Graphs III - Network Flow

Flow network setup

- graph $G=(V,E)$
- edge capacity $w(u,v) \geq 0$
  - if edge does not exist, then $w(u,v)=0$
- special vertices: source vertex s; sink vertex t
  - no edges into s and no edges out of t
- Assume every vertex v is on a path from source to sink ($s \rightarrow v \rightarrow t$)
  - vertices v that are not on such path can be ignored (deleted) along with all connecting edges

Flow network

- flow is a function f:VxV->R such that
  - flow from u to v: f(u,v) ≤ w(u,v)
  - symmetry f(u,v) = -f(v,u)
  - flow is conserved on all nodes except source s and sink t

$$\sum_{v \in V} f(u,v) = 0$$

- total flow (from the sorce)

$$|f| = \sum_{v \in V} f(s,v)$$

Maximum Flow Problem

- determine the flow f that realizes the maximum total flow

More on Flows

- f(u,u)=0
- total net flow into/out-to a vertex is 0

$$\sum_{v \in V} f(v,u) = 0$$

  - except for source s and sink t

- if edge (u,v) is missing in G, there can be no net flow from u to v

More on Flows

- positive net flow entering v

$$\sum_{u \in V; f(u,v) > 0} f(u, v)$$

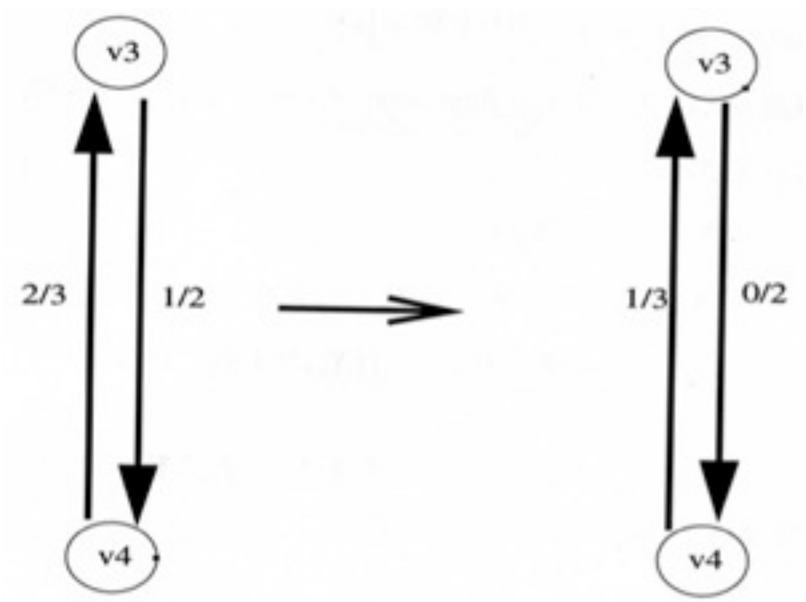
- positive net flow leaving v

$$\sum_{u \in V; f(v,u) > 0} f(v, u)$$

- these two are equal

Cancellation

- positive flow on (u,v) cancels positive flow on (v,u) until only one is positive (the other becomes 0)
  - both flows decrease, so they still satisfy capacity constraints
  - flow conservation satisfied since both flow reduced by the same amount



Ford-Fulkerson

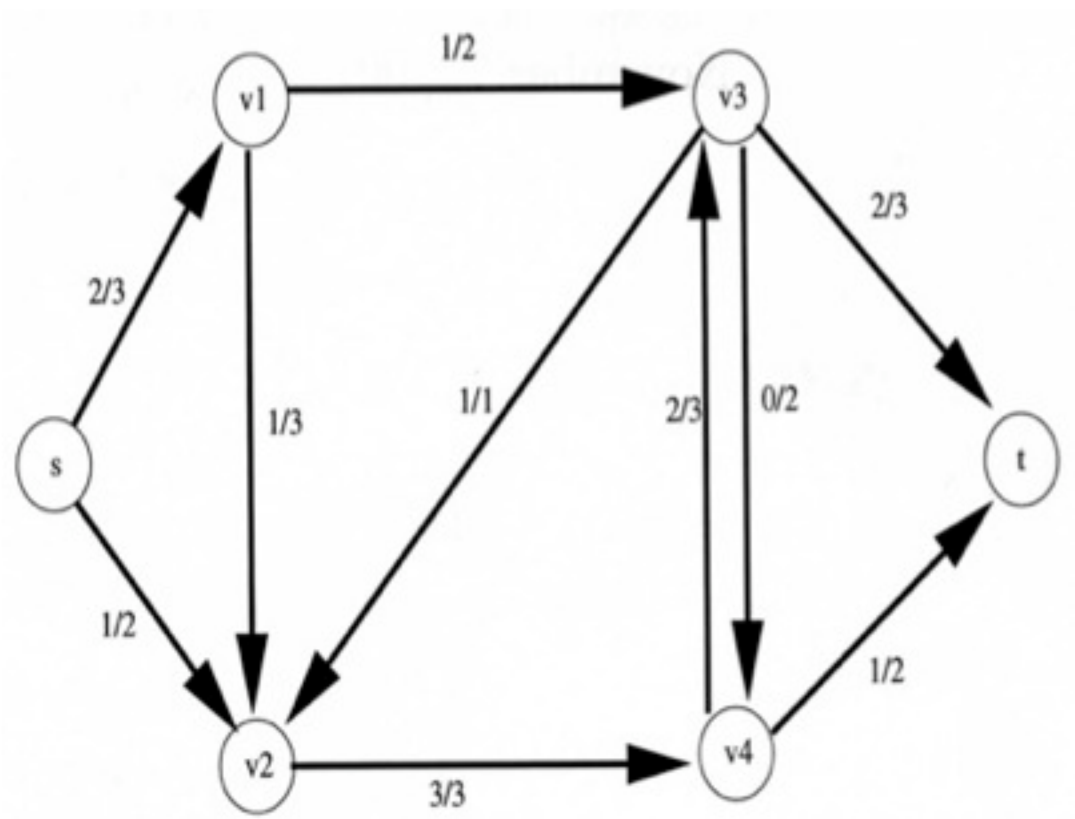
- want the max flow for source s to sink t
  - a class of algorithms, not a single one

