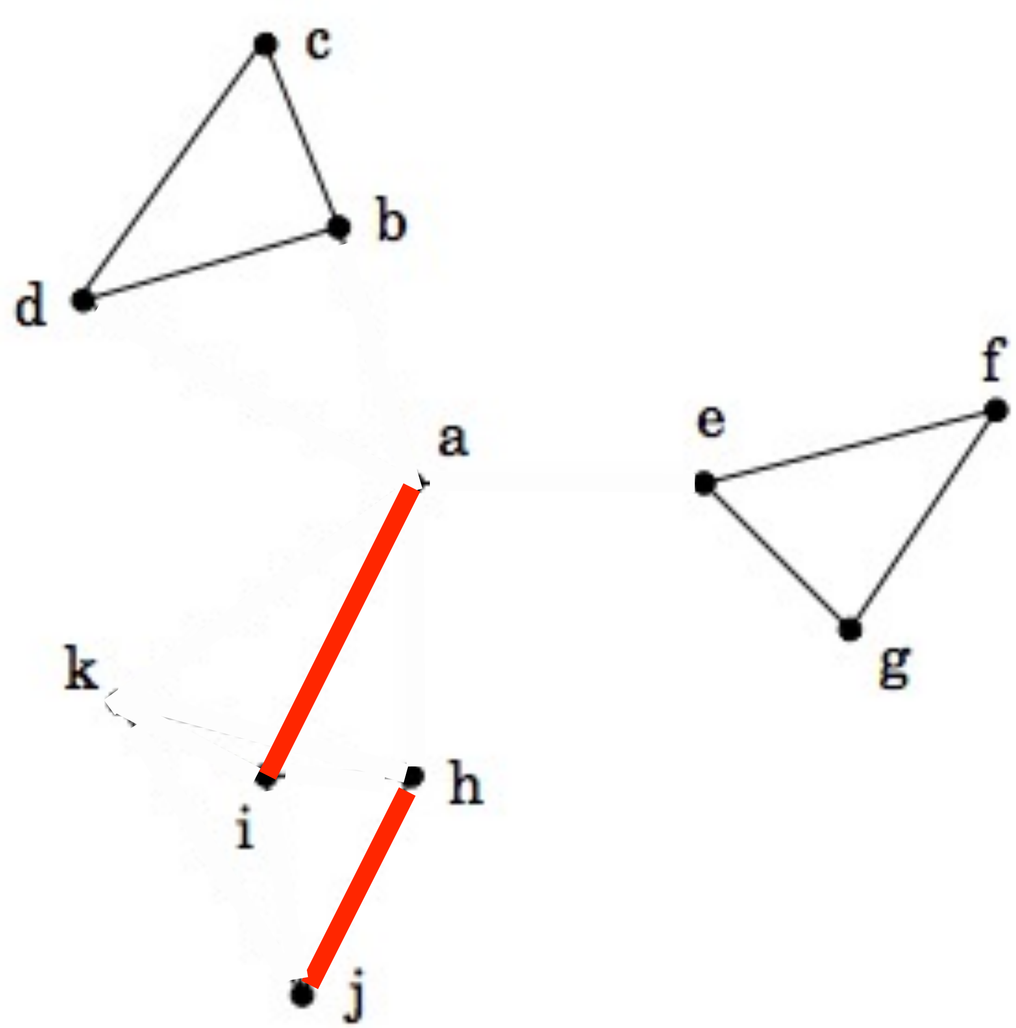

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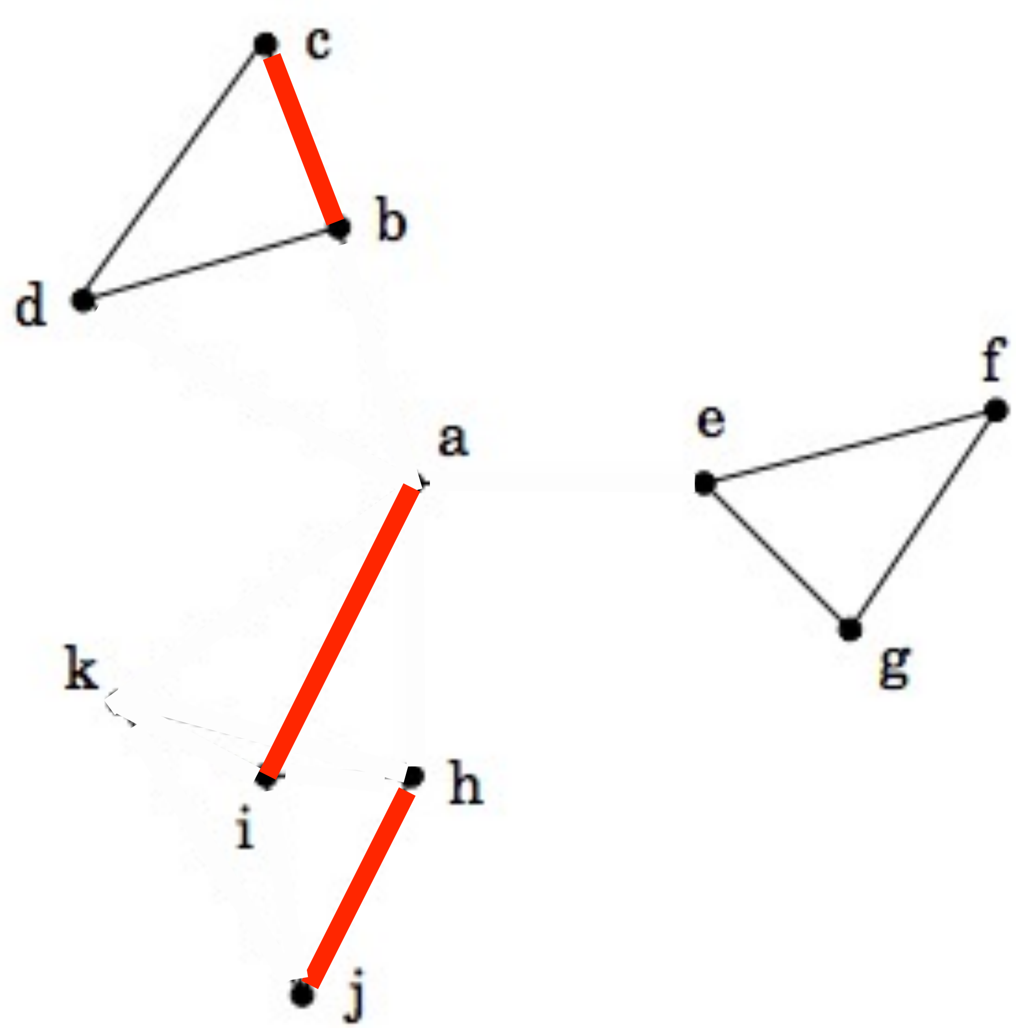

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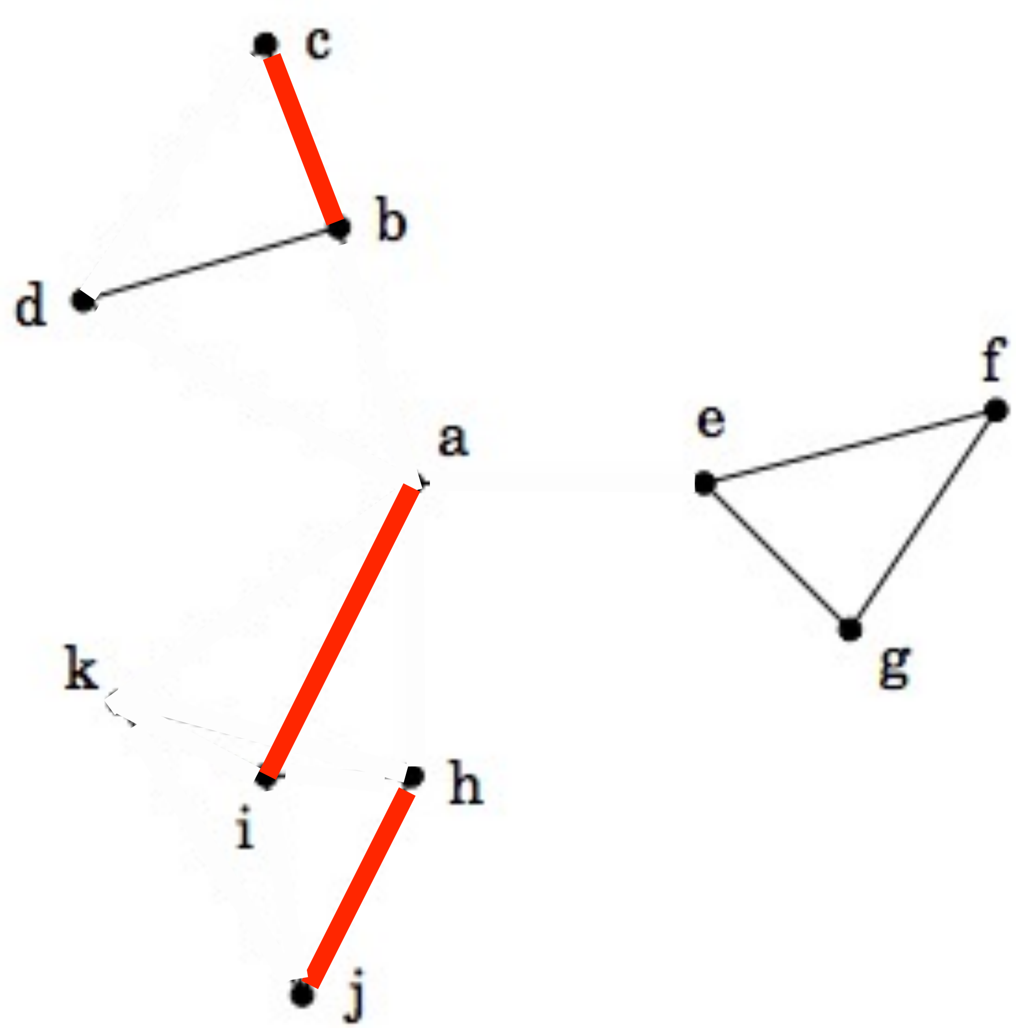

