CS1800

Discrete Structures

Fall 2017

Lecture 21

10/25/17

Last time

- Finish counting
  - examples

Today

- Start Probability

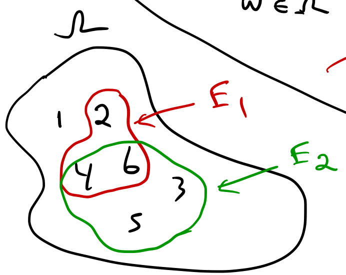
Next time

- Continue probability
  - examples

Probability

- Random experiment
- generates outcomes : $\omega \in \Omega$ i.e. "omega" i.e. "omega"
- sample space : set of all possible outcomes Ω
- event : subset of sample space
- $p : \Omega \to \mathbb{R}$ probability measure
  - $0 \le p(\omega) \le 1 \quad \forall \omega \in \Omega$
  - $\sum_{\omega \in \Omega} p(\omega) = 1$

$\Rightarrow$ we will assume for now that $p(\omega) = \frac{1}{|\Omega|} \quad \forall \omega \in \Omega$



Example

- roll a fair six sided die
- rolled a 5
- $\{1, 2, 3, 4, 5, 6\} = \Omega$
- $E_1 = $ "even" $= \{2, 4, 6\}$
- $E_2 = $ "≥ 3" $= \{3, 4, 5, 6\}$
- $p(1) = p(2) = \cdots = p(6) = 1/6$

$$p(E) = \sum_{\omega \in E} p(\omega)$$

e.g. $p(E_1) = p(2) + p(4) + p(6)$

If $p(\omega) = \frac{1}{|\Omega|} \quad \forall \omega \in \Omega$

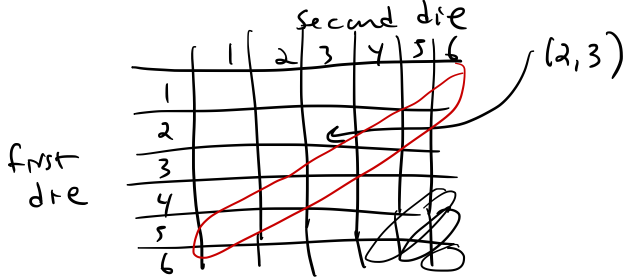
$$p(E) = \frac{|E|}{|\Omega|}$$

e.g. $p(E_1) = 3/6$

Examples

① Roll one fair die

$\Rightarrow \Omega = \{1,2,3,4,5,6\}$



② Roll two fair dice

$\Rightarrow \Omega = \{(1,1),(1,2),(1,3)\ldots,(2,1),(2,2)\ldots,(6,6)\} = \{1,2,\ldots 6\} \times \{1,2,\ldots 6\}$

$|\Omega| = 36$

$E_1 =$ total is 7

$E_1 = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$

$$p(E_1) = \frac{|E_1|}{|\Omega|} = \frac{6}{36} = \frac{1}{6}$$

$E_2 =$ total is greater than 8

$= 9$ or 10 or 11 or 12

$$\begin{array}{ccccccc} 12 & & 11 & & 10 & & 9 \\ \downarrow & & & & & & \\ 1 & + & 2 & + & 3 & + & 4 \end{array}$$

$|E_2| = 1 + 2 + 3 + 4$

$= 10$

$$p(E_2) = \frac{|E_2|}{|\Omega|} = \frac{10}{36} = \frac{5}{18}$$

Cards : Standard deck of cards : 4 suits Hearts, Diamonds, Clubs, Spades

within each suit $2, 3, 4, \ldots, 10, J, Q, K, A$

13 total per suit

- Rand. Exp. : draw one card from deck

- Sample space : $\Omega = \{ 2H, 3H, \ldots, AH, 2D, 3D, \ldots, AS \}$

$|\Omega| = 52$

3 face cards per suit

4 suits

- E_1 = face card (Jacks, Queen, King) $|E_1| = 3 \cdot 4 = 12$

$$p(E_1) = \frac{|E_1|}{|\Omega|} = \frac{12}{52} = 3/13$$

- E_2 = card is between 2 & 10 (a number)

$$|E_2| = 9 \cdot 4 = 36$$

$$p(E_2) = \frac{|E_2|}{|\Omega|} = \frac{36}{52} = 9/13$$

Urn Problems : 15 red balls
10 blue balls

- Rand. Exp. : draw one ball from urn

- $\Omega = \{ R1, R2, \dots R15, B1, B2, \dots B10 \}$ $|\Omega| = 25$

- $E_1 = \text{red}$ $|E_1| = 15$

$$P(E_1) = \frac{|E_1|}{|\Omega|} = \frac{15}{25} = 3/5$$

- Rand. exp. : draw 3 balls at once (sampling w/o replacement)