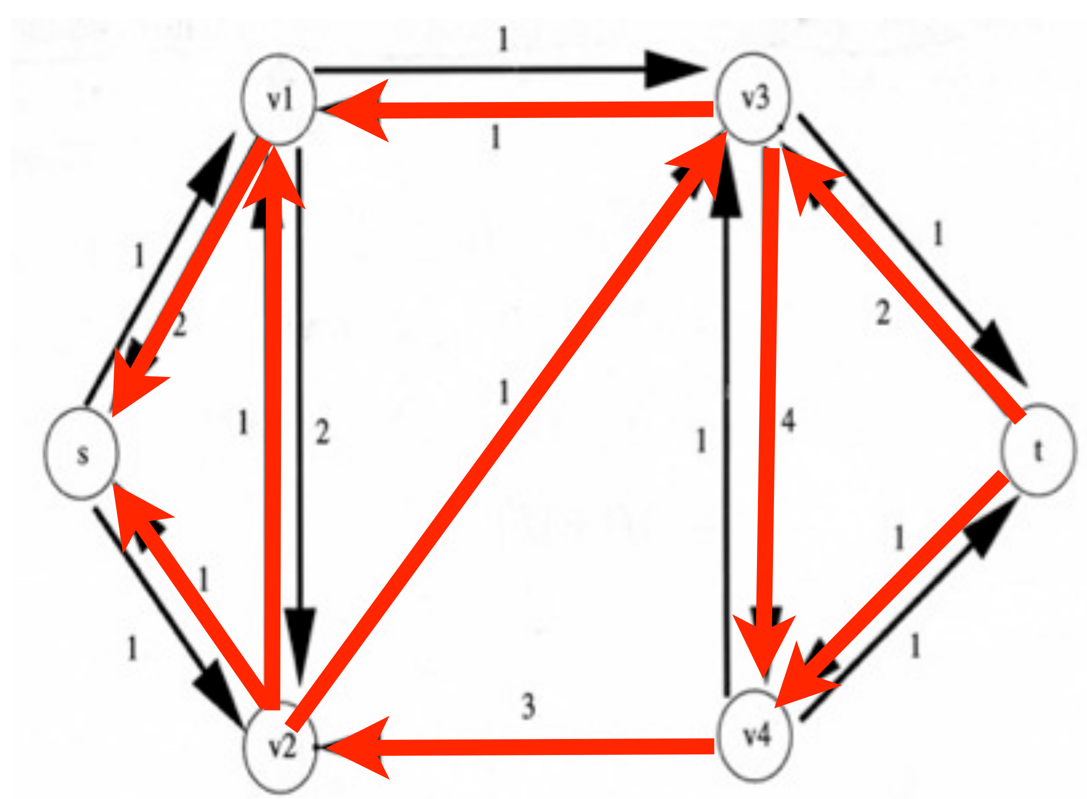
```
▸ initialize flow with 0;
▸ repeat
    ▸ find an augmenting path from s to t (that admits
      more flow)
    ▸ send more flow on that path
▸ until no augmenting path exists
```

- have to prove that this termination condition implies the flow is max.
  - if an augmenting path exists, sending more flow to it increases the value of the existing flow

Residual network

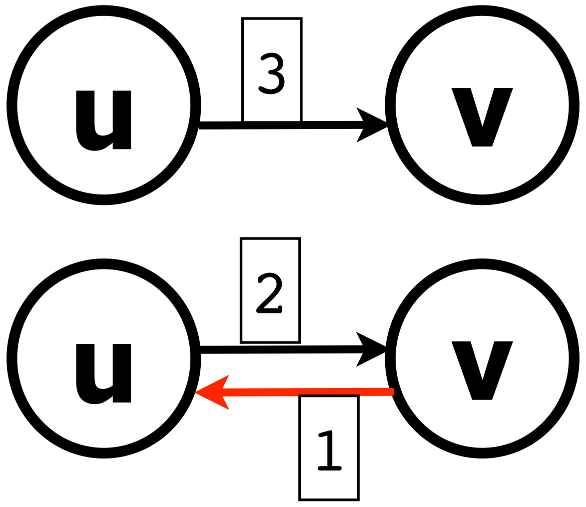
- after sending some flow on a path from s to t, the graph essentially changes
  - existing flow edges will have a different capacity in the residual network (because the flow uses some)
  - new edges can appear (in red) : the possibilities of reversing the existing flow





Residual network

- residual capacity of edge (u,v) : r(u,v) = w(u,v)−f(u,v)
- residual network "R" induced by f is given by the set of edges also called "R" with positive residual capacity
  - edges set R = { (u,v) : r(u,v)>0 }
- note that some new edges appear !
  - example (u,v)∈E; w(u,v)=3, f(u,v)=1
  - then r(u,v) = 3−1 = 2
  - edge (v,u) not in the original graph
  - but r(v,u) = 0 − f(v,u) = 0− (−1)=1; therefore edge (v,u) is now part of the residual network.



- edge (v,u) can be part of the residual newtwork only if either (u,v) or (v,u) are edges in the original graph
  - thus |R| ⩽ 2|E|

Augmenting paths

- any path $p = s \to t$ in the residual network R
- the residual capacity of the path p is the minimum ("bottleneck") edge residual capacity
  - $r(p) = \min \{ r(u,v) : (u,v) \in p \}$
- add the path p as additional flow f_p of size r(p)
  - to the existing flow f that created R
  - new flow $f' = f + f_p$
  - increases the flow total by r(p). Proof in the book.

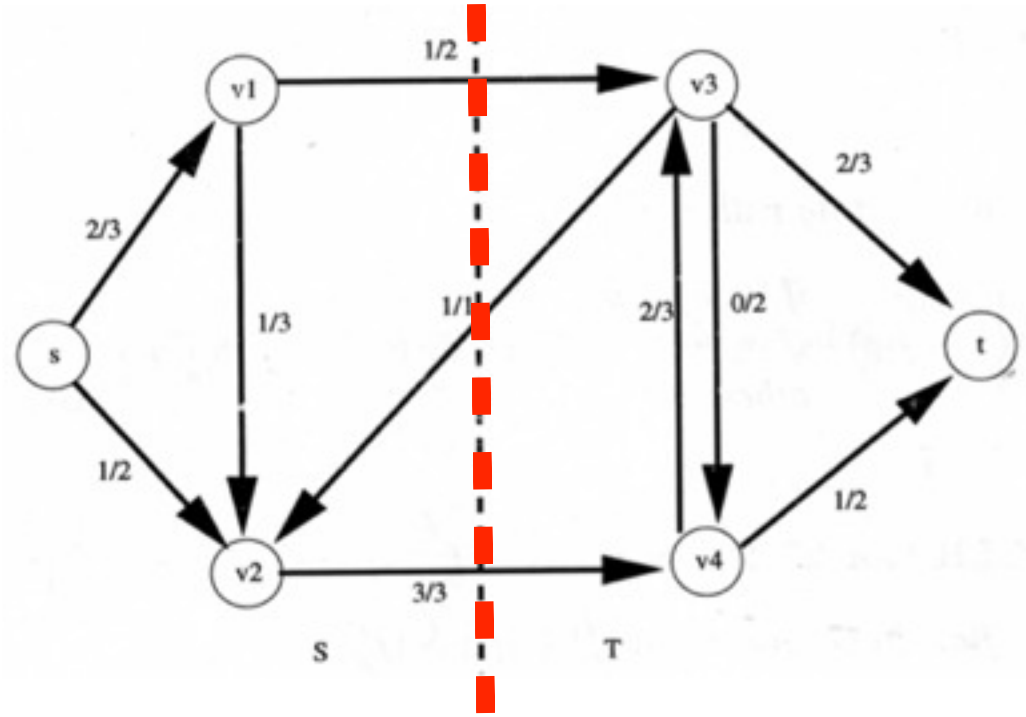
Cuts in flow network

- Cut C = (S,T) is a partition of vertices
  - $S \cup T = V$; $S \cap T = \emptyset$; $S = V \setminus T$
  - S contains the source and T contains the sink $s \in S$; $t \in T$
- Net flow across cut is f(S,T), the sum of all flows on edges from S to T