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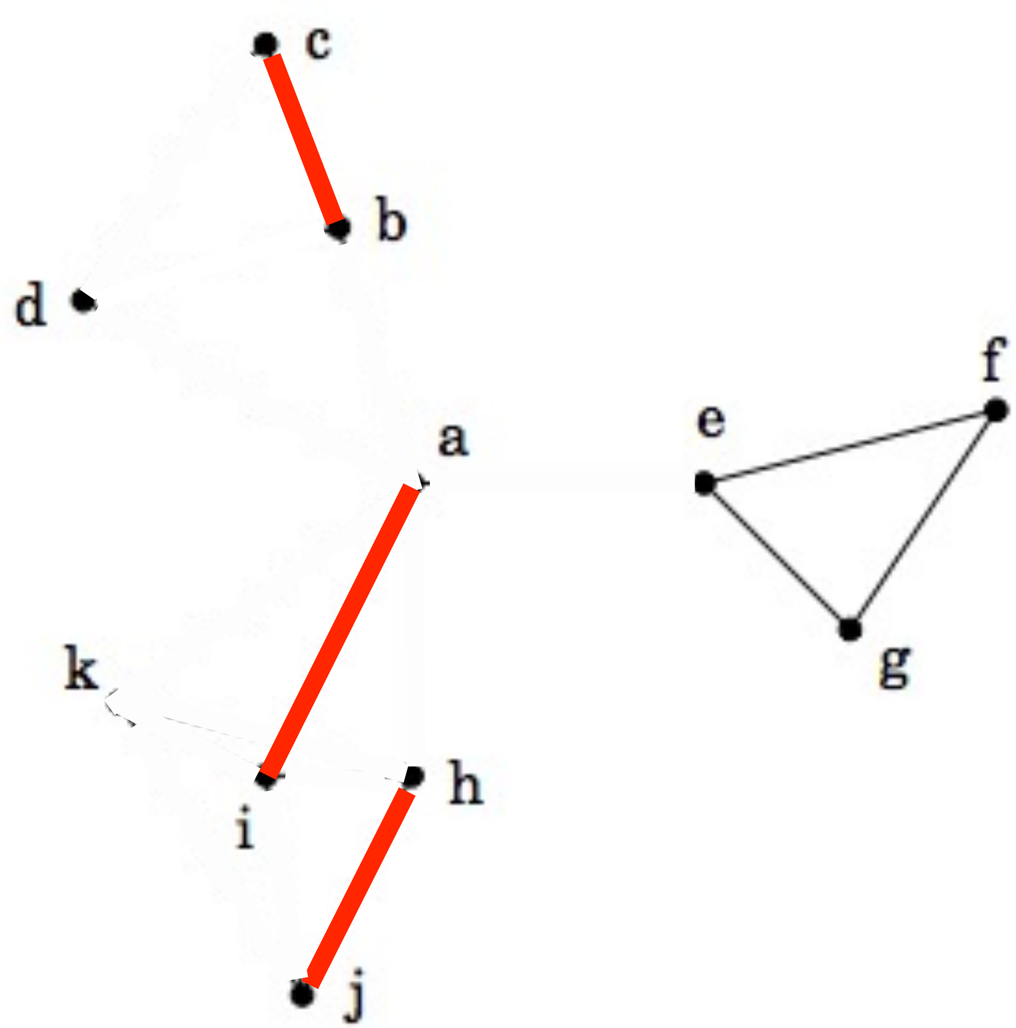

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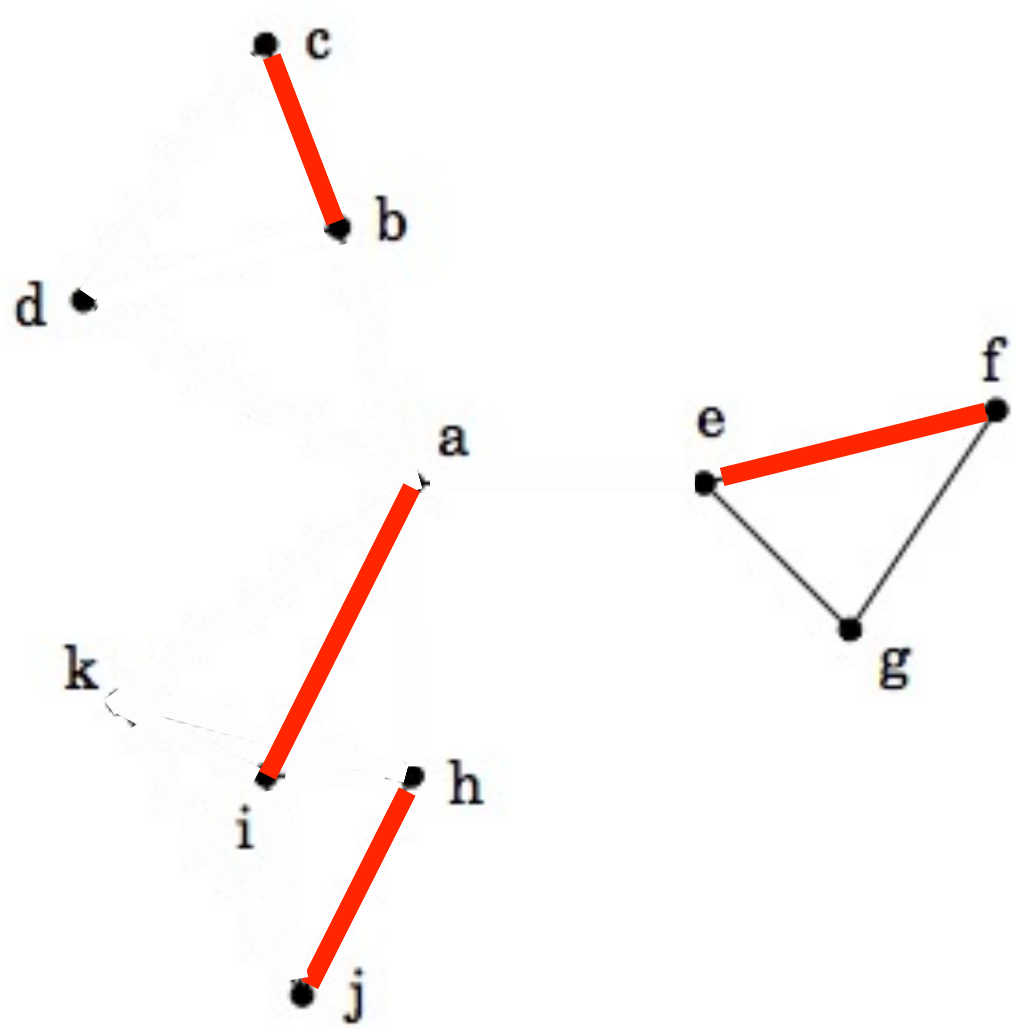

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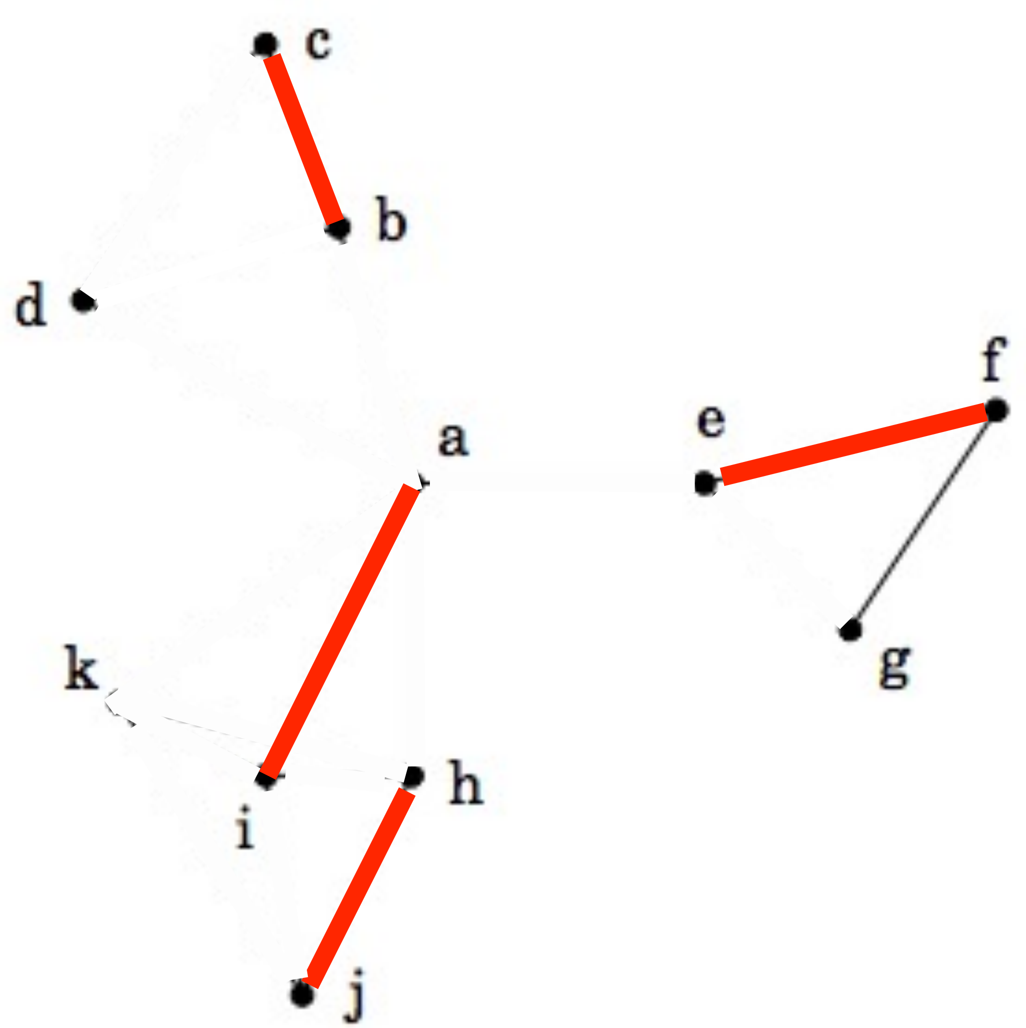

