CS1800

Discrete Structures

Fall 2019

Lecture 9

10/4/19

Last time

- Finish GPHP
- Permutations
- Combinations

Today

- Finish counting
  - Balls-in-bins
  - Binomial theorem
  - Examples

Next time

- Start probability

Balls-in-bins

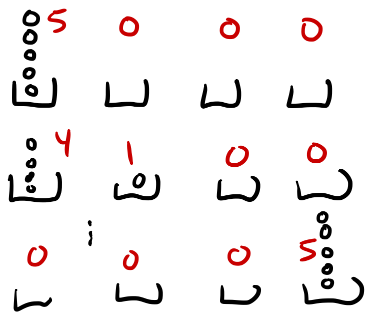
E.g. - 4 bins
- 5 balls (indistinguishable)
- Q: How many ways to place 5 balls in 4 bins?

00000111

00001011

⋮

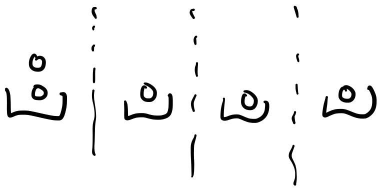
11100000



algebraicly ...

$$x_1 + x_2 + x_3 + x_4 = 5$$

where each $x_i \in \mathbb{N}$



\# ways: $\binom{8}{3}$

00 | 0 | 0 | 0 ⇒ 00|0|0|0 ⇒ 00101010

- bit strings - length 8 < 5 balls, 3 dividers
- need to <u>choose</u> 3 1s out of 8 positions

arrangements of n balls into k bins is

$$\binom{n+k-1}{k-1} = \binom{n+k-1}{n} \qquad \binom{n}{r} = \binom{n}{n-r}$$

Q: 5 brands of soda, buy 15 cans total, how many ways?

5	3	4	2	1
coke	pepsi	MD	sprite	DP

"balls" – cans of soda

"bins" – brands

$$\binom{15+(5-1)}{(5-1)} = \binom{19}{4} = 3{,}876$$

Binomial theorem: $(x+y)^n$

$(x+y)^0 = 1$

$(x+y)^1 = x+y$

choose x or y　　choose x or y

$(x+y)^2 = x^2+2xy+y^2 = (x+y)(x+y)$

	x	y
x	x^2	xy
y	yx	y^2

$(x+y)^3 = ?$

F	$x\cdot x$	$\to$	x^2
O	$x\cdot y$	$\to$	xy
I	$y\cdot x$	$\to$	yx
L	$y\cdot y$	$\to$	y^2

$\Rightarrow x^2+2xy+y^2$

$(x+y)^4 = ?$

$\vdots$

$(x+y)^n = \overbrace{(x+y)\cdot(x+y)\cdot(x+y) \text{ --- } (x+y)}^{n}$

Claim: # of y's chosen $\to$ dictates final term, e.g. xy^2 or x^2y

# of ways of doing so $\to$ dictates coefficient in front of term

$3xy^2$ vs $6xy^2$

$(x+y)\cdot(x+y)\cdot(x+y)$

	# ways	term
0 y's	$1 = \binom{3}{0}$	x^3
1 y	$3 = \binom{3}{1}$	x^2y
2 y's	$3 = \binom{3}{2}$	xy^2
3 y's	$1 = \binom{3}{3}$	y^3

$\Rightarrow x^3+3x^2y+3xy^2+y^3$

1　3　3　1

$$(x+y)^4 = (x+y)\cdot(x+y)\cdot(x+y)\cdot(x+y)$$

How many y's ?

	# ways	
0	$1 = \binom{4}{0}$	x^4
1	$4 = \binom{4}{1}$	$x^3 y$
2	$6 = \binom{4}{2}$	$x^2 y^2$
3	$4 = \binom{4}{3}$	$x y^3$
4	$1 = \binom{4}{4}$	y^4

$$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

$$(x+y)^n = \underbrace{(x+y) \cdots (x+y)}_{n}$$

# y's	# ways	term
0	$\binom{n}{0}$	x^n
1	$\binom{n}{1}$	$x^{n-1}\cdot y$
2	$\binom{n}{2}$	$x^{n-2}\, y^2$
$\vdots$		$\vdots$
n	$\binom{n}{n}$	y^n

$$\binom{n}{0}\cdot x^n + \binom{n}{1}\cdot x^{n-1}\cdot y + \binom{n}{2}\cdot x^{n-2}\cdot y + \cdots + \binom{n}{n} y^n$$

$$= \boxed{\sum_{j=0}^{n} \binom{n}{j} x^{n-j}\cdot y^j}$$

Applications & Consequences

① What is $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots\binom{n}{n} = \, ? \, = 2^n$