$$f(S,T) = \sum_{u \in S; v \in T} f(u,v)$$

- capacity of a cut is the sum of edges capacity from S to T

$$w(S,T) = \sum_{u \in S; v \in T} w(u,v)$$



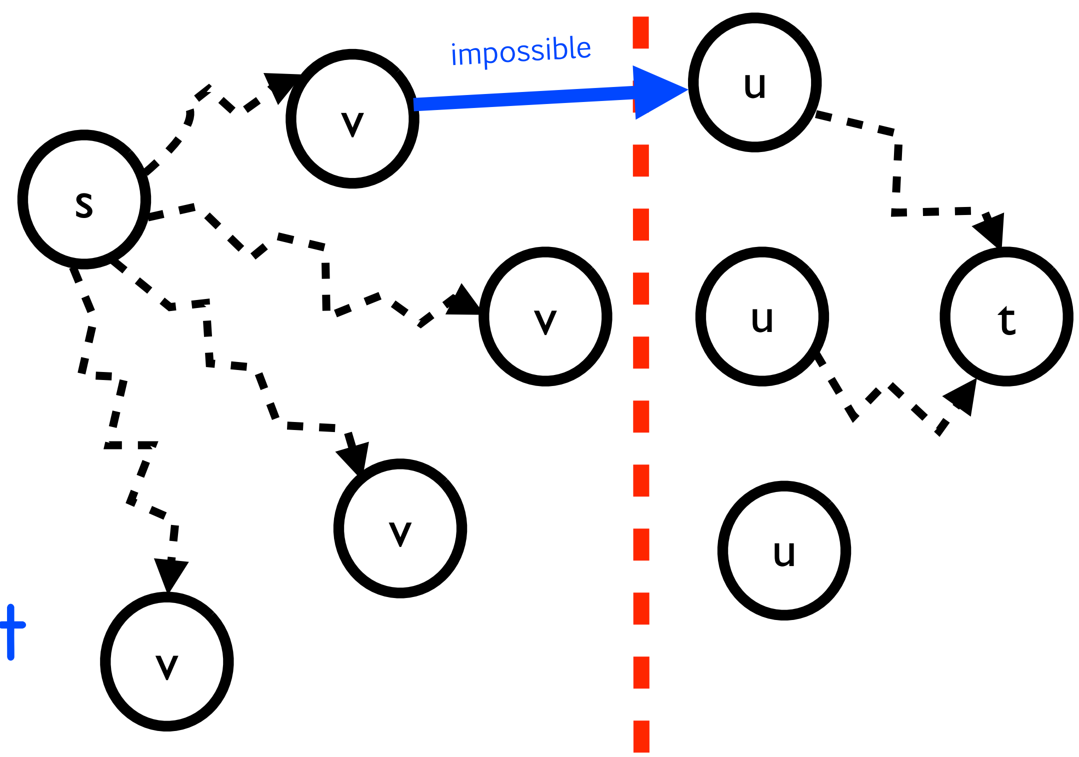
f(S,T)=3 ; w(S,T)=5

Max Flow – Min Cut theorem

- (S,T) is a cut, f a flow with total value |f|. Then
  - f(S,T) = |f| (the total flow value)
  - consequently $|f| \le w(S,T)$: flow value is smaller than the capacity of any cut
- MAX-FLOW MIN-CUT theorem . The following are equivalent:
  - (a) f is a max flow
  - (b) residual network $R=R_f$ has no augmenting paths
  - (c) there is a cut (S,T) such that $|f| = w(S,T)$

Max Flow – Min Cut proof intuition

- (a)=>(b) already discussed
- (b)=>(c): consider S = { v | ∃ path s⇝v in residual R}
  - s∈S
  - T =V∖S; t∈T. If t∈S, then there would be a augmenting path in R
  - R cant have an edge (v∈S, u∈T) because that would mean u∈S
  - thus existing flow saturates the cut (S,T)



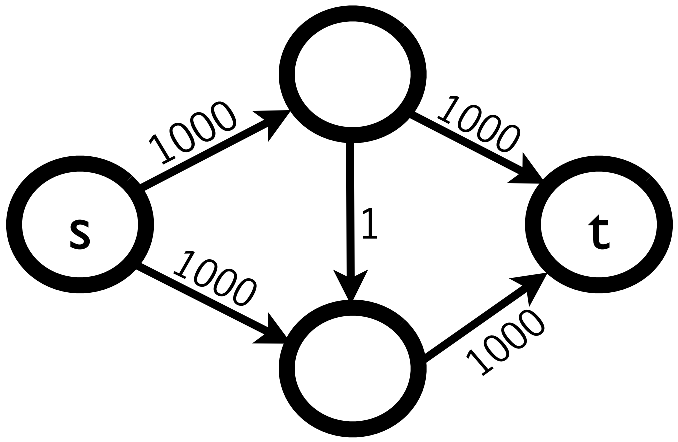
- (c)=>(a): no flow can be bigger than capacity of a cut, so f must be a maximum flow (since it saturates the cut described above)

Ford-Fulkerson

- for each edge (u,v)
  - init: f(u,v)=0; f(v,u)=0
- R = G
- while exists path p(s⇝t)in residual R
  - c(p) = min { r(u,v); (u,v) ∈ p }//*path capacity, used as new flow*
  - for each (u,v)∈p
    - f(u,v) = f(u,v) + c(p) ; f(v,u)= - f(u,v)
  - recompute residual network $R=R_f$

Ford-Fulkerson

- Running time with integer capacities
  - fing a path in R is O(E) (say with DFS)
  - |f|=total flow value
  - at most |f| iterations; every iteration increases the flow by 1 or more
  - total O(E*|f|)
- problem: |f| can be very large, thus the algorithm very slow
  - for real-value edge cacpacities, Ford-Fulkerson can be arbitrary slow



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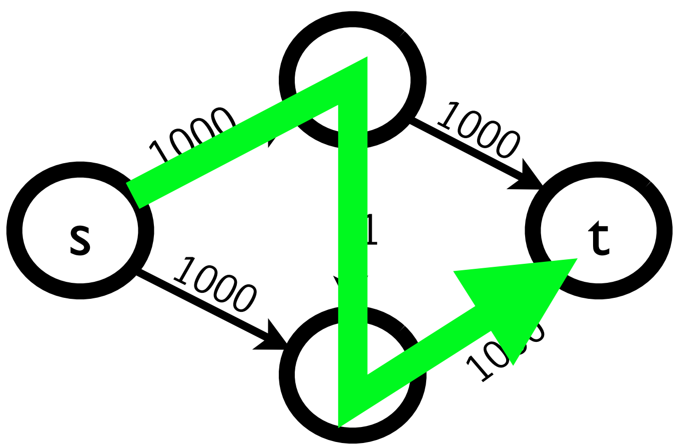