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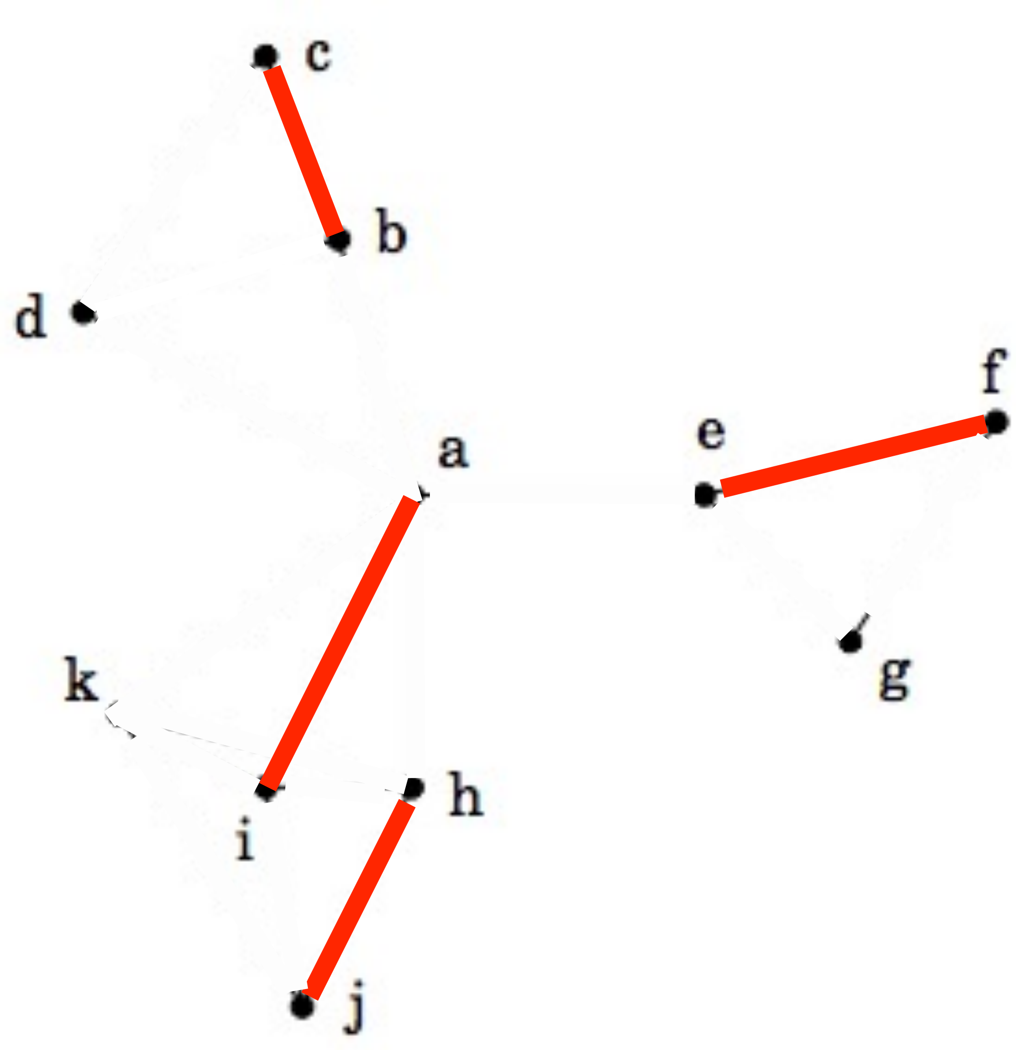

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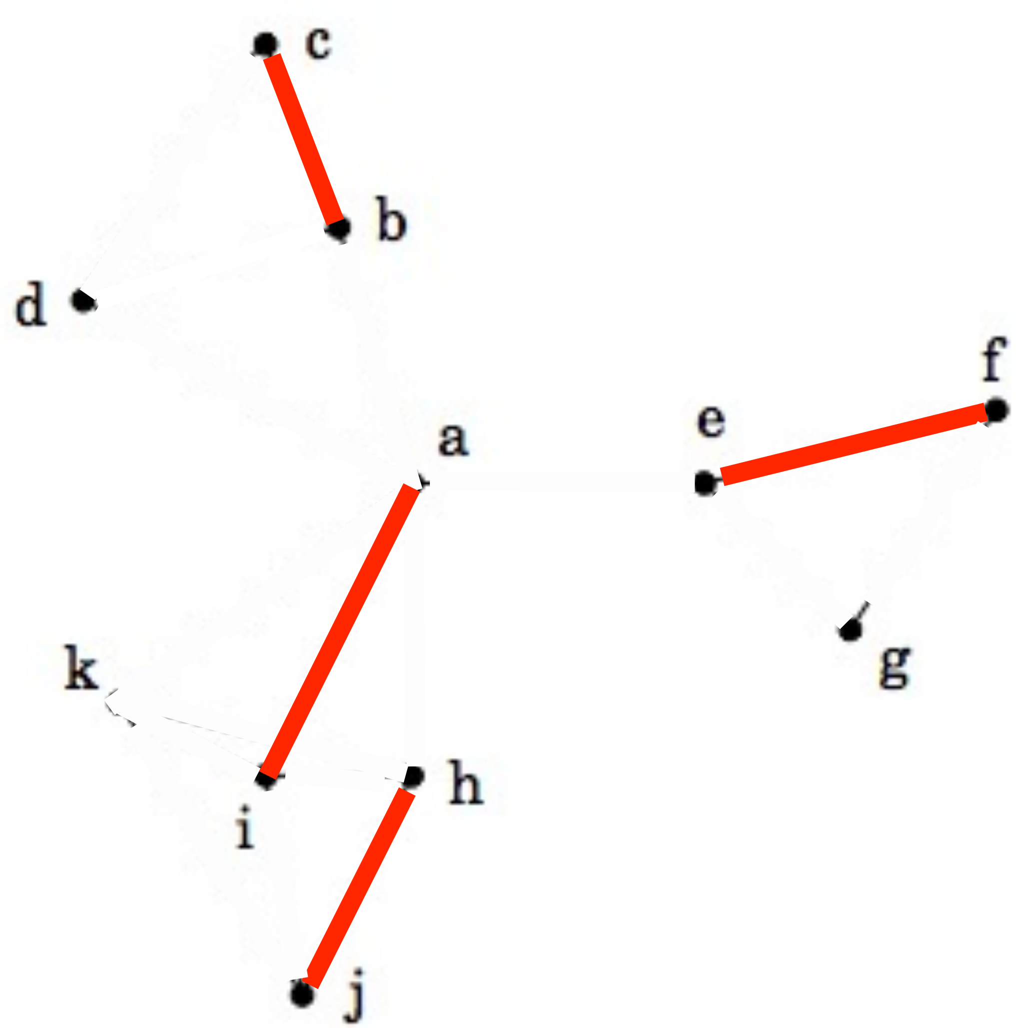
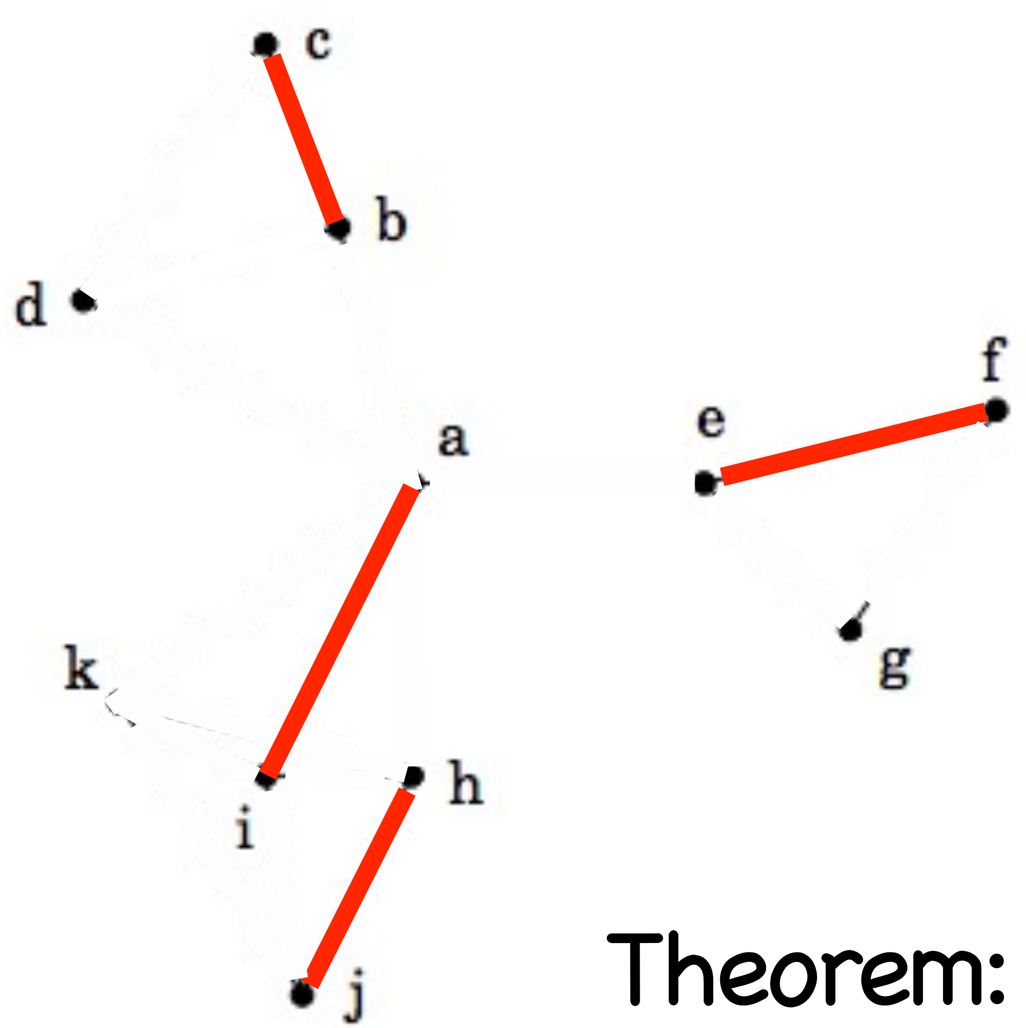


- choose an edge (u,v)
  - add u,v to VCover
  - delete all edges with ends in u or v
- repeat until no edges left
- for the example in the picture:
  - (a,i)
  - (h,j)
  - (b,c)
  - (e,f)
- VC_approx={a,i,h,j,b,c,e,f}
- VC_OPTIM={b,d,e,g,k,i,h}

