Recap: Joint / Conditional Probabs.

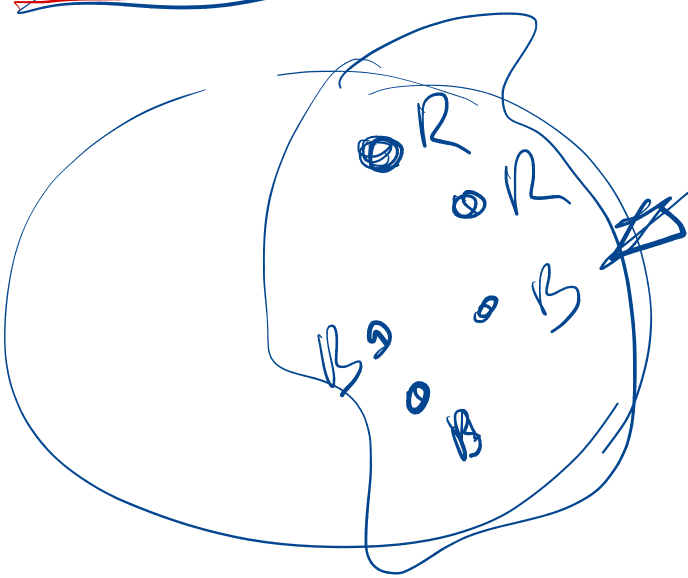
Objects : $S = $ (R.V.) Shape $\in \{\square, \bigcirc\}$ $\Omega(S)$

$C = $ (R.V.) Color $\in \{$Red, Blue, Green$\}$ $\Omega(C)$

2 R.V. $\Rightarrow$ joint probability (distribution)

Events (def) : $S = \square$, $C = $ Red ; $C = R \vee C = B$

$C \in \{R, B\}$



$|\Omega(S)| = 2$ $|\Omega(C)| = 3$

$\Rightarrow$ total options (Cartesian)

$\Rightarrow$ Joint table

$\sum_{S,C} P(S, C) = 1$

$\nearrow$ joint

COLOR

SHAPE	Red 0.42	Blue 0.14	Green 0.44
□ 0.56	$P(R,\square)$ 0.25	$P(B,\square)$ 0.10	$P(G,\square)$ 0.21
○ 0.44	$P(R,\circ)$ 0.17	$P(B,\circ)$ 0.04	$P(G,\circ)$ 0.23

$$P(\square,R)=P(S=\square \wedge C=R)=\boxed{P(R,\square)}\in[0,1]$$

joint probab

= "proportion of Red Squares out of total in ALL box"

$$\sum_{s,c} P(S=s,C=c) = P(R,\square)+P(B,\square)+P(G,\square)+P(R,\circ)+P(B,\circ)+P(G,\circ)=1$$

$$P(\square)=P(S=\square)=P(R,\square)+P(B,\square)+P(G,\square) \quad \text{sum rule}$$

$$P(\circ)=P(S=\circ)=P(R,\circ)+P(B,\circ)+P(G,\circ)$$

$$P(B)=P(C=B)=P(B,\square)+P(B,\circ)=0.14$$

Conditional $P(\square \mid R) = P(S=\square \mid C=R)$ restriction of the space

given given

= "proportion of Red squares out of all Reds"

$$= \frac{P(\square, R)}{P(R)}$$

joint cond

$$P(R, \square) = P(R) \cdot P(\square \mid R)$$

$$= P(\square) \cdot P(R \mid \square)$$

$P(R \mid \square)$ = "proportion of Red sq out of all sq"

$$P(R) \cdot P(\square \mid R) = P(\square) \cdot P(R \mid \square)$$

Th Bayes

$$P(\square \mid R) = \frac{P(\square) \cdot P(R \mid \square)}{P(R)}$$

RECITATION 6

$\sum \text{all} = 1$

$7x = 1$

$x = 1/7$

Faces	1	2	3	4	5	6
prob	x	x	x	x	2x	x

Question 2. non uniform

Suppose we define a *weighted* six-sided die to be rigged such that the outcome of rolling a 5 is twice as likely as any other outcome. All the remaining outcomes are equally likely.