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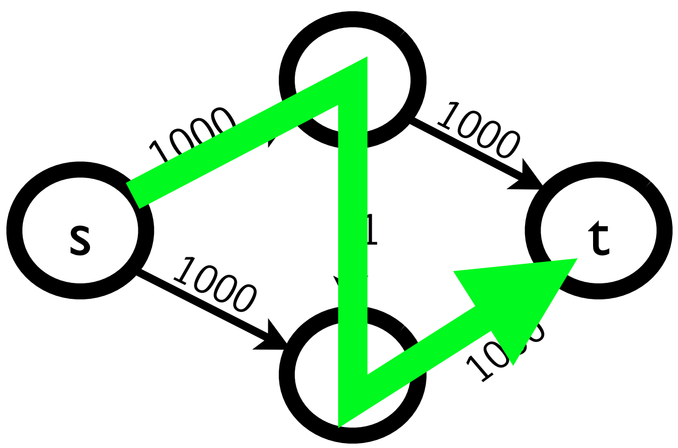

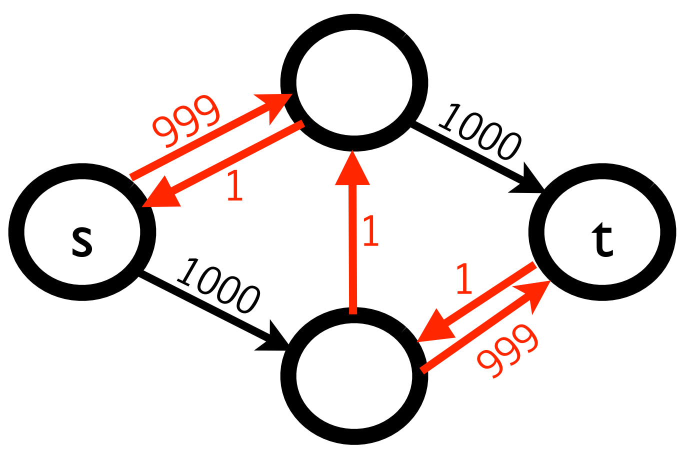

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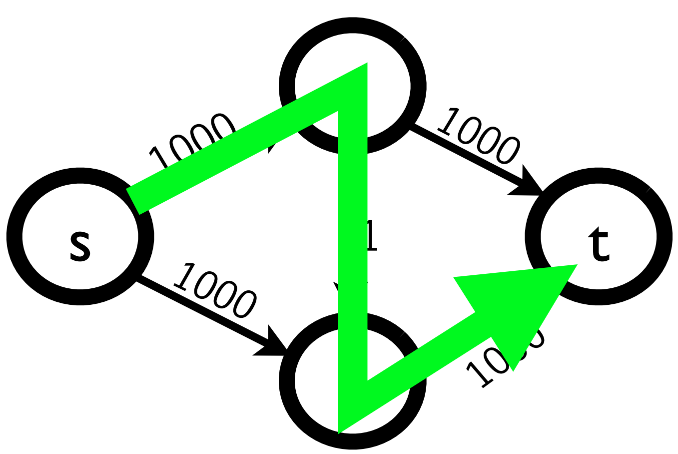

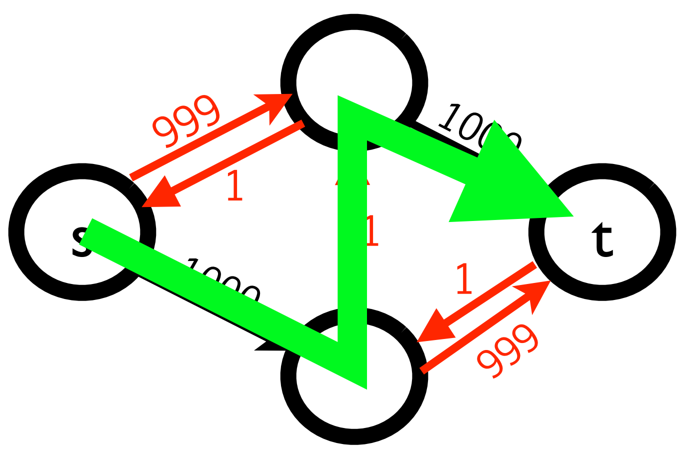

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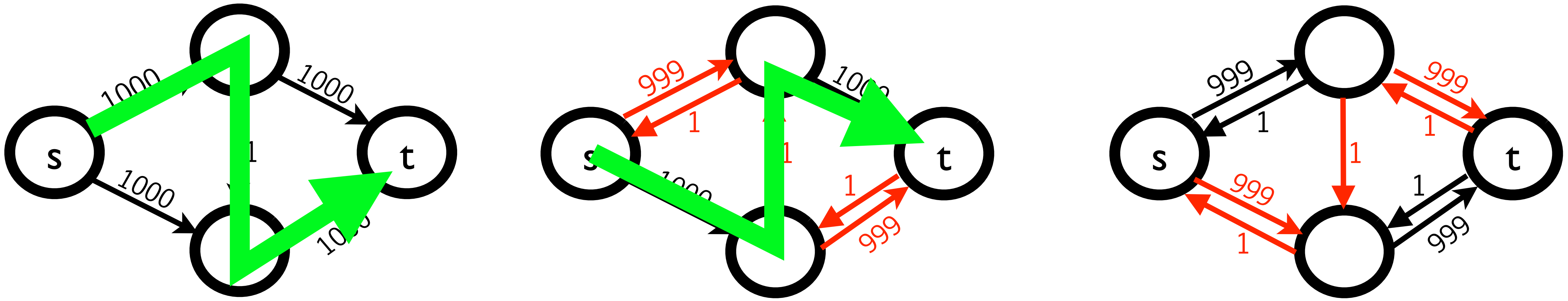

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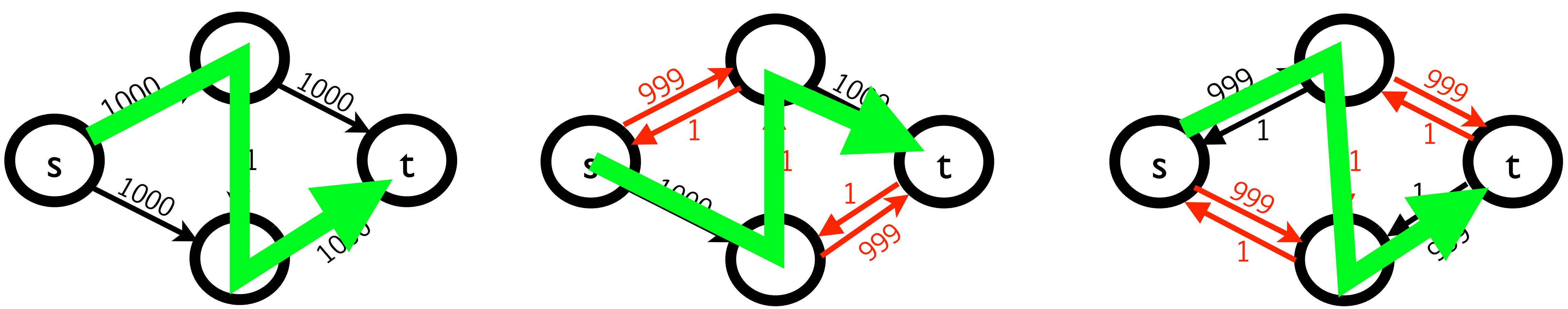