Vertex Cover approx algorithm



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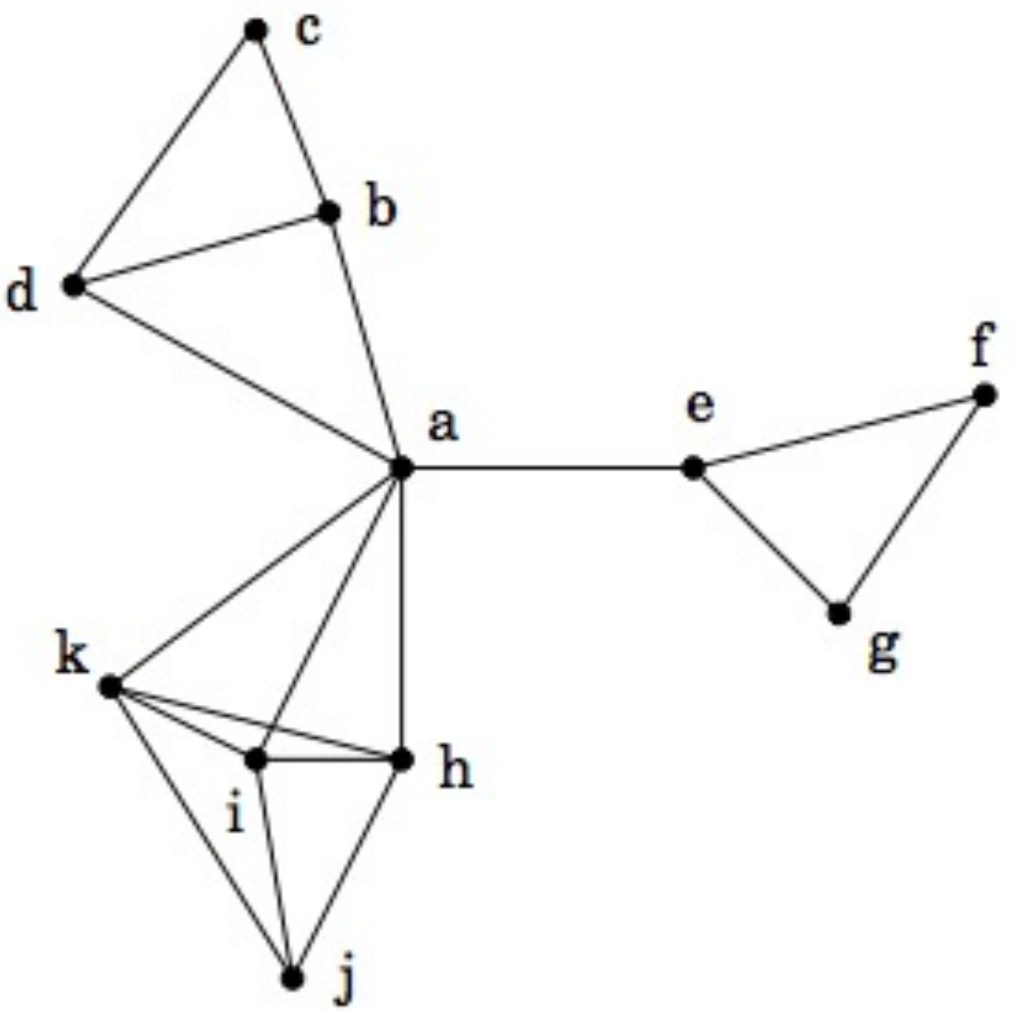
Theorem:

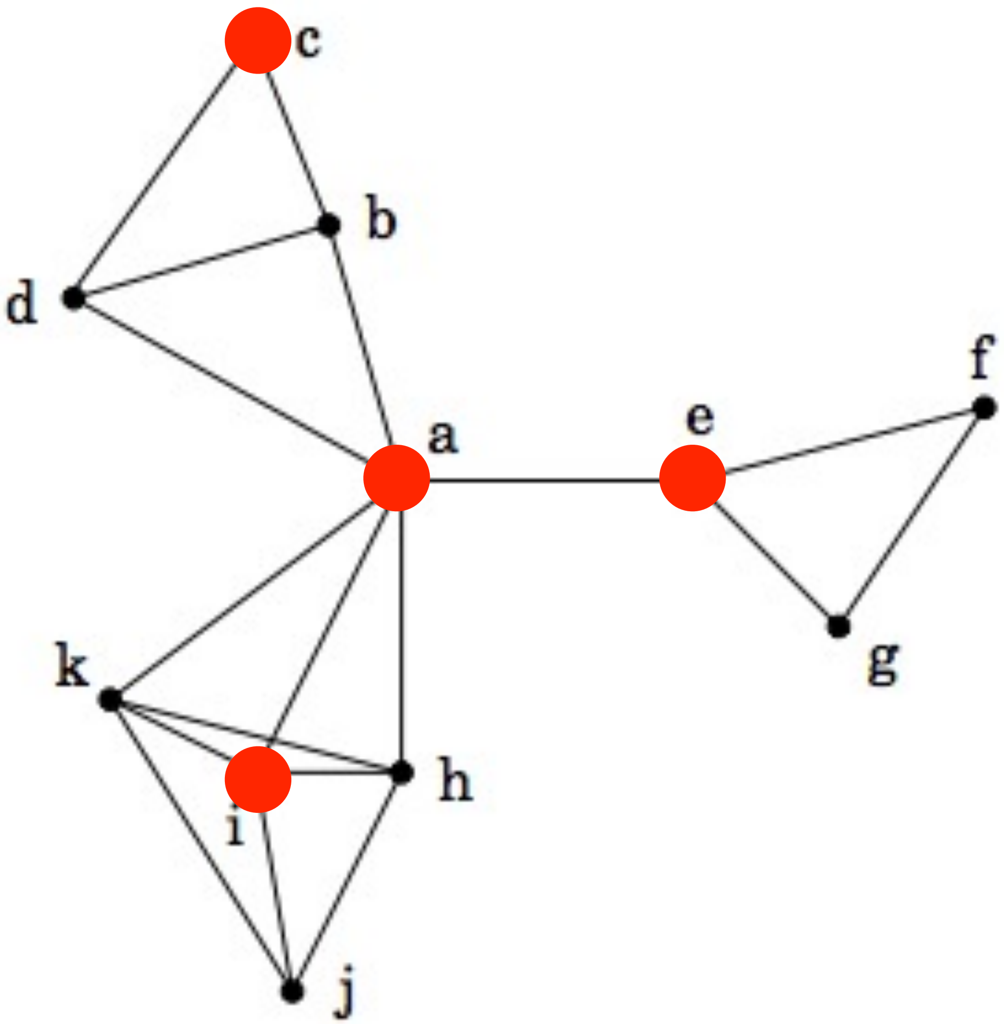
- size(VC_gredy) ≤ size(VC_optim) * 2
  - approx ratio of 2

Set Cover problem



- set of towns S = {a,b,c,d,…,k}
- edge(u,v) : distance(u,v)<10miles
- Set Cover SC⊂S : a set of towns such that every town is within 10 miles of some town in SC

Set Cover problem



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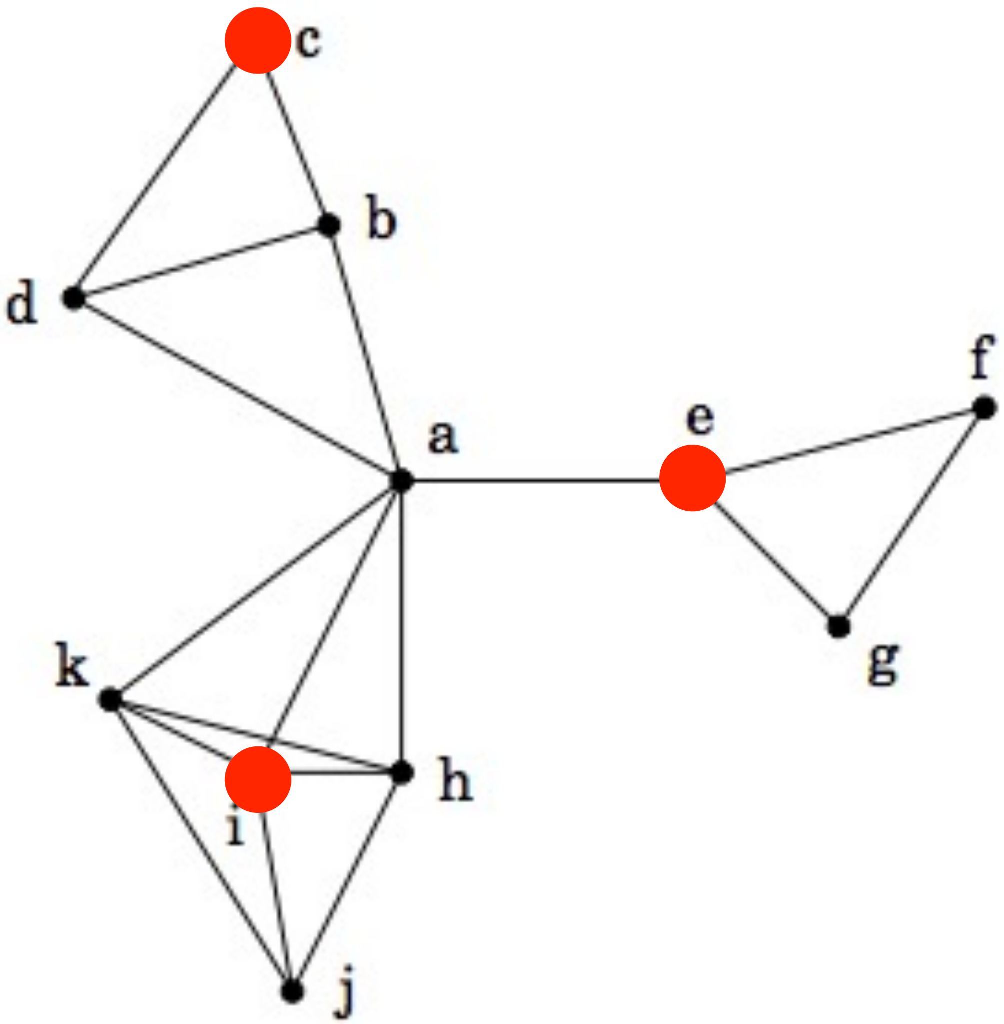

