Problem 5 - 12 points

MIDTERM 2019 Fall $A = \{1,2,3,4,5,6,7,8,9,10\}$

The following questions all consider the set of natural numbers between 1 and 10 inclusive.

i. (2 points) Using set builder notation, write this set. $A = \{x \in \mathbb{N} \mid 1 \leq x \leq 10\}$

ii. (4 points) How many subsets of size three contain only even numbers

iii. (3 points) How many subsets of size three contain only even numbers, but not 2 and 4 both?

iv. (3 points) How many sequences of length 15 with elements from the set are never decreasing? For example, 1,1, 2, 2, 2, 3, 5, 5, 5, 5, 5, 6, 7, 8, 10 is one such sequence. (Hint: Notice that a sequence is determined only by count and value, the example is two 1s, three 2s, one 3, five 5s, one 6, one 7, one 8, and one 10.)

II subsets of size 3 out of k elements set $\Rightarrow \binom{k}{3}$

not entire A

B = even elements of A set, $B = \{2,4,6,8,10\}$ $|B| = k = 5$

$\boxed{\text{Answer is } \binom{5}{3}}$

III all subsets in part II, except the ones that fail the restriction

Solution1 $\rightarrow$ $\binom{5}{3}$ − # subsets that contain both 2 and 4.

$\{2, 4, x\}$ $x \in \{6, 8, 10\}$

3 such subsets

$\binom{5}{3} - 3$

Solution 2 (III): $B = \boxed{\{\underbrace{2,4}_{B1}\} \cup \{\underbrace{6,8,10}_{B2}\}}$

want subsets of size 3 excluding ones with both 2 and 4.

disjoint
- one element from B1 (2 or 4) + 2 elem from B2 $\binom{2}{1}\binom{3}{2}$
- no elem from B1, choose all 3 from B2 $\binom{3}{3} = 1$ subset

Total (sum Rule) $2 \cdot 3 + 1 = 7$

Sanity check: Solution 1 $\binom{5}{3} - 3 = \frac{5!}{3!\,2!} - 3 = \frac{4 \cdot 5}{2} - 3 = 7$ ✓

$A = \{1,2,3,4,5,6,7,8,9,10\}$

iv. (3 points) How many sequences of length 15 with elements from the set are never decreasing? For example, 1,1, 2, 2, 2, 3, 5, 5, 5, 5, 5, 6, 7, 8, 10 is one such sequence. (Hint: Notice that a sequence is determined only by count and value, the example is two 1s, three 2s, one 3, five 5s, one 6, one 7, one 8, and one 10.)

15 elements in sequence, fixed order (increasing)

chunks

"1"	"2"	"3"	"4"	"5"	"6"	"7"	"8"	"9"	"10"
2times	3	1	0	5	1	1	1	0	1

total count = 15

Bin1 Bin 2 Bin 3 no4 Bin 4 . . . no9 Bin 10

15 items balls seq values → 10 Bins valid elements from A

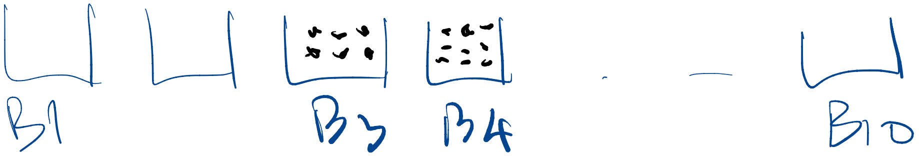
$$\boxed{\binom{\overset{\text{balls}}{15}+\overset{\text{bins}}{10}-1}{10-1}}$$

✓ • every valid sequence can be obtained by balls-into-bins ✓

✓ • balls into bins procedure only produces valid sequences

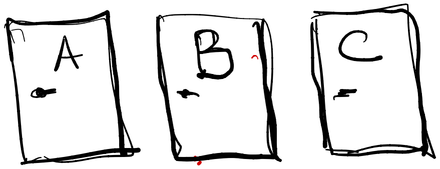
→ any arrangement balls→bins that is NOT valid sequence?

✓ • double counting? 2 balls→bins arrang same sequence.



→ 333333|444444444

Monty Hall Puzzle



3 Doors : 2 empty
1 with treasure

- You choose one at random initially (A)
- Monty opens a diff door (B) without treasure at random (if many)
- Do you stick to initial door (A), or switch to the other door (C)?

intuition: why would C be more likely than A? A → 1/3 chance

- Monty does not give info on door A. But it does (impl) on door C C → 2/3 chance

R.V R = door with the treasure ($\Omega(R) = \{A, B, C\}$ initially uniform)