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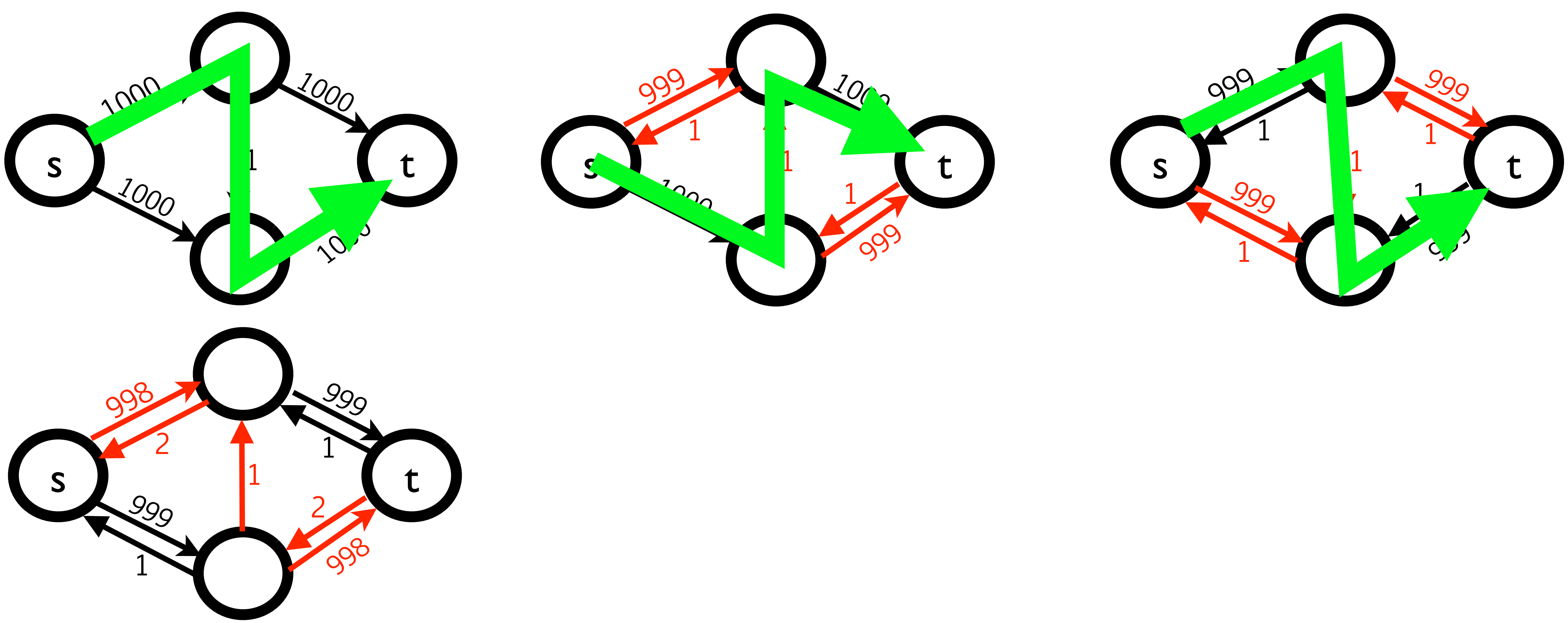

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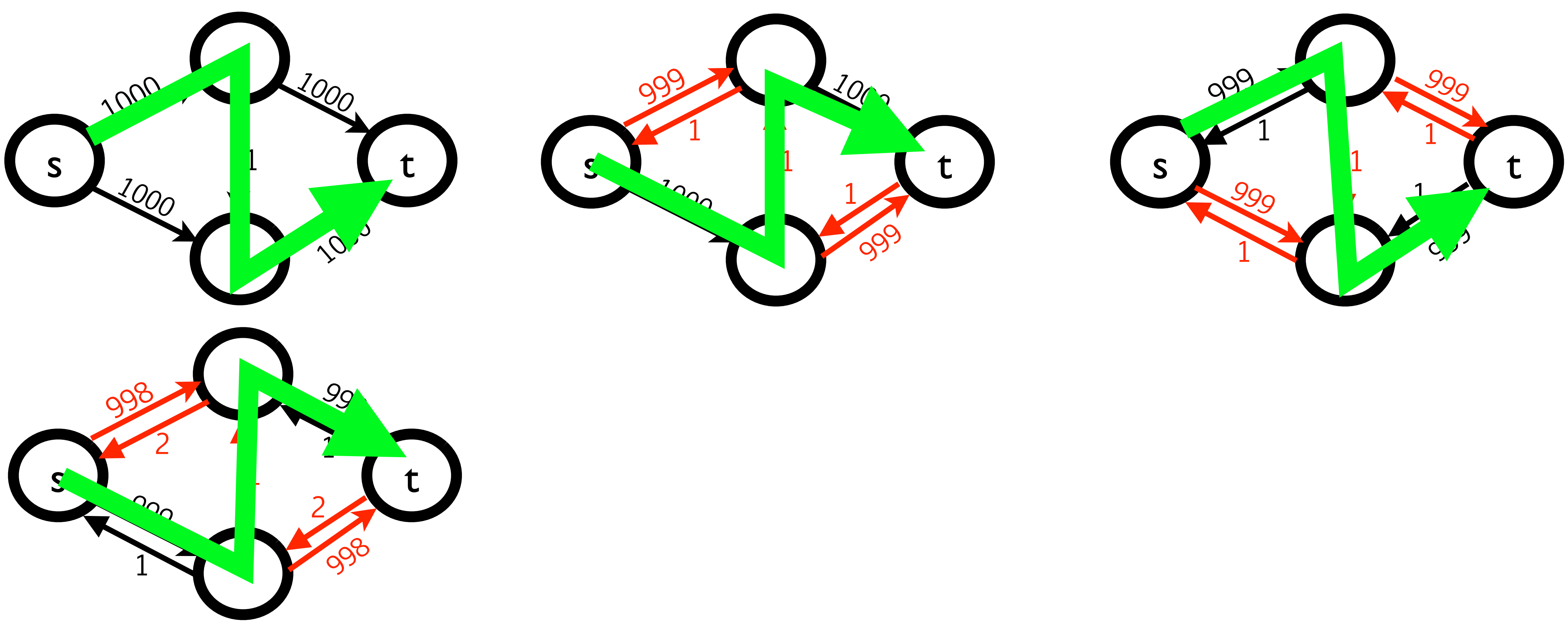