(a) What is the probability of rolling a 5 on a weighted die? What is the probability of rolling any other number? $P(5) = 2x$

(b) Suppose you roll two weighted dice. Die no. 1 comes up five. What is the probability that their sum is greater than 7, given this new information?

a = outcome first die
b = outcome 2nd die

$P(a+b>7 \mid a=5) = P(5+b>7) = P(b>2) = P(b=3)+P(b=4)+P(b=5)+P(b=6) = 5x = 5/7$

(c) Suppose you roll a weighted die 3 times. What is the probability of seeing exactly...

- 3 fives? $\left(\frac{2}{7}\right)^3$ $P(5)\cdot P(5)\cdot P(5) = 2x$
- 2 fives? $P(5)\cdot P(5)\cdot P(\bar{5})$ 3
- 1 five?
- 0 fives?

What can you say about the sum of those four numbers?

$P(a+b>7 \mid a=5 \text{ OR } b=5)$

$\dfrac{P(a+b>7 \wedge a=5 \text{ or } b=5)}{P(a=5 \text{ or } b=5)} = \dfrac{16x^2}{24x^2} = \dfrac{16}{24}$

(d) Let X be the random variable assigned to the number of fives that we see. What is the expected value of X if the die is rolled 3 times?

$x_1 = \begin{cases} 1 & \text{if 1st die} = 5 \\ 0 \end{cases}$

$x_2 = \begin{cases} 1 & \text{if 2nd die} = 5 \\ 0 \end{cases}$

$x_3 = \begin{cases} 1 & \text{if 3rd} = 5 \\ 0 & \text{other} \end{cases}$

$E[\#\text{of } 5] = E[x_1+x_2+x_3] = E[x_1] + E[x_2] + E[x_3] =$

$P(1\text{st die} = 5) = \frac{2}{7} + \frac{2}{7} + \frac{2}{7} = 6/7$

a \ b	1	2	3	4	5	6
1					$2x^2$	
2					$2x^2$	
3					$2x^2$	
4					$2x^2$	
5	$2x^2$	$2x^2$	$2x^2$	$2x^2$	$4x^2$	$2x^2$
6					$2x^2$	

R.V. $X = \#$ of dice that roll 5

$X \in \{0, 1, 2, 3\}$

$P(X=v) = \left(\frac{2}{7}\right)^v \cdot \frac{5}{7}^{(3-v)}$

$v=2$: $P(X=2) = \left(\frac{2}{7}\right)^2 \cdot \left(\frac{5}{7}\right)^1 \cdot \binom{3}{2}$

$P(5)$ $P(\text{not } 5)$

I	II	III
5	5	Z
5	Z	5
Z	5	5

v=1 5 4 6

6 = 3! ways to permute

Question 3.

indicator r.v. $X_i = \begin{cases} 1 & \text{if } ✓ \\ 0 & \text{if } ✗ \end{cases}$

Please watch this important video before starting this question:

https://www.youtube.com/watch?v=naQeNNaZYoA

Jimmy and my boy Neil Patrick Harris started out with a carton of 12 eggs. 8 were boiled, 4 were raw. Note that the initial probability of getting a raw egg is 1/3, but that changes once they start cracking them on their faces.

Suppose a single person (could be Jimmy or NPH) cracks 3 eggs. What is the expected number of raw eggs that get smashed on that person's face?

$E[\sum x_i] = \sum E[x_i]$

brute force

X = # of cracked raw eggs

$P(X=0) =$

$P(X=1) =$

$P(X=2) =$

$P(X=3) = \frac{4}{12} \cdot \frac{3}{11} \cdot \frac{2}{10} = \frac{24}{1320}$

dep

$E[X] = 0 \cdot P(X=0) + 1 \cdot P(X=1) + 2 \cdot P(X=2) + 3 \cdot P(X=3)$

Linearity of E[ ]