$\Omega = \{ \{R_1, R_2, R_3\}, \{R_1, R_2, R_4\}, \dots \{R_1 R_2, B_1\} \dots \{B8, B9, B10\} \}$

$|\Omega| = \binom{25}{3} = 2300$

- $E_1 = \text{all reds}$ $|E_1| = \binom{15}{3}$

$$P(E_1) = \frac{|E_1|}{|\Omega|} = \frac{\binom{15}{3}}{\binom{25}{3}} = \frac{455}{2300} = \frac{91}{460} \approx 19.8\%$$

- $E_2 =$ 2 red, 1 blue

$$|E_2| = \underbrace{\binom{15}{2}}_{\text{ways to get 2 reds}} \cdot \underbrace{\binom{10}{1}}_{\text{ways to get one blue}} = 1050$$

$$P(E_2) = \frac{\binom{15}{2}\binom{10}{1}}{\binom{25}{3}} = \frac{1050}{2300} \approx 45.7\%$$

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Lecture 22
10/26/17

Last time

- Start probability

Today

- Continue Probability
  - more examples

- Start expectation & variance

Next time

- Finish expectation & variance

Sampling w/ replacement: 25 balls, 15 red & 10 blue

$\Rightarrow$ Draw 3 balls, one at a time, put back between draws

$\Omega = \{ (R1, R1, R1), (R1, R1, R2) \ldots \quad (B10, B10, B10) \}$

$|\Omega| = 25^3$

- E_1 = all red $\quad |E_1| = 15^3$

$$p(E_1) = \frac{|E_1|}{|\Omega|} = \frac{15^3}{25^3} = \left(\frac{15}{25}\right)^3 = \left(\frac{3}{5}\right)^3 = \frac{27}{125} \approx 21.6\%$$

- E_2 = 2 red, 1 blue $\quad |E_2| = \binom{3}{1} \cdot 10 \cdot 15^2$

which blue

which reds

when did I draw blue?

$$p(E_2) = \frac{|E_2|}{|\Omega|} = \frac{\binom{3}{1} \cdot 10 \cdot 15^2}{25^3}$$

$$= \frac{3 \cdot 2 \cdot \cancel{5} \cdot 3^2 \cdot \cancel{5^2}}{5^3 \cdot \cancel{5^3}} = \frac{54}{125} = 43.2\%$$

- Bit strings (Bytes)
- Rand. Exp.: · flip fair coin 8 times
  - heads → output 1
  - tails → output 0

  ⟹ generates a byte.

- $\Omega = \{ 00000000, 00000001, \ldots, 11111111 \}$

  $|\Omega| = 2^8$

- $E_1 =$ byte has exactly 4 1s.

  $|E_1| = \binom{8}{4} = 70$

  pick positions for 1s

  (fixes positions for 0s)

  $P(E_1) = \frac{70}{2^8} = \frac{70}{256} \approx 27.34\%$

E_2 = byte that does not contain consecutive 1s.

- generate 2 bits at a time

  00
  01 } allowed
  10
  ~~11~~

- left-to-right
  - if 01, then the next two bits can't be 10

first 2 bits	next 2 bits	next 2 bits	last 2 bits	
00, 10	00, 10	00, 10	00, 01, 10	$\rightarrow 2\cdot 2\cdot 2\cdot 3 = 24$
		01	00, 01	$\rightarrow 2\cdot 2\cdot 1\cdot 2 = 8$
	01	00	00, 01, 10	$\rightarrow 2\cdot 1\cdot 1\cdot 3 = 6$
		01	00, 01	$\rightarrow 2\cdot 1\cdot 1\cdot 2 = 4$
01	00	00, 10	00, 01, 10	$\rightarrow 1\cdot 1\cdot 2\cdot 3 = 6$
		01	00, 01	$\rightarrow 1\cdot 1\cdot 1\cdot 2 = 2$
	01	00	00, 01, 10	$\rightarrow 1\cdot 1\cdot 1\cdot 3 = 3$
		01	00, 01	$\rightarrow 1\cdot 1\cdot 1\cdot 2 = 2$

