

**Lecture Notes for
CS105 Algorithms**
Network Flow – Edmonds-Karp

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Edmonds-Karp Algorithm

So Ford-Fulkerson just says we should find repeatedly find augmenting paths and push flow along that path.

- So depending on how you go about finding augmenting paths, you'll get algorithms with potentially different running times (and may never terminate).
- The Edmonds-Karp algorithm says find the shortest augmenting path from s to t in the residual network using BFS.
↳ fewest # of edges

Same algorithm just make sure you take a shortest path in G_f when **edge weights are all 1**:

Algorithm 0.0: Edmonds-Karp(G, s, t).

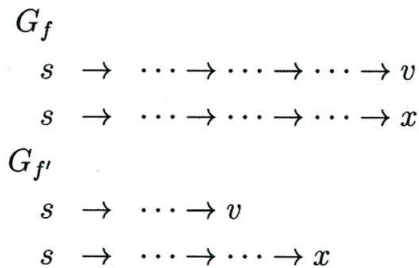
1. for each $(u, v) \in E$
2. do $f[u, v] \leftarrow 0$
3. $f[v, u] \leftarrow 0$
4. while BFS finds a shortest path p from s to t in G_f
↳ fewest # of edges
5. do $c_f(p) = \min\{c_f(u, v) : (u, v) \in p\}$
6. for each $(u, v) \in p$
7. do $f[u, v] \leftarrow f[u, v] + c_f(p)$
8. $f[v, u] \leftarrow -f[u, v]$
9. return f .

Definition 0.1 Let $\delta_f(s, v)$ be the number of edges on the shortest path from s to v in G_f .

Lemma 0.2 (CLR 27.8) Using Edmonds-Karp, $\delta_f(s, v)$ in G_f is nondecreasing with each flow augmentation.

Proof.

- Suppose there is a flow augmentation that caused the shortest path from s to at least one vertex to decrease.
- Let f be the flow before this augmentation
- Let f' be the flow after this augmentation
- Let $X = \{x : \delta_{f'}(s, x) < \delta_f(s, x)\}$.
- Let v be a vertex in X such that $\delta_{f'}(s, v) \leq \delta_{f'}(s, x)$ for all $x \in X$. (i.e. v is the vertex in X with the shortest path from s after the augmentation)



- Consider a shortest path from s to v in $G_{f'}$.
- Let u be the vertex preceding v on this path.

$G_{f'}$:

$$s \rightarrow \dots \rightarrow u \rightarrow v$$

- $u \notin X$, since u is closer to s than v in $G_{f'}$.
- So $\delta_f(s, u) \leq \delta_{f'}(s, u)$ since otherwise $u \in X$.

2 cases to consider:

1. $(u, v) \in E_f$
2. $(u, v) \notin E_f$

Case 1: $(u, v) \in E_f$

- v got closer to s but u didn't
- yet u now precedes v on the shortest path from s to v .

$$G_f$$
$$s \rightarrow \dots \rightarrow \dots \rightarrow u \rightarrow v$$

The last line follows since u precedes v on the shortest path from s to v in $G_{f'}$.

So we must have $(u, v) \notin E_f$.

Case 2: $(u, v) \notin E_f$

- Let p be the augmenting path that produces $G_{f'}$.
- p is a shortest path from s to t .
- p must contain (v, u) (in the direction from v to u) since $(u, v) \in E_{f'}$ and $(u, v) \notin E_f$.

$$p: s \rightarrow \dots \rightarrow v \rightarrow u \rightarrow \dots \rightarrow t$$

- So we have:

$$\begin{array}{ll} G_f & \\ s \rightarrow \dots \rightarrow \dots \rightarrow v \rightarrow u & \text{if } u \text{ is even reachable} \\ G_{f'} & \\ s \rightarrow \dots \rightarrow \dots \rightarrow u \rightarrow v & \text{is now a shortest path} \end{array}$$

- this doesn't make sense.

Formally:

$$\begin{aligned} \delta_f(s, v) &\leq \delta_f(s, u) + 1 \\ &\leq \delta_{f'}(s, u) + 1 \\ &= \delta_{f'}(s, v) \quad \text{contradiction} \end{aligned}$$

The last line follows since u precedes v on the shortest path from s to v in $G_{f'}$.
So we must have $(u, v) \notin E_f$.

Case 2: $(u, v) \notin E_f$

- Let p be the augmenting path that produces $G_{f'}$.
- p is a shortest path from s to t .
- p must contain (v, u) (in the direction from v to u) since $(u, v) \in E_{f'}$ and $(u, v) \notin E_f$.

$$p: s \rightarrow \dots \rightarrow v \rightarrow u \rightarrow \dots \rightarrow t$$

- So we have:

$$\begin{array}{l} G_f \\ s \rightarrow \dots \rightarrow \dots \rightarrow v \rightarrow u \end{array} \quad \text{is a shortest path in } G_f$$

$$\begin{array}{l} G_{f'} \\ s \rightarrow \dots \rightarrow u \rightarrow v \end{array} \quad \text{is now a shortest path}$$

- So v has moved closer, u has not, yet u moved in front of v on a shortest path.
- This is impossible:

Formally:

$$\begin{aligned} \delta_f(s, v) &= \delta_f(s, u) - 1 && (v, u) \text{ is on a shortest path in } G_f \\ &\leq \delta_{f'}(s, u) + 1 \\ &= \delta_{f'}(s, v) - 2 && (u, v) \text{ is on a shortest path in } G_{f'} \\ &< \delta_{f'}(s, v) && \text{contradiction} \end{aligned}$$

Since we have contradictions in either case, no flow augmentation using Edmonds-Karp can cause the shortest path from s to any vertex to decrease. █