RV

$Y_1 = 1/0$ if first cracked egg is raw

$Y_2 = 1/0$ second cracked egg = raw

$Y_3 = 1/0$ = third cracked egg = raw

$E[\#\text{ of cracked raw}] =$

$= E[Y_1 + Y_2 + Y_3] = E[Y_1] + E[Y_2] + E[Y_3]$

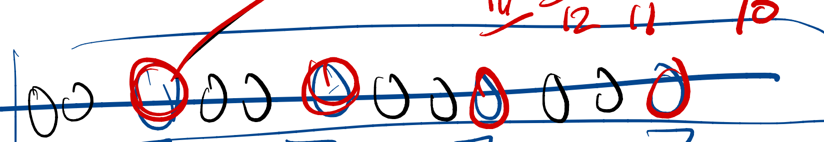
$= \frac{4}{12} + \frac{4}{12} + \frac{4}{12}$

$= 1.$

$\frac{1}{12} +$ $\quad \frac{1}{12}$

$\frac{11}{12} \cdot \frac{1}{11} +$ $\quad \frac{1}{12}$

$\frac{11}{12} \cdot \frac{10}{11} \cdot \frac{1}{10} +$ $\quad \frac{1}{12}$



Viral Linearity by raw eggs:

Z_1: first raw egg, 1 = cracked, 0 = not crack

Z_2: 2nd raw, 1/0

Z_3: 3rd raw, 1/0

Z_4: 4th raw egg, 1/0

$E[\#\text{ cracked raw}] =$

$E[Z_1 + Z_2 + Z_3 + Z_4] =$

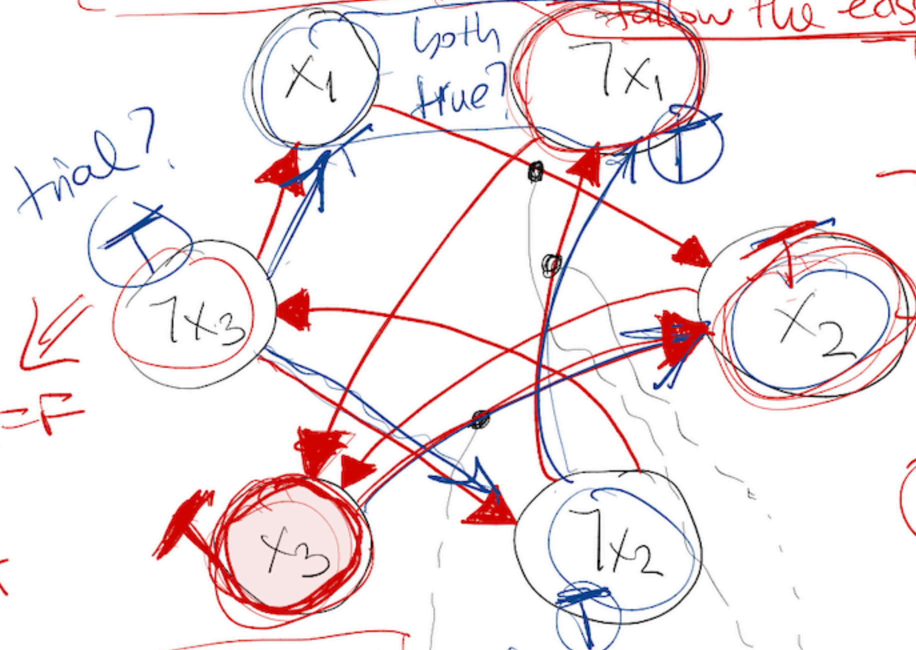
$= E[Z_1] + E[Z_2] + E[Z_3] + E[Z_4] = 1$

$\frac{3}{12} \quad \frac{3}{12} \quad \frac{3}{12} \quad \frac{3}{12}$

if give contradiction?

$\neg x_3 \Rightarrow x_1 \Rightarrow x_2 \Rightarrow x_3 \Longrightarrow \begin{cases} x_3 = T \\ \neg x_3 = F \end{cases}$

Trial and error: start with ~~...~~ = T
follow the edges



1) input formula
each clause $(X \vee Y)$

2) transform each clause
into 2 implications

3) Graph:
nodes: all literals
and all $\neg$literals

4) Make sure all
implications are true.