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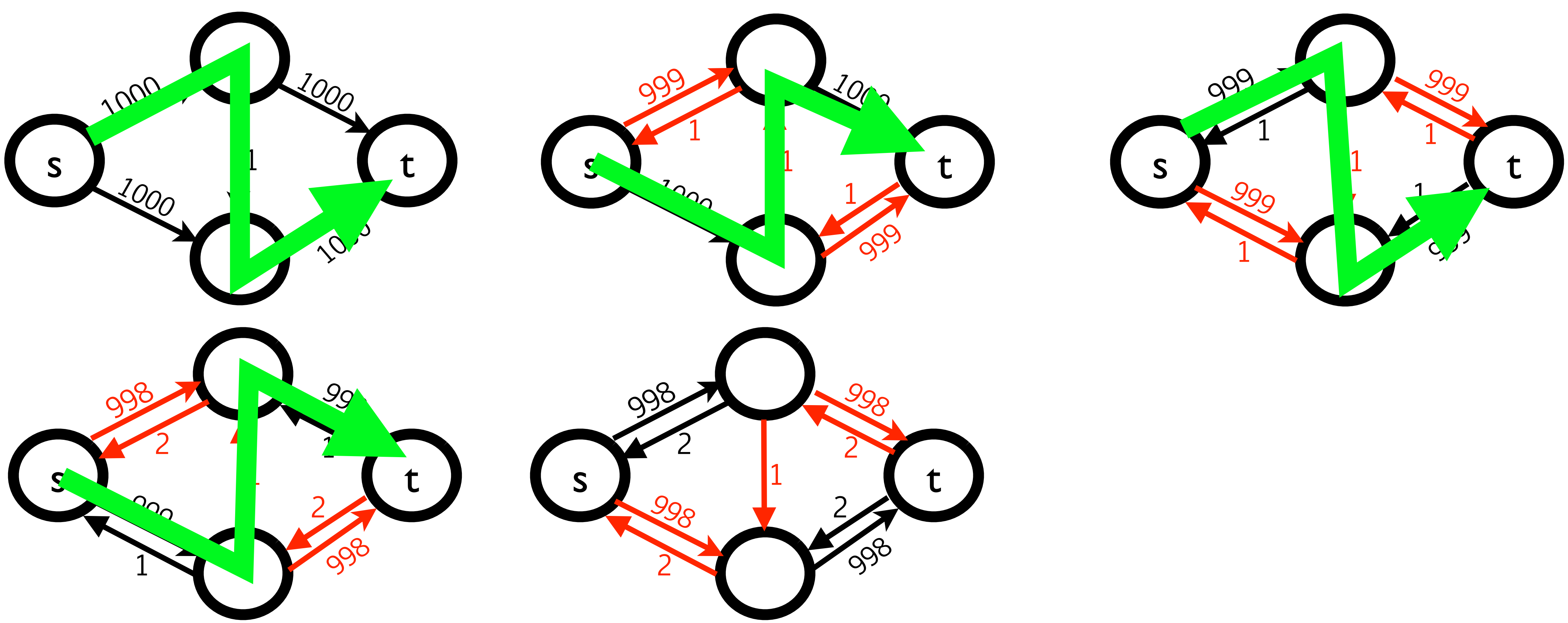

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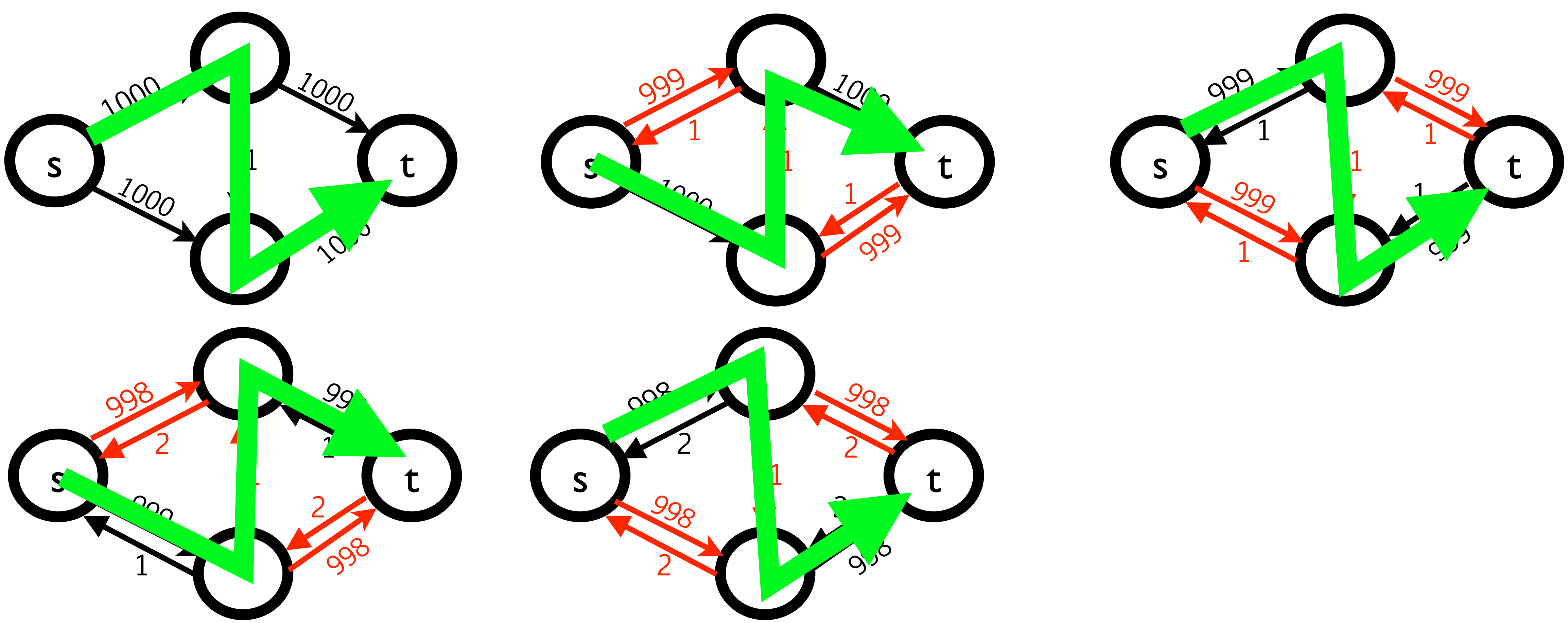