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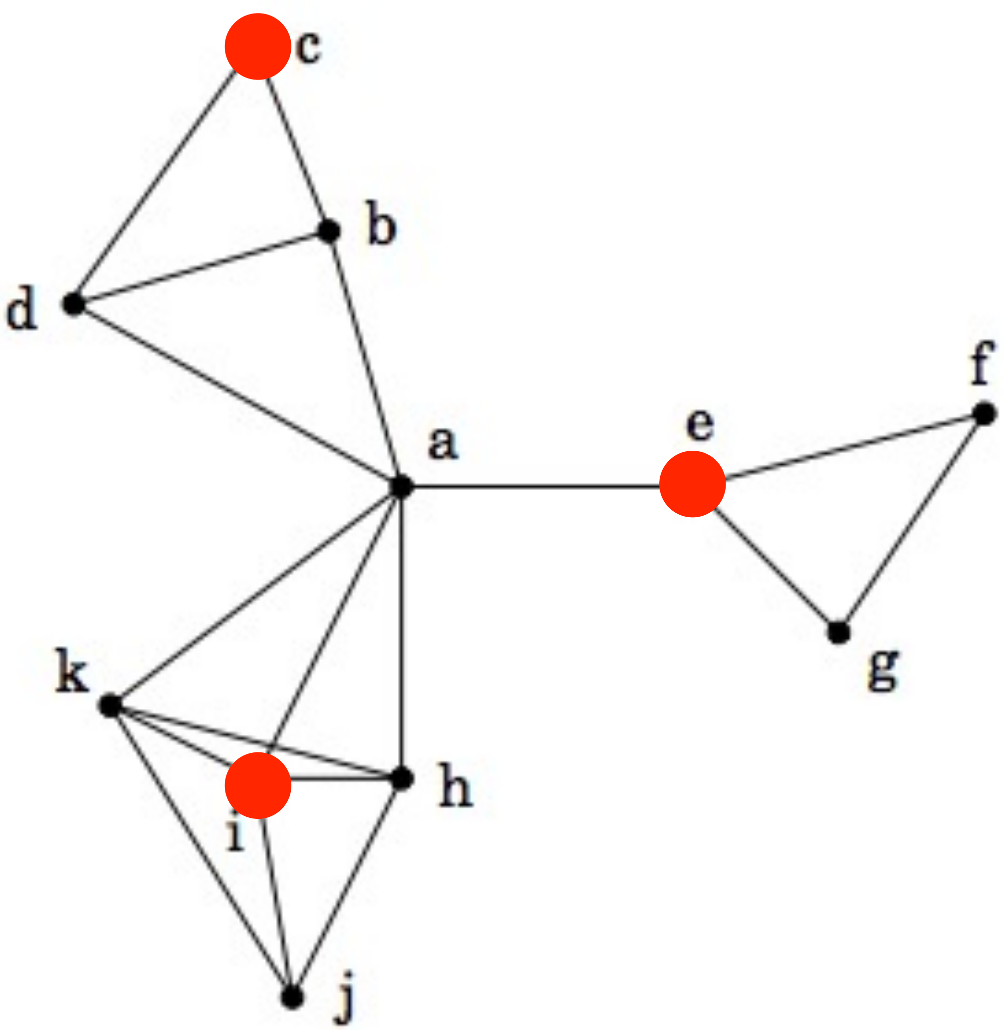

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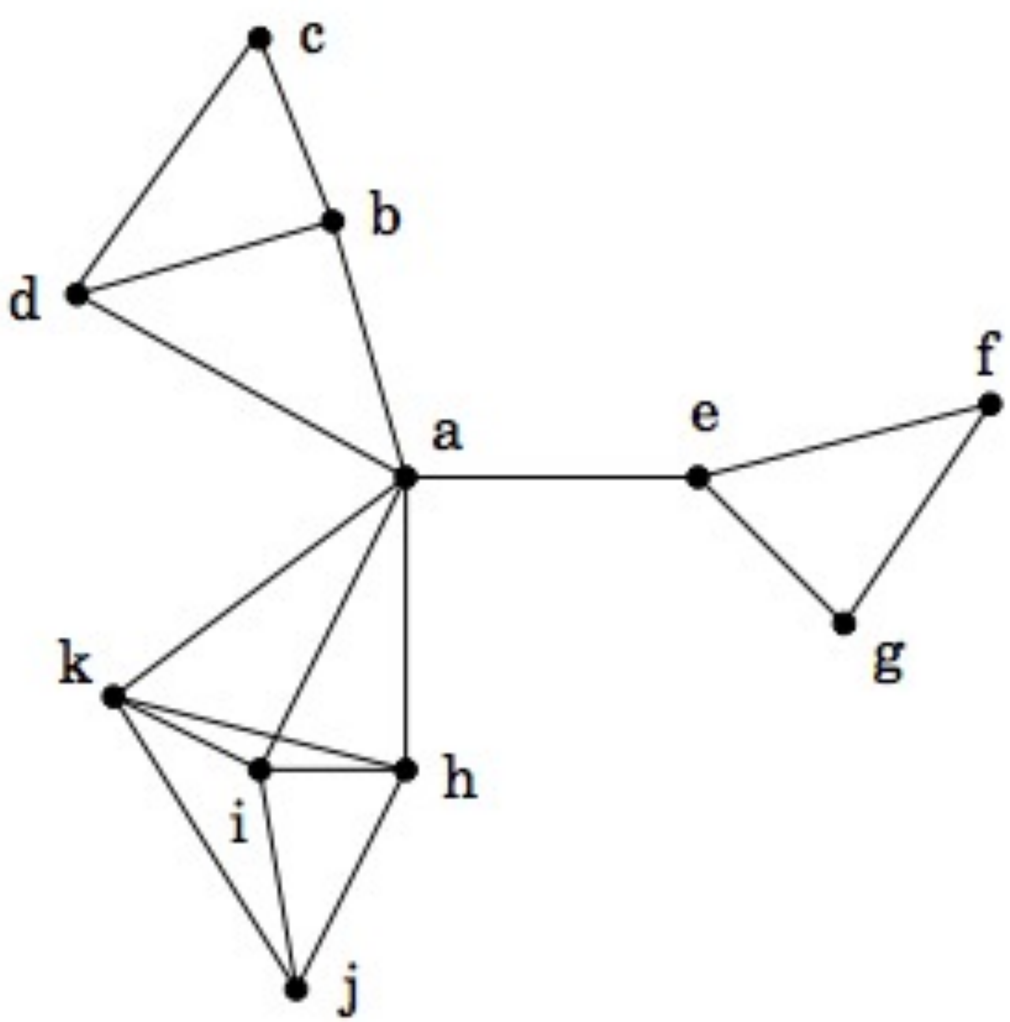
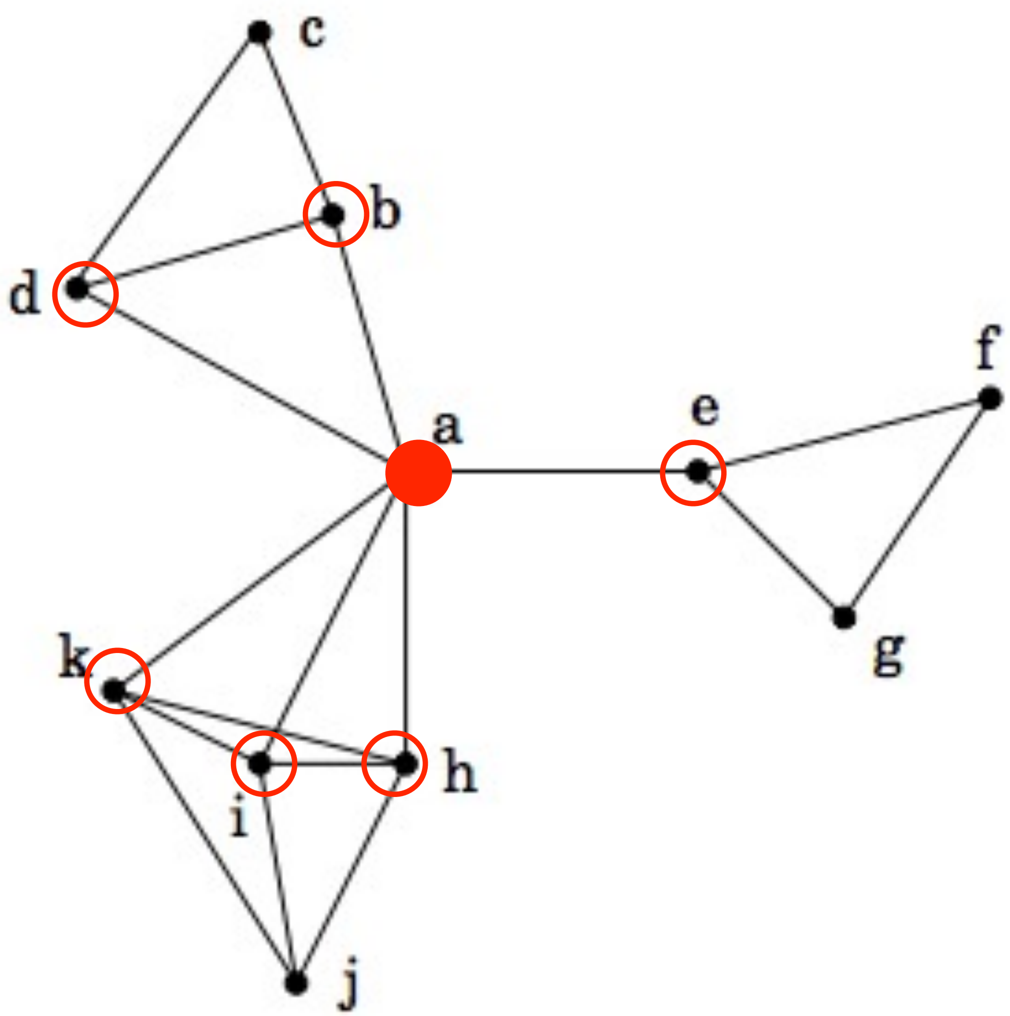
- S= {i,e,c} a smaller set cover
- TASK: find minimum size SetCover
  - NP complete
  - general version of Vertex Cover