$\neg x_3 = F$

$\Updownarrow$

$x_3 = T$

$(x_2 \vee \neg x_3) \wedge (x_3 \vee x_1) \wedge (x_2 \vee \neg x_1) \wedge (x_3 \vee \neg x_2)$

$x_3 \Rightarrow x_2$	$\neg x_3 \Rightarrow x_1$	$\neg x_2 \Rightarrow \neg x_1$	$\neg x_3 \Rightarrow \neg x_2$
$\neg x_2 \Rightarrow \neg x_3$	$\neg x_1 \Rightarrow x_3$	$x_1 \Rightarrow x_2$	$x_2 \Rightarrow \cancel{\neg x_3}$

t part (thinking)

aph + red arrows (edges)

$X \rightarrow Y$ cannot be $T \rightarrow F$ (anything else ok)

etween $\{x_i, \neg x_i\} = \{T, F\}$

Alternative (1) Assign: set all x_i some initial value
change as you go

(2) Set $x_3 =$ ~~value either~~ T — trial & error
Trial
Follow arrows
keep going ...

CONTRADICT → $x_3 = F$ FINAL
~~$x_3 = T$~~

OR

WORKS OUT (not only original set x_3)

assignments so far final?

#steps is polynomial ($2m^2$, $2m^3$, $2m \cdot n$ . . .)

not exponential (2^n)
(truth table)

Informal proof (but complete)

- "WORKS OUT" $\Rightarrow$ Found assignment (partially)
  those edges I don't have to go through again

- CONTRAD $\Rightarrow$ I might have to go through edges one more time
  $\Downarrow$
  ex. $\neg x^3 = T$

- how many edges max
  How many time do I have to traverse them?

blue shirt. Red's too loud.

- Each group must be accompanied by exactly one person wearing a gold shirt.

How many possible groups can the team send with this additional restriction? You can leave your answer in choice notation (i.e. $\binom{3}{2} + \binom{5}{2}$). (Hint: Focus on the number of red shirts in each groups to find all the cases. Then combine them with the sum rule.)

Problem 6 [Hard]**: Counting Set Partitions**

Let S and $S_1, S_2, \cdots S_n$ be sets. $S_1, S_2, \cdots, S_n$ *partition* S if and only if

- $\bigcup_{i=1}^{n} S_i = S$ and
- Every pair of subsets share no elements

$|S| = 4$

Disjoint cases by part. sizes

We call $S_1, S_2, \cdots S_n$ a *partition* of S.
For example, let $S = \{1, 2, 3, 4\}$. Then

- $\{1, 2\}, \{3, 4\}$ partition S.
- $\{1\}, \{2, 4\}, \{3\}$ also partition S.
- However, $\{1, 3, 4\}, \{2, 4\}$ do *not* partition S.

1 1 1 1 → 1 way

1 1 2 →

2 2 →

Finally, two partitions are the same if they have the same subsets. For example, $\{1, 2\}, \{3, 4\}$ is the same partition as $\{3, 4\}, \{1, 2\}$.

Say we have a set $A = \{1, 2, 5, 14, 42\}$.

Two sorted sequences lengths 9 and 7 are given: (1,2,3,...9) and (a,b,c,d,e,f,g). We want to interleave them into a sequence of length 16 such that numbers 1-9 remain in relative order, and also literals a-g remain in relative order. How many ways are there to do this? Example valid sequences are 1a2bc34d56efg789, 12345abc678de9fg, and a1bcdef23456789g.

Problem 5 [Medium]: Alien Contact

Trained contact teams have been dispatched to meet the aliens. A team of 41 people discovers a far off planet inhabitanted by aliens.

i. The team needs to send 3 groups of size 5,5, and 7 down to the planet. Two groups are the same if they have the same group members. How many possible groups can be created from the crew?

5y RG dist

Greece

ii. The groups scared the inhabitants due to the colors of their shirts. The team consists of 23 people wearing red shirts, 15 people wearing blue shirts, and 3 people wearing gold shirts. The inhabitants ask for the following changes to the groups:

7 5

3R3B 2R2B

2R2B 2R

- Send only 2 groups, one of 5 and another of 7.
- In every group, a person wearing a red shirt must be paired with a person wearing a blue shirt. Red's too loud.
- Each group must be accompanied by exactly one person wearing a gold shirt.

How many possible groups can the team send with this additional restriction? You can leave your answer in choice notation (i.e. $\binom{3}{2} + \binom{5}{2}$). (Hint: Focus on the number of red shirts in each groups to find all the cases. Then combine them with the sum rule.)

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