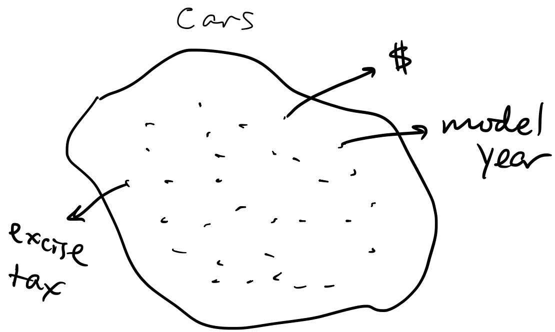
$= 55$

$$P(E_2) = \frac{55}{256} \approx 21.48\%$$

Expectation

- Random variable

$X : \Omega \to \mathbb{R}$



$$E[X] = \sum_{x} x \cdot \Pr[X = x]$$

$$E[X] = \sum_{\omega \in \Omega} X(\omega) \cdot p(\omega)$$

expected
or
"average"
value

1/2 cars	27 000
1/3 cars	10 000
1/6 cars	15 000

$$E[X] = \sum_{x} x \cdot \Pr[X = x] = \tfrac{1}{2} \cdot 27\,000 + \tfrac{1}{3} \cdot 10\,000 + \tfrac{1}{6} \cdot 15\,000$$

– weighted average

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Lecture 23
10/30/17

Last time

- Finished Probability Examples
- Started Expectation

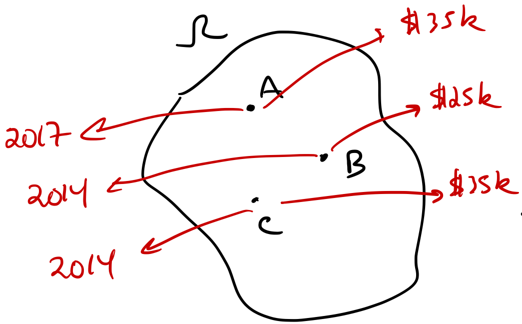
Today

- Expectation & Variance

Next time

- Entropy

Underlying probability measure induces distribution over range of rand. var.



- A – Alice's car
  - \$35,000
  - 2017
- B – Bob's car
  - \$25,000
  - 2014
- C – Charlie's
  - \$35,000
  - 2014

$p(A) = p(B) = p(C) = 1/3$

probability measure

X_1 = purchase price

$\text{range}(X_1) = \{25000, 35000\}$

ind. dist.

$D(25000) = \Pr\{X_1 = 25000\} = 1/3$

$D(35000) = \Pr\{X_1 = 35000\} = 2/3$

$$E\{X_1\} = \sum_x x \cdot \Pr\{X_1 = x\}$$
$$= 25000 \cdot 1/3 + 35000 \cdot 2/3$$
$$= 31666.67$$

X_2 = model year

$\text{range}(X_2) = \{2014, 2017\}$

$D(2014) = \Pr\{X_2 = 2014\} = 2/3$

$D(2017) = \Pr\{X_2 = 2017\} = 1/3$

$$E\{X_2\} = \sum_x x \cdot \Pr\{X_2 = x\}$$
$$= 2014 \cdot 2/3 + 2017 \cdot 1/3$$
$$= 2015$$

$$E\{X_1\} = \sum_{\omega \in \Omega} X_1(\omega) \cdot p(\omega)$$
$$= X_1(A) \cdot p(A) + X_1(B) \cdot p(B) + X_1(C) \cdot p(C) = 35000 \cdot 1/3 + 25000 \cdot 1/3 + 35000 \cdot 1/3 = 31666.67$$

① · Roll one fair six-sided die $\{\text{⚀}, \text{⚁}, \text{⚂}, \text{⚃}, \text{⚄}, \text{⚅}\}$

$X =$ value of die face

$$X(\text{⚀}) = 1$$
$$X(\text{⚁}) = 2$$
$$\vdots$$
$$X(\text{⚅}) = 6$$

What is $E[X]$?

prob. meas: $p(\text{⚀}) = p(\text{⚁}) = p(\text{⚂}) = \cdots = p(\text{⚅}) = 1/6$ ← prob. meas. dictacted by rand. exp.