M = door that Monty opens: not the one picked, not the one with treasure

Say initial pick is $\boxed{A}$ $\Rightarrow \Omega(M) = \{B, C\}$ OBSERVE $M=B$

before M=R.v after M=B, var

given

$$P(R=A \mid M=B) = \frac{P(M=B \mid R=A)\, P(R=A)}{P(M=B)} = \frac{1/2 \cdot 1/3}{1/2} = \boxed{1/3}$$

init prior

across all choices by player

$$P(R=B \mid M=B) = \frac{P(M=B \mid R=B) \cdot P(R=B)}{P(M=B)} = \frac{0 \cdot 1/3}{1/2} = 0$$

0 as Monty never opens the door with treasure

$$P(R=C \mid M=B) = \frac{P(M=B \mid R=C) \cdot P(R=C)}{P(M=B)} = \frac{1 \cdot 1/3}{1/2} = \boxed{\frac{2}{3}}$$

$1 - P(R=A \mid M=B)$

⇒ better to switch to door C, prob treasure = $\frac{2}{3}$

$$
\begin{aligned}
P(M=B) = {} & P(M=B \mid R=A) \cdot P(R=A) + & \quad 1/2 \cdot 1/3 \\
& + P(M=B \mid R=B) \cdot P(R=B) & \quad 0 \\
& + P(M=C \mid R=C) \cdot P(R=C) & \quad 1 \cdot 1/3 \\
\hline
& & 1/6 + 1/3 = 1/2
\end{aligned}
$$

Zika Virus FL 2016

based on true story

- Prevalence of zika in FL $10^{-5} = p(\text{zika})$
- accuracy of blood test is 99%
  - $P(\text{test}=\text{pos} \mid \text{zika}) = 99/100$
  - $P(\text{test}=\text{pos} \mid \text{no zika}) = 0.01$

2 diff pieces of information.

Jimmy tests positive! what is the chance Jimmy actually has zika?

Z = have zika T = positive test

want $P(Z \mid T) = \dfrac{P(T \mid Z) \cdot P(Z)}{P(T)} = \dfrac{99/100 \cdot 1/100{,}000}{P(T \mid Z) \cdot P(Z) + P(T \mid \bar{Z}) \cdot P(\bar{Z})}$

$P(\bar{Z}) \rightarrow 1 - P(Z)$

$$= \frac{99/100 \cdot 1/100{,}000}{99/100 \cdot 1/100{,}000 + 1/100 \cdot (1 - 10^{-5})} = \frac{99/10^{+7}}{99/10^{7} + 99999/10^{7}}$$

$\overset{?}{\simeq} 0.00099 \simeq 0.1\% = 1$ in 1000 maybe not to worry yet

or at least repeat the test.

Balls into Bins Statistics

m balls are indep thrown into one of the k bins, uniformly.

① What is the expected #balls falling into a bin?

② what is the expected # of empty bins?

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① Define R.V. $X_{ij} = \begin{cases} 1 & \text{if ball } i \to \text{bin } j \\ \boxed{0 \text{ if not}} & \end{cases}$ Total $m \times k$ R.V.

$$E[\#\text{ of balls in bin } t] = E\Big[\underbrace{\sum_{i=1}^{m} X_{it}}_{\text{"1" for every ball} \to \text{bin } t}\Big] = \sum_{i=1}^{m} E[X_{it}] = m \cdot \frac{1}{k}$$

pick at $1 \le t \le k$

$$E[X_{it}] = 1 \cdot p(X_{it}=1) + 0 \cdot pr(X_{it}=0) = P(X_{it}=1) = 1/k$$

prob of ball i → bin t

unif over k bins

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② Fix a bin t $1 \le t \le k$

R.V. $Y_t = \begin{cases} 1 & \text{if all balls miss bin } t \Rightarrow \text{empty} \\ 0 & \text{if not} \Rightarrow \text{not empty} \end{cases}$

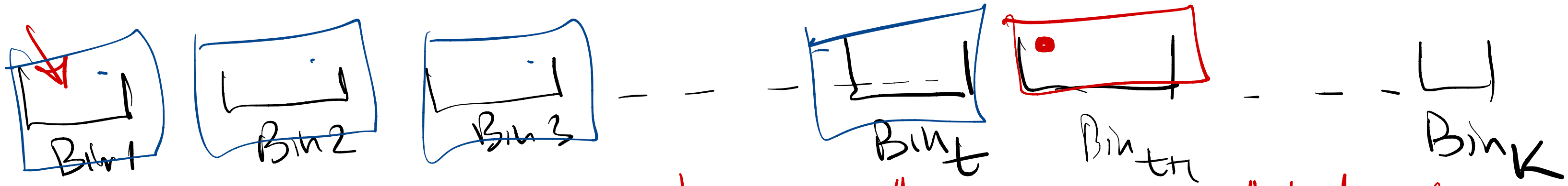
$$P[Y_t = 1] \;\underset{\text{indep}}{\overset{\text{m throws}}{=}}\; \prod_{i=1}^{m} \boxed{\text{ball } i \text{ miss bin } t} = \prod_{i=1}^{m} Pr[X_{it} = 0] = \prod_{i=1}^{m} \left(1 - \frac{1}{k}\right)$$

$$= \left(1 - \frac{1}{k}\right)^m.$$

$$E[\#\text{empty bins}] = E\Big[\sum_{t=1}^{k} Y_t\Big] = \boxed{\sum_{t=1}^{k} E[Y_t] = k \cdot \left(1 - \frac{1}{k}\right)^m \sim k \cdot e^{-m/k}}$$

optional: "Coupon Collector PB"

③✱ Say k if fixed (#bins), but $\boxed{m}$ is a random variable: we throw m balls until all the bins are non-empty. What is the E[m] ?