Set Cover approx algorithm

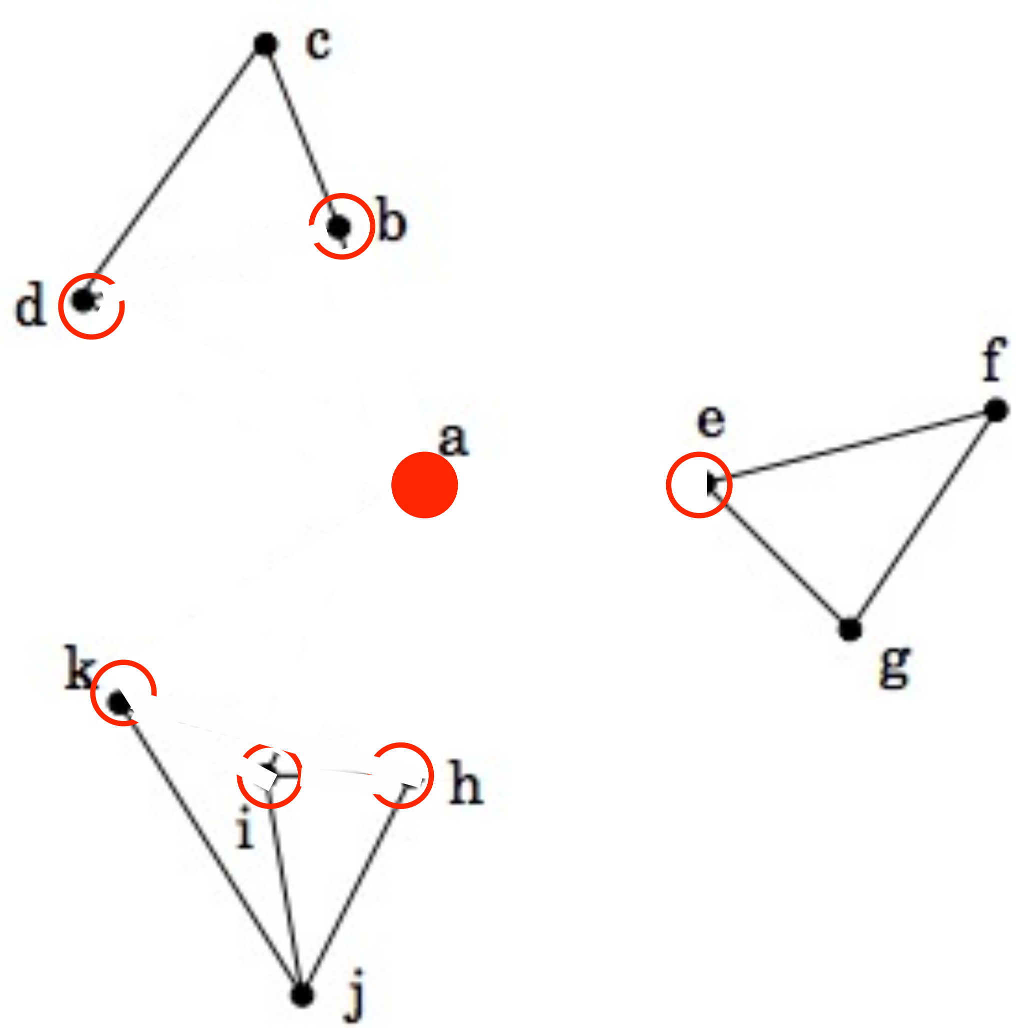


Set Cover approx algorithm



- pick the vertex with most connections/degree
  - deg(a)=6
  - eliminate "a" and all "a"-neighbors

Set Cover approx algorithm



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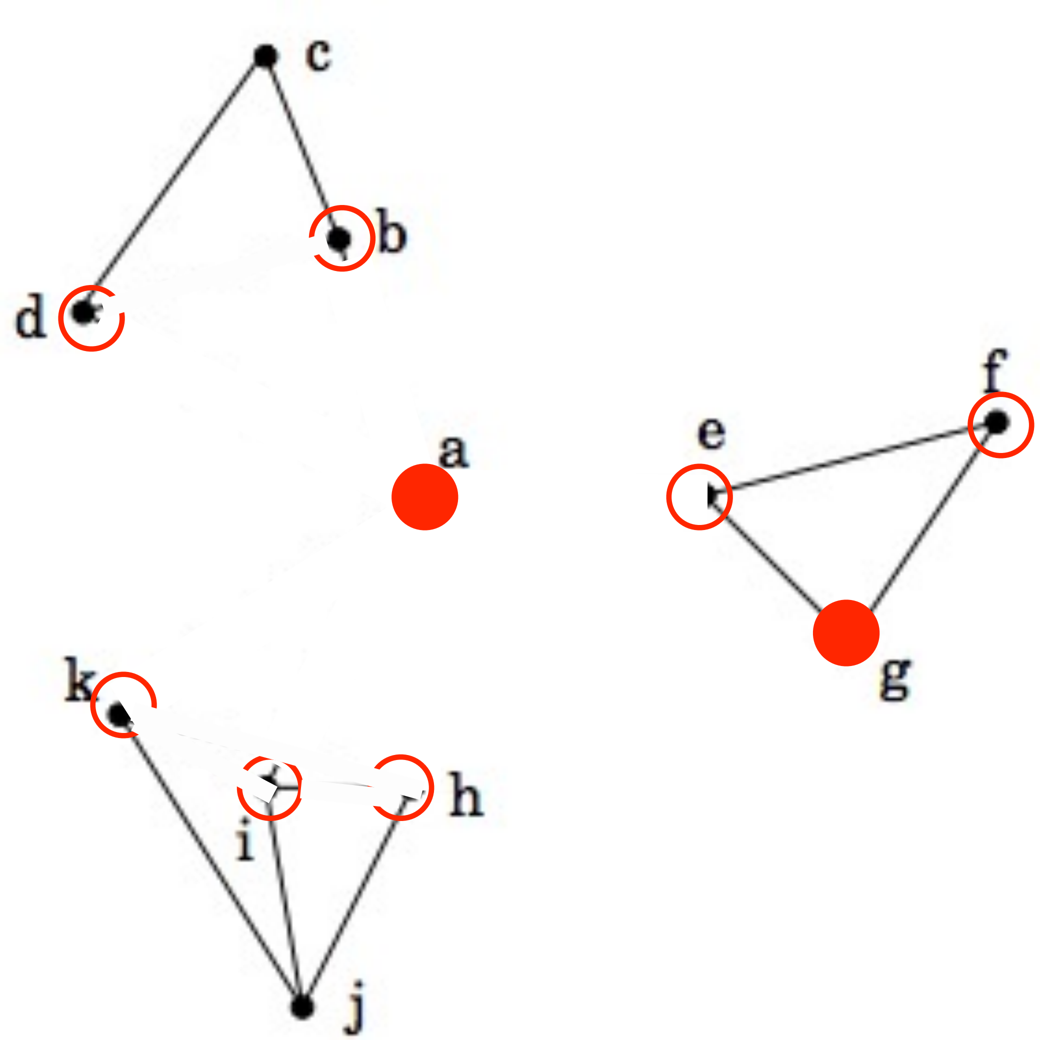

- pick the vertex with most connections/degree
  - deg(a)=6
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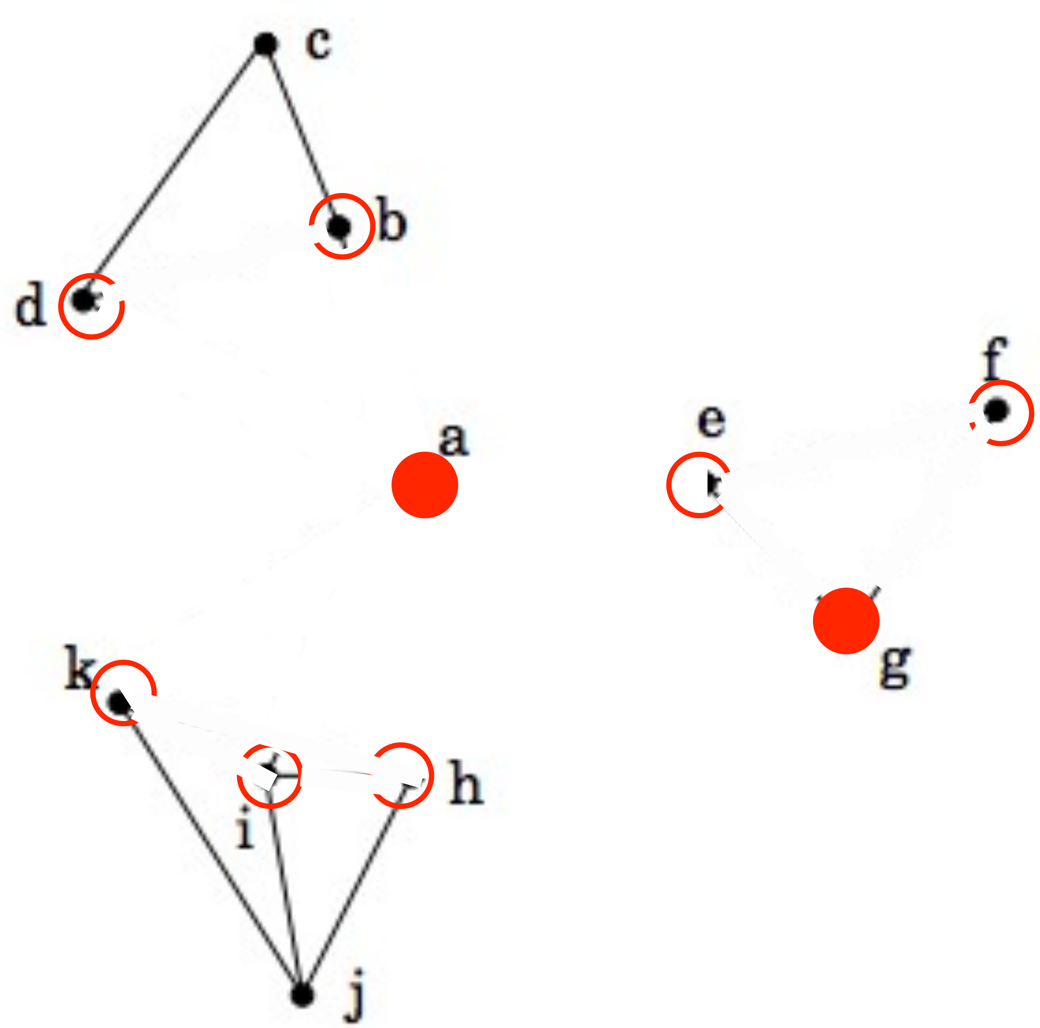
- pick the next vertex with most connections to uncovered towns
  - deg_now(g)=1
  - eliminate g and g-neighbors