$\Rightarrow$ $\Pr\{X=1\} = \Pr\{X=2\} = \cdots = \Pr\{X=6\} = 1/6$ ← induced dist. over range of r.v.

$$E[X] = \sum_{x} x \cdot \Pr\{X=x\} = \sum_{i=1}^{6} i \cdot \Pr\{X=i\} = 1 \cdot \Pr\{X=1\} + 2 \cdot \Pr\{X=2\} + \cdots$$
$$= 1 \cdot 1/6 + 2 \cdot 1/6 + \cdots \; 6 \cdot 1/6$$
$$= 1/6 \cdot (1+2+3+4+5+6) = \boxed{3.5}$$

$$E[X] = \sum_{\omega \in \Omega} X(\omega) \cdot p(\omega) = X(\text{⚀}) \cdot 1/6 + X(\text{⚁}) \cdot 1/6 + X(\text{⚂}) \cdot 1/6 + \cdots + X(\text{⚅}) \cdot 1/6$$
$$= 1 \cdot 1/6 + 2 \cdot 1/6 + \cdots + 6 \cdot 1/6$$
$$= 3.5$$

② Roll two fair six-sided die

$X =$ sum of values of die faces

what is $E[X]$

X:

die 2

die 1	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

die1 = 3
die2 = 5
$(3,5)$

$p(-,-) = 1/36$

$$E[X] = \sum_x x \cdot \Pr\{X = x\}$$

$X = 2 \quad \Pr\{X=2\} = 1/36$

$X = 3 \quad \Pr\{X=3\} = 2/36$

$X = 4 \quad \Pr\{X=4\} = 3/36$

$\vdots$

$X = 12 \quad \Pr\{X=12\} = 1/36$

$$E[X] = \sum_{\omega \in \Omega} X(\omega) \cdot p(\omega)$$

$$= \sum_{\omega \in \Omega} X(\omega) \cdot 1/36 = 1/36 \cdot \sum_{\omega \in \Omega} X(\omega)$$

$$= \frac{1}{36}\left(1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 + \cdots + 1 \cdot 12\right)$$

$$E[X] = \sum_x x \cdot \Pr\{X = x\}$$

$$= 2 \cdot 1/36 + 3 \cdot 2/36 + 4 \cdot 3/36 + \cdots + 12 \cdot 1/36$$

$$= 7$$

Linearity of Expectation

$X_1 =$ first die value

$X_2 =$ second die value

$$E\{X_1 + X_2\} = E\{X_1\} + E\{X_2\}$$

$$= 3.5 + 3.5 = 7$$

(3)

- Pay \$6 to play
- Roll 2 6-sided die
- payout is sum of die faces <u>except</u> if doubles, then 0.

	1	2	3	4	5	6
1	-6	-3	-2	-1	0	1
2	-3	-6	-1	0	1	2
3	-2	-1	-6	1	2	3
4	-1	0	1	-6	3	4
5	0	1	2	3	-6	5
6	1	2	3	4	5	-6

$X =$ winnings (profit)

$$E\{X\} = \sum_x x \cdot \Pr\{X=x\}$$

$X = -6 \qquad \Pr\{X=-6\} = 6/36 = 1/6$

$X = -3 \qquad \Pr\{X=-3\} = 2/36 = 1/18$

$\vdots$

$X = +5 \qquad \Pr\{X=5\} = 2/36 = 1/18$

$$E\{X\} = \sum_x x \cdot \Pr(X=x)$$

$$= (-6)\cdot 6/36 + (-3)(2/36) + \cdots + 5(2/36)$$

$$= -0.1\overline{66}$$

lose 16.6 ¢ per play or about 2.8%

$$E\{X\} = \sum_{\omega\in\Omega} X(\omega)\cdot P(\omega)$$

$$= \sum_{\omega\in\Omega} X(\omega)\cdot 1/36$$

$$= 1/36 \cdot \sum_{\omega\in\Omega} X(\omega)$$

$$= \frac{1}{36}\cdot\left((-6)\cdot 6 + (-3)\cdot 2 + \cdots + 5\cdot 2\right)$$

$$= \frac{1}{36}\cdot(-6) = -0.1\overline{66}$$