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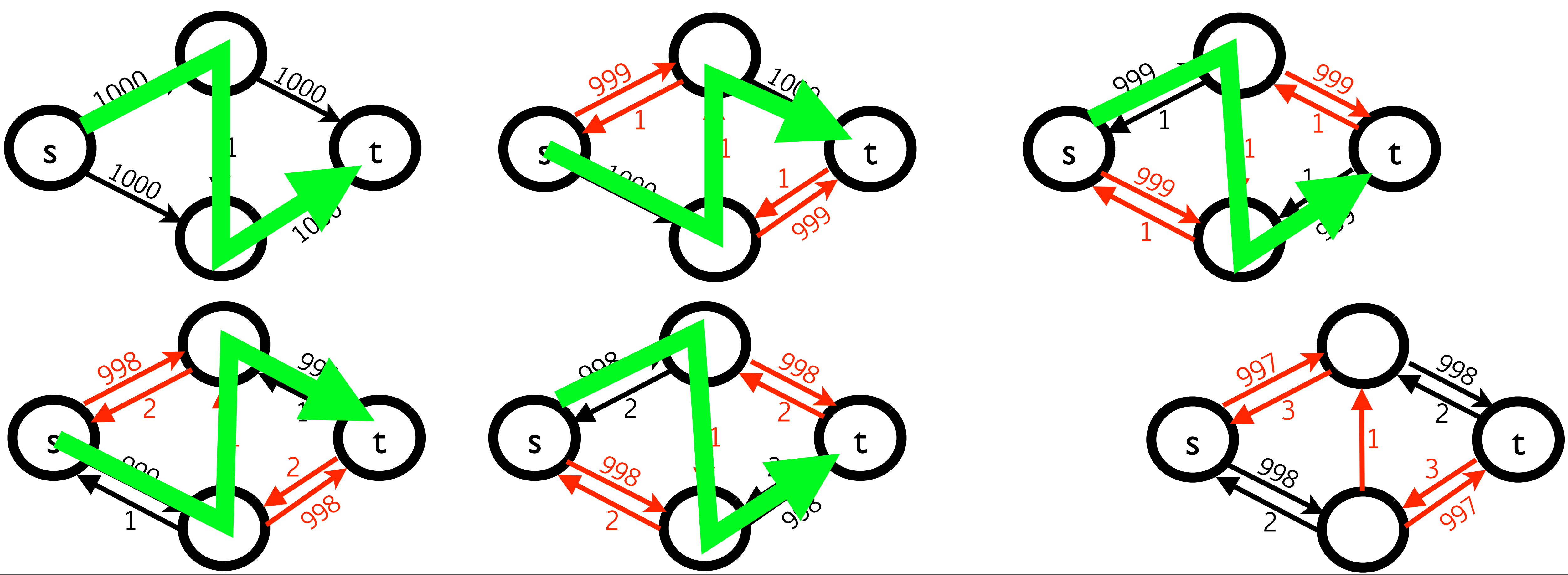

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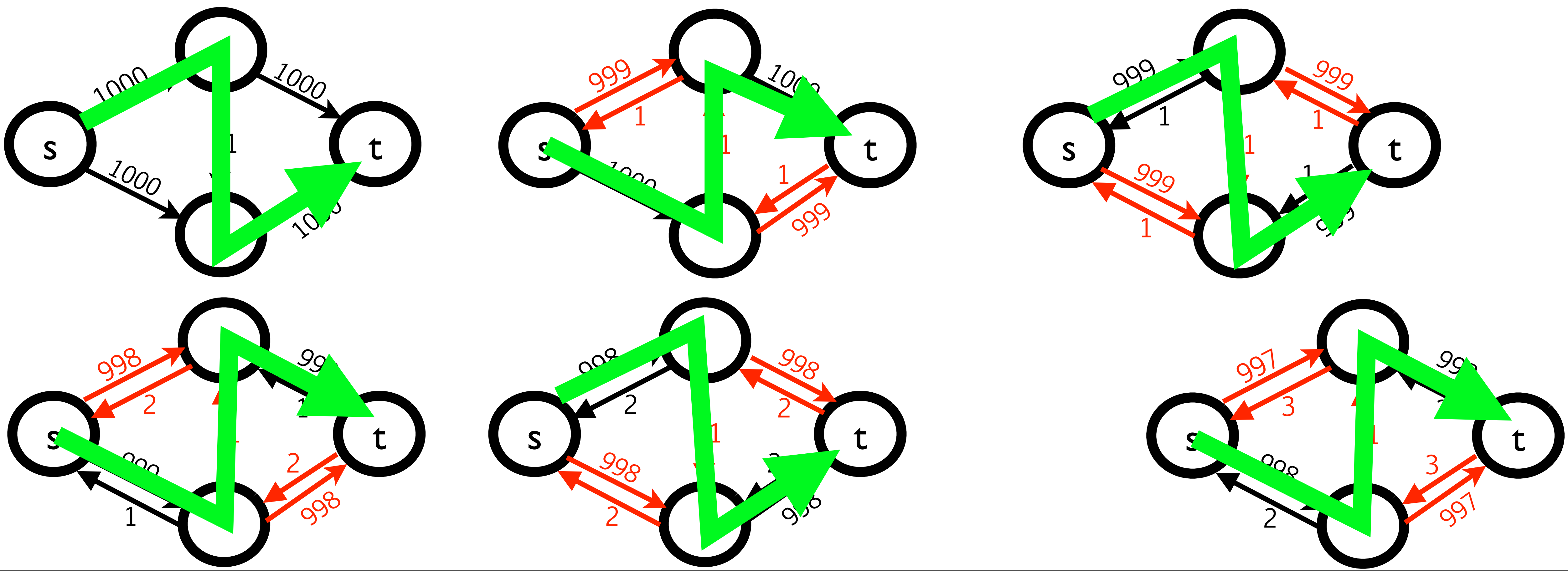


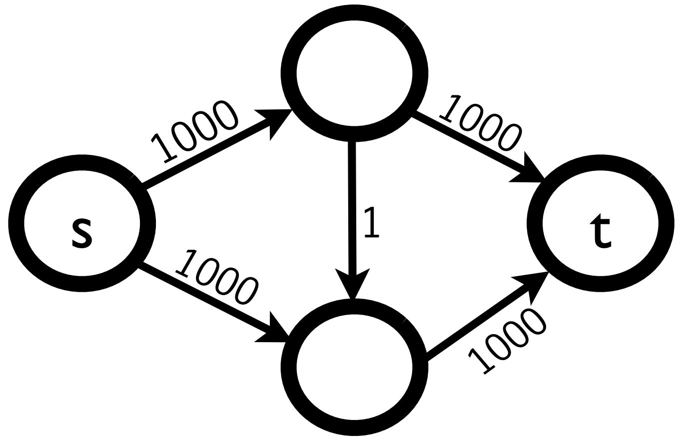
Edmonds-Karp

- same as FF, but find the augmenting path with BFS

```
for each edge (u,v)
    init: f(u,v)=0; f(v,u)=0
R = G
while BFS finds path p(s⇝t)in residual R
    c(p) = min { r(u,v); (u,v) ∈ p }//path capacity, used as new flow
    for each (u,v)∈p
        f(u,v) = f(u,v) + c(p) ; f(v,u)= - f(u,v)
    recompute residual network R=Rf
```

Analysis of Edmonds-Karp

- BFS will find the augmenting path with fewest number of edges
- note that previus toy bad example would find max flow after two iterations



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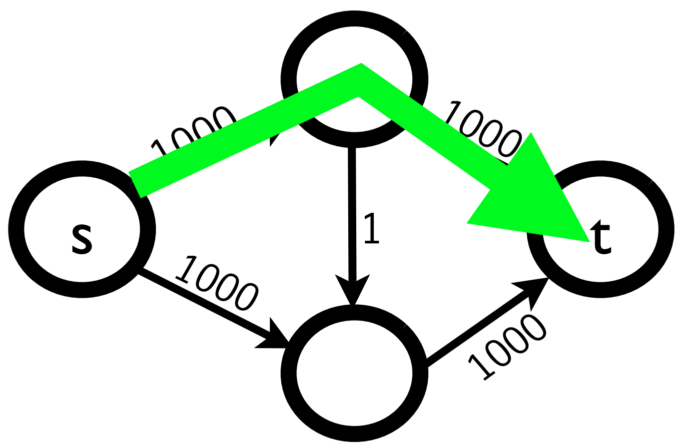