Set Cover approx algorithm



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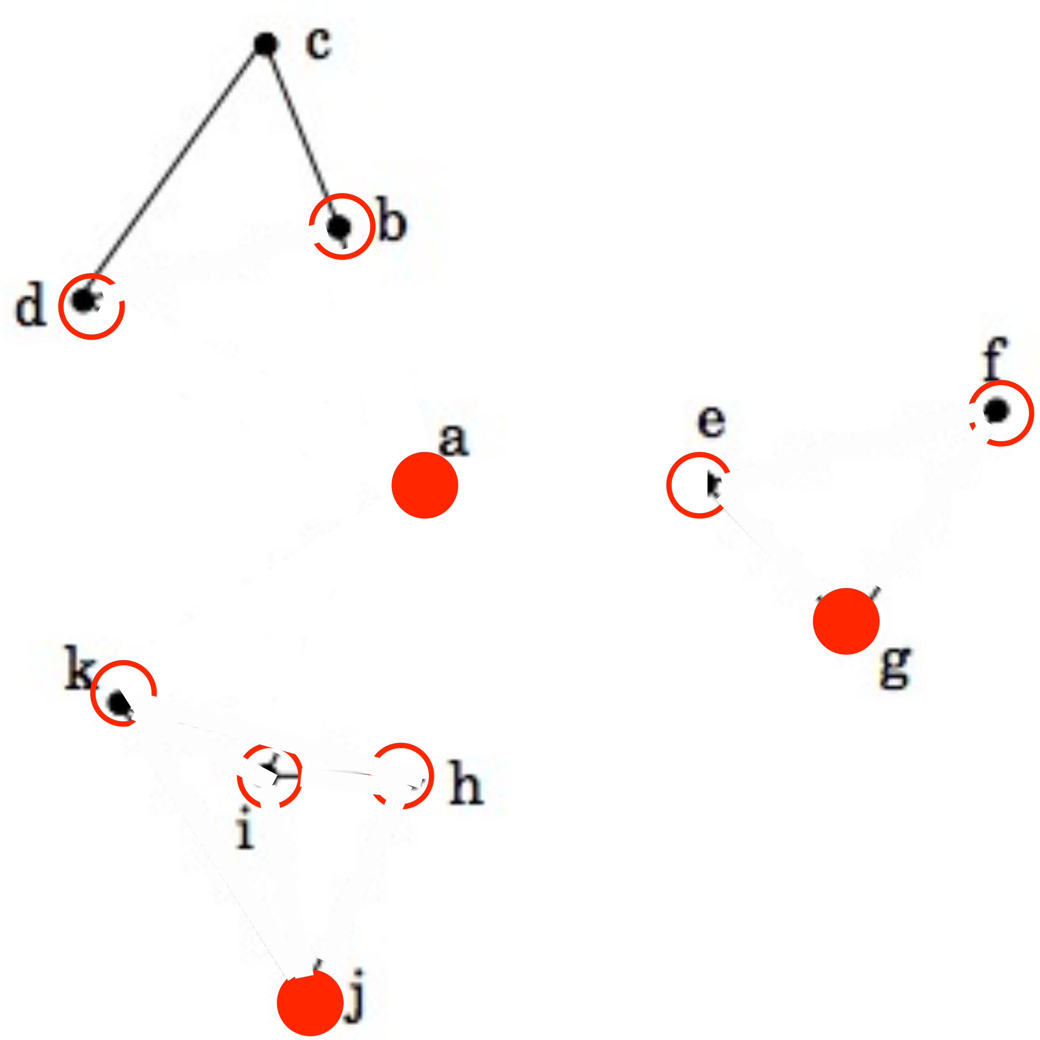

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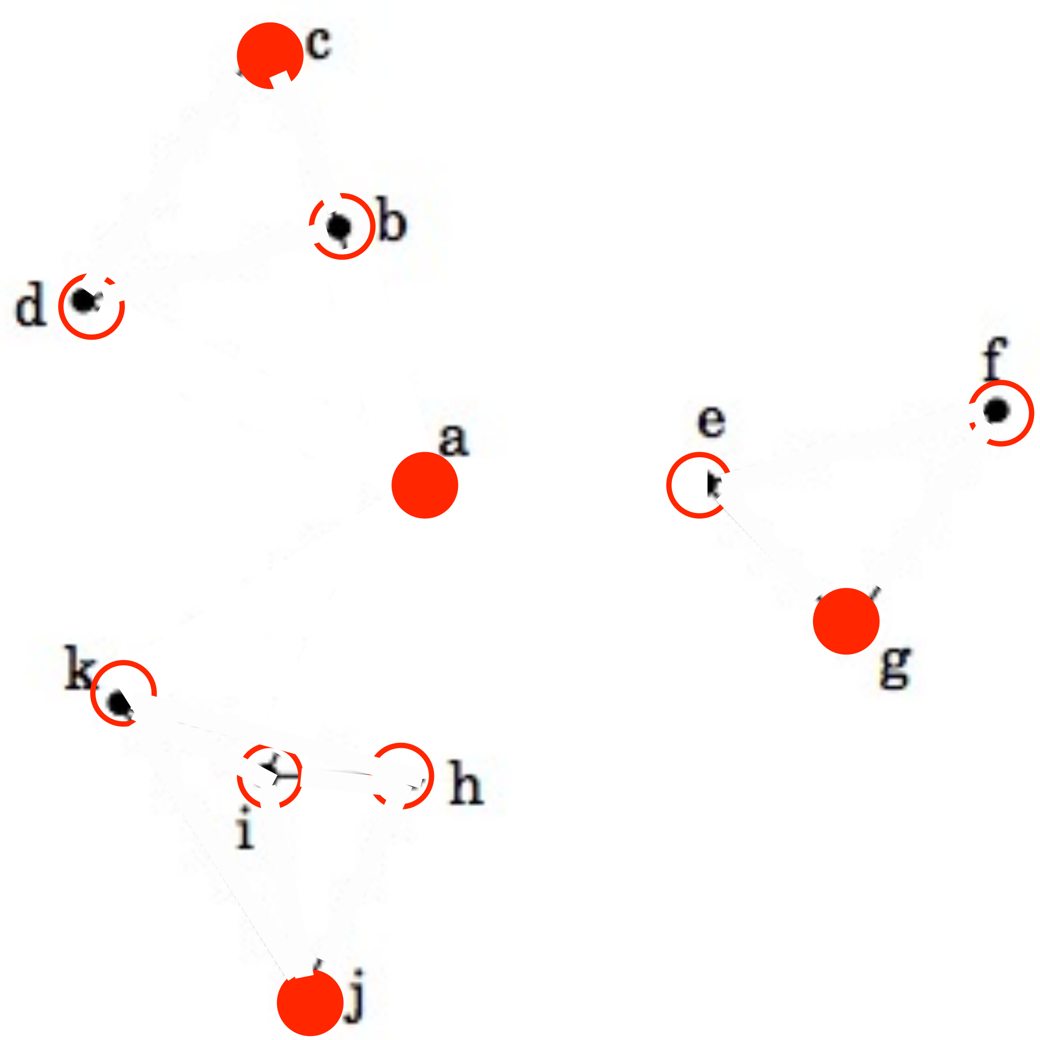

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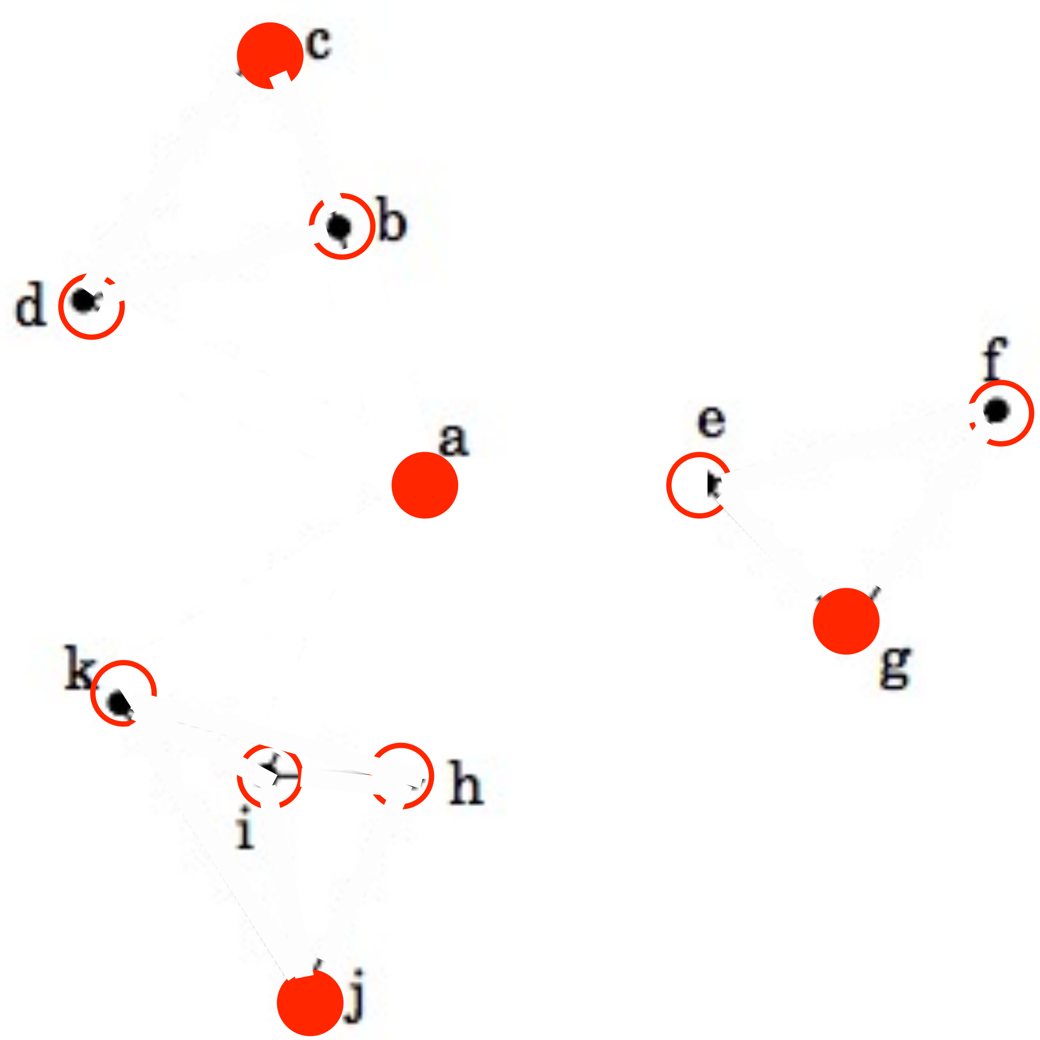