Consider $S=\{a,b,c\}$

$$P(S) = \begin{array}{llll} 0 & \phi & 1 & = \binom{3}{0} \\ 1 & \{a\},\{b\},\{c\} & 3 & = \binom{3}{1} \\ 2 & \{a,b\},\{a,c\},\{b,c\} & 3 & = \binom{3}{2} \\ 3 & \{a,b,c\} & 1 & = \binom{3}{3} \end{array}$$

subset sizes

$$2^3 = |P(S)| = \binom{3}{0}+\binom{3}{1}+\binom{3}{2}+\binom{3}{3}$$

Claim: $\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n} = 2^n$

$(x+y)^n$

$$\boxed{\sum_{j=0}^{n}\binom{n}{j}x^{n-j}\cdot y^j}$$

$$2^n = (1+1)^n = \sum_{j=0}^{n}\binom{n}{j}\cdot 1^{n-j}\cdot 1^j = \sum_{j=0}^{n}\binom{n}{j}$$

$$= \binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n}$$

$$11^0 = 1$$
$$11^1 = 11$$
$$11^2 = 121$$
$$11^3 = 1331$$
$$11^4 = 14641$$

$$11^n = (\overset{x}{1} + \overset{y}{10})^n = \sum_{j=0}^{n} \binom{n}{j} \cdot 1^{n-j} \cdot 10^j$$
$$= \sum_{j=0}^{n} \binom{n}{j} \cdot 10^j$$
$$= \binom{n}{n} \cdot 10^n + \cdots + \binom{n}{1} \cdot 10 + \binom{n}{0} \cdot 10^0$$

$$11^3 = (1+10)^3 = \binom{3}{3} \cdot 10^3 + \binom{3}{2} \cdot 10^2 + \binom{3}{1} \cdot 10^1 + \binom{3}{0} \cdot 10^0$$
$$= 1 \cdot 10^3 + 3 \cdot 10^2 + 3 \cdot 10^1 + 1$$
$$= 1331$$

1

1 1

1 2 1

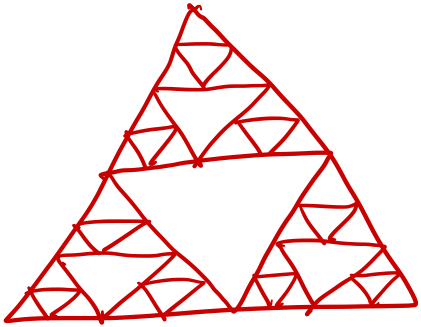
1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

⋮



Examples

(617) 373- 8169

Telephone #s: Former rules in US & Canada

508

312

Area codes: 3- digits First is **not** 0 or 1

Second digit must be 0 or 1

Exchange: 3- digits First & second digit **not** 0 or 1

Line #: 4- digits, not all zeroes

How many?

$$\overbrace{(8\cdot 2\cdot 10)}^{\text{area code}}\cdot\overbrace{(8\cdot 8\cdot 10)}^{\text{exchange}}\overbrace{(10^4-1)}^{\text{line\#}}$$

product product (all poss. − violations)

$$= 1{,}023{,}897{,}600$$

Dinner Party

16 people circular table

Q: How many unique circular arrangements of people?

two ways to think about it

- 16! permutations
- but many are rotations of each other
- there 16 rotations

$$\Rightarrow \frac{16!}{16} = 15!$$

- sit first person down anywhere
  - one way, since chairs are indistinguishable in circular arrang.
- arrange other 15 around that person

$$\Rightarrow 1 \cdot 15! = 15!$$

A C D B
D B A C

A C D B
C D B A
D B A C
B A C D

- Same setup: 16 people, circular table
    - same 16 people
    - I attend
    - I always talk to 2 people next to me

Q: How many dinner parties until <u>guaranteed</u> that I talk to same person twice?

A: - pigeonhole principle
   - 15 people besides me → 15 slots
   - each party – talk to 2 people
   - at 8^{th} party, will have talked to 16 people total
   - only 15 people besides me

⟹ by PHP, must have talked to someone twice

Halloween Candy

- n packs of M&Ms, give to k children, $n \geq k$
- how many ways? balls-in-bins

$$\binom{n+k-1}{k-1}$$

- suppose instead each child must get at least one pack
  - now how many ways?

10 packs
6 children

→ distribute k packs, one per child
→ $n-k$ left, balls-in-bins

$$\binom{(n-k)+(k-1)}{(k-1)} = \binom{n-1}{k-1}$$