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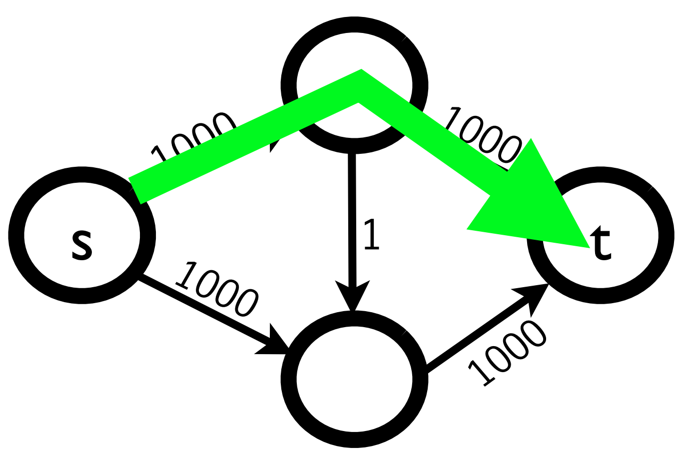

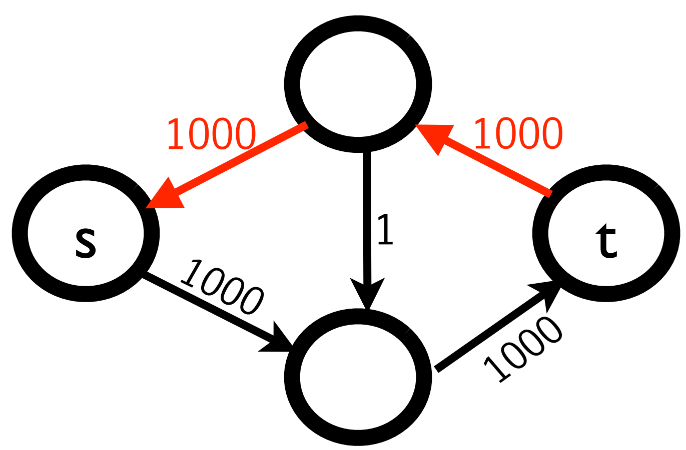

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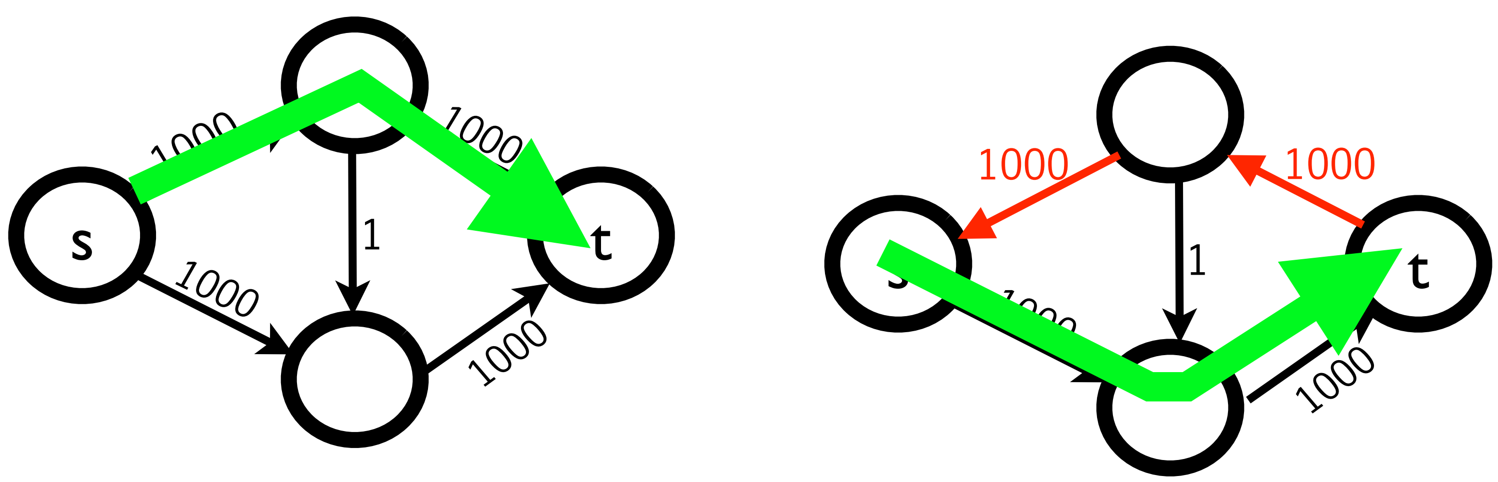

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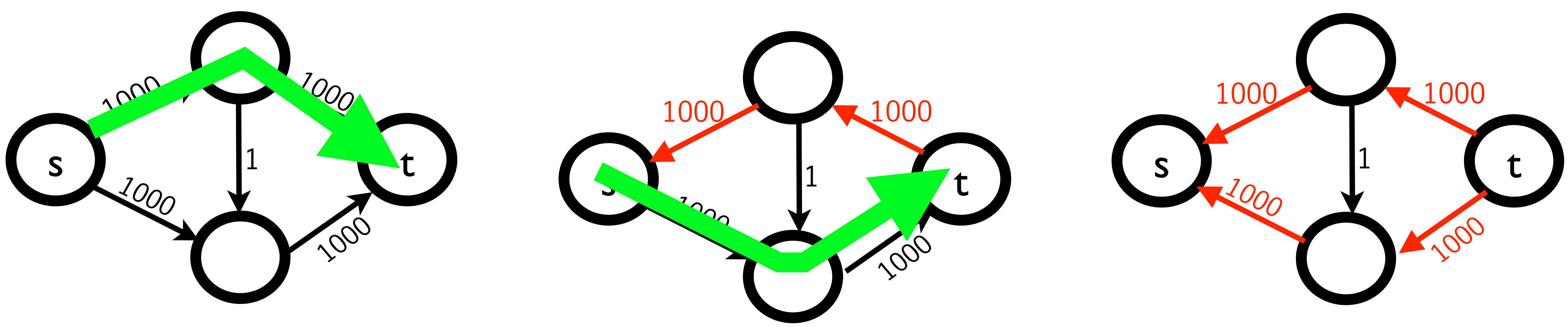

Analysis of Edmonds-Karp

- How many augmenting paths can EK find?
  - augmenting path p has critical edge (u,v), if (u,v) is the minimum residual capacity edge on the path
  - any edge can be critical at most |V| times during EK. Proof in the book
  - there are E edges, so at most |V|*|E| critical edges for the entire execution
  - thus at most O(VE) augmenting paths (each path has at least one critical edge)
- BFS takes O(E) to find each augmenting path
- total $O(VE^2)$

Push-Relabel (Optional reading)

- Advanced material not covered
  - optional reading from book
- intuition : flood the network, using vertex heights
  - nodes can accumulate flow
  - the more flow they accumulate, the "higher" they go
  - flow goes downhill
- practical / fast implementation: $O(V^3)$ running time.