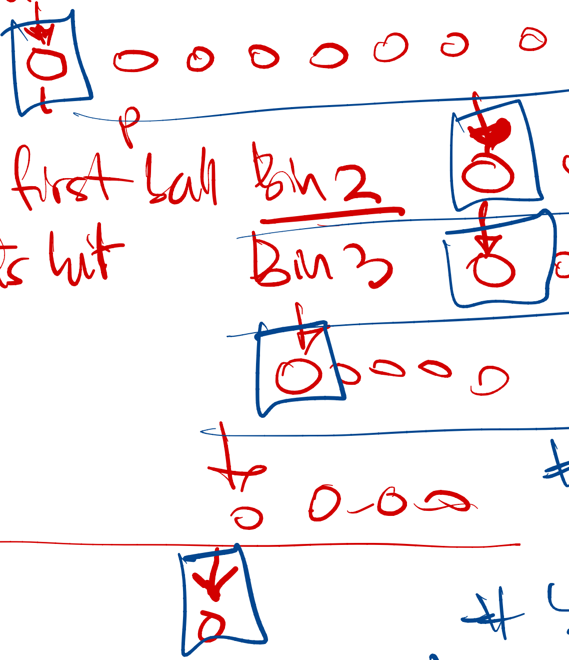


Bins are now in "unit event order" → coupon collected order.

- First ball → Bin 1.
- Next bin that gets its first ball Bin 2
- Next empty bin gets hit Bin 3
- Next Bin 4

Bin t

Bin t+1



#balls going to bin 1 X_1

#balls until we hit bin 3 X_2

#balls until we hit bin 4 X_3

#balls until we hit (first time) bin t X_t

balls until we hit bin t+1 X_{t+1}

X_t = #balls after hitting bin t (first time) until we hit bin t+1 (first time)

P(X = ball) to hit bin t+1 = 1 − P(X = ball falls into one of the non-empty t bins)

$= 1 - \frac{t}{K}$ "chance of success" fixed

⟹ geometric distribution
(exercise)

$$E(X_t] = E\left[X = \text{geometric dist} \quad \text{success} = 1 - \frac{t}{k}\right]$$

$x \in [0,1]$

$1 + x + x^2 + x^3 + x^4 \dots = \frac{1}{1-x}$

$$= \frac{1}{\text{success rate}} = \frac{1}{1 - t/k} = \frac{k}{k-t+1}$$

$$\# \text{balls} = 1 + \sum_{t=1}^{k} X_t$$

= #balls till bin 1 → 1
#balls till bin 2
#balls till bin k

$$= \sum_{t=1}^{k} \frac{k}{k-t+1} = k \sum_{t=1}^{k} \frac{1}{k-t+1} = k \cdot \sum_{t=1}^{k} \frac{1}{t} \simeq k \cdot H_k$$

Harmonic
H_k

$$H_k = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k}$$

Passwords = 8 capital letters (A-Z)   3 spots not that letter

Count all passwords with a letter occurring ≥5 times.

Jimmy Incorrect solution:

1	2	3	4	5	6	7	8
Z	B	B	A	B	B	C	B

- choose letter to repeat → 26 choices (say "B")
- choose 5 places to put it out of 8 → $\binom{8}{5}$ choices (say I get {2,3,5,6,8})
- fill the other 3 spots with any 3 letters → 26^3 choices. (say $\overset{1}{Z}$, $\overset{4}{A}$, $\overset{7}{C}$)

what is wrong: DOUBLE COUNTING! Last step Jimmy could get another "B"

⇒ 6th B (ok for validity)   B B B A B B C B

≥5 times, 6 times ok.

this password is counted 6 times!

First step: B BB_ BB_B | First step: B B B _ BB_ B

step two first B | Last step get the last B

2 diff ways (by Jimmy) same password.

SOLUTION: DISJOINT CASES by # occurrences ≥5

	choose letter		choose __		Choose the rest ≠ letter
K=5:	26	×	$\binom{8}{5}$	×	25^3
K=6:	26	×	$\binom{8}{6}$	×	25^2
K=7:	26	×	$\binom{8}{7}$	×	25
K=8:	26	×	1	×	1

$$\sum_{K=5}^{8} 26\times\binom{8}{K}\times 25^{8-K}$$