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- pick the next vertex with most connections to uncovered towns
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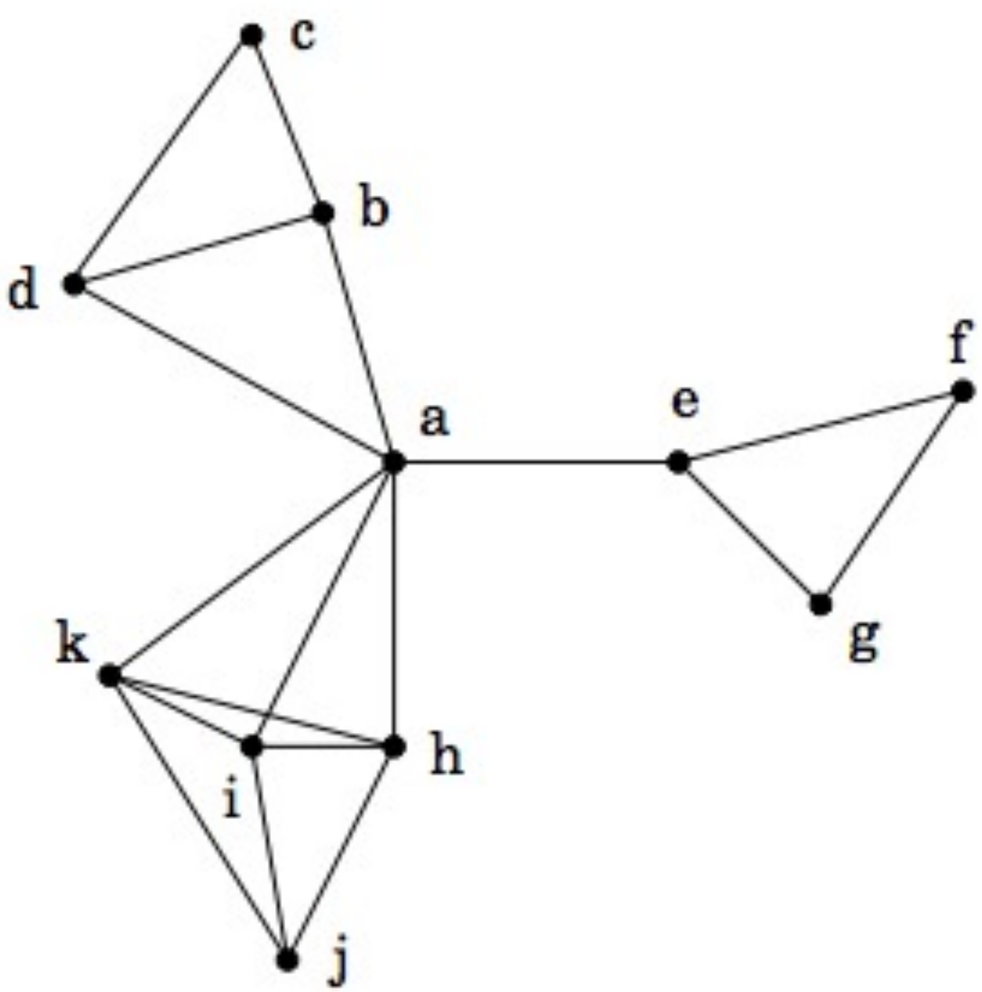
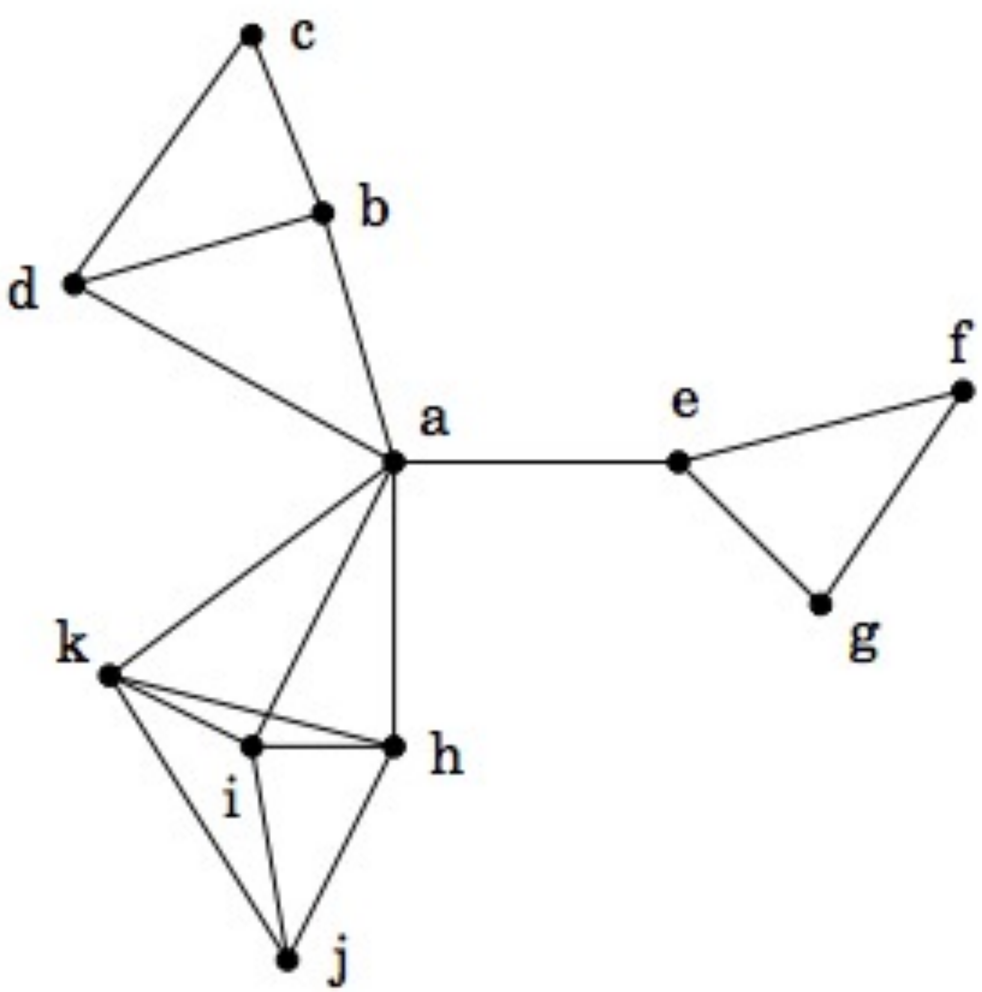
  - eliminate g and g-neighbors
- repeat for j then for c
- VertexCover = {a,g,j,c}, size 4

Set Cover approx algorithm



- SetCover_approx = {a,j,c,g}, size 4
- SetCover_optimal = {b,i,e}, size 3

Set Cover approx algorithm



- SetCover_approx = {a,j,c,g}, size 4
- SetCover_optimal = {b,i,e}, size 3

- Theorem:

  size(SetCover_greedy)≤ size(SetCover_optim)* log(|V|)

- approx ratio is log(n)

CLIQUE approximation

- much harder to approximate CLIQUE than VECTOR-COVER
- see wikipedia CLIQUE page
  - http://en.wikipedia.org/wiki/Clique_problem#Approximation_algorithms
- there can be no polynomial time algorithm that approximates the maximum clique to within a factor better than $O(n^{1-\varepsilon})$, for any $\varepsilon > 0$

3SAT approximation algorithm

- simple algorithm: assign each literal to TRUE or FALSE randomly, independently
- success: for any 3SAT clause $(a \vee b \vee c)$ the probability of evaluating FALSE is computed as the probability of all three literals to be FALSE
  - $p[(a \vee b \vee c) = FALSE] = 1/2 * 1/2 * 1/2 = 1/8$
- we can expect about 7/8 of the clauses to be satisfied and 1/8 to be not satisfied
- approx rate (expected) 8/7

SUBSET-SUM problem

- Given a set of positive integers S={a1,a2,..,an} and an integer size T
  - Task: find a subset of numbers from S that sum to t
- Idea: while traversing the array, keep a list with all partial sums
  - index 0: L_0={0}
  - index 1: L_1= {0, a1}
  - index 2: L_2= {0, a1, a2, a1+a2}
  - index 3: L_3= {0, a1, a2, a3, a1+a2, a1+a3, a2+a3, a1+a2+a3}
- at index n, verify if T is in the final list

SUBSET SUM exact algorithm

EXACT-SUBSET-SUM(S, t)

$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ **to** n
    $L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$
    remove from L_i every element that is greater than t
return the largest element in L_n

- exponential running time !
  - because the list L_i size can become exponential
- exercise: compare with DP solution based on discrete Knapsack

SUBSET SUM approx algorithm

APPROX-SUBSET-SUM(S, t, ϵ)

$n = |S|$
$L_0 = \langle 0 \rangle$
for $i = 1$ **to** n
    $L_i = \text{MERGE-LISTS}(L_{i-1}, L_{i-1} + x_i)$
    $L_i = \text{TRIM}(L_i, \epsilon/2n)$
    remove from L_i every element that is greater than t
let z^* be the largest value in L_n
return z^*

- TRIM(L, ε /2n) truncates long lists to avoid exponential list size
  - values truncated are closely approximated by the values staying in the list
- (1+ ε) approximation rate, for a given ε
- ε is a parameter of the TRIM function