

Last time

Induction

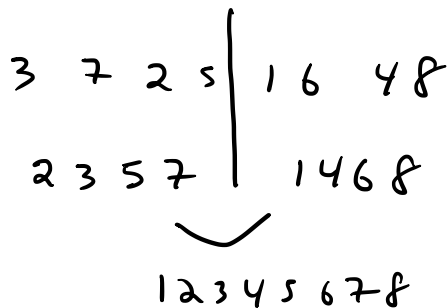
Today

- Recurrence
- Growth of functions

Next time

- Finish GOF
- Order notation

Merge Sort



$$\begin{array}{c} T(n) \\ \parallel \\ T(n/2) + T(n/2) \\ n \end{array}$$

Reality: $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 15n + 6$

$$\boxed{T(n) = 2 \cdot T(n/2) + n}$$

but... asymptotically and in terms of order notation, this is equivalent to just

$$T(n) = 2 \cdot T(n/2) + n$$

$$T(n) = \begin{cases} 1 & \text{if } n=1 \\ 2T(n/2) + n & \text{if } n \geq 2 \end{cases}$$

$$\underline{T(n) = 2T(n/2) + n \quad ; \quad T(1) = 1}$$

$$T(\boxed{n}) = \boxed{n} + 2T(\boxed{n}/2)$$

↑ pattern

$$T(n) = n + 2T(n/2)$$

$$= n + 2 \left[n/2 + 2 \cdot T(n/2^2) \right]$$

$$= n + n + 2^2 \cdot T(n/2^2)$$

$$= 2 \cdot n + 2^2 T(n/2^2)$$

$$\sim = 2 \cdot n + 2^2 \cdot \left[n/2^2 + 2 \cdot T(n/2^3) \right]$$

$$= 2n + n + 2^3 \cdot T(n/2^3)$$

$$= 3n + 2^3 \cdot T(n/2^3)$$

$$\sim = 4n + 2^4 \cdot T(n/2^4)$$

⋮

k times

$$= kn + 2^k \cdot T(n/2^k)$$

conjecture for pattern for any
k iterations.

Q: For what value of k is $n/2^k = 1$

$$\Leftrightarrow n = 2^k$$

$$\Leftrightarrow k = \log_2 n$$

→ at this point,
recursive procedure
terminates with
 $T(1) = 1$

$$T(n) = k \cdot n + 2^k \cdot T(n/2^k)$$

at
 $k = \log_2 n$

$$= (\log_2 n) \cdot n + 2^{\log_2 n} \cdot T(n/2^{\log_2 n})$$

$$= n \log_2 n + n \cdot T(1)$$

$$\log \frac{a}{b} = \log a - \log b$$

$$= n \log_2 n + n \cdot T(1)$$

$$= n \log_2 n + n \cdot 1$$

$$= n \log_2 n + n \quad \checkmark$$

$$T(n) = n \log_2 n + n$$

$$T(n) = 2 \cdot T(n/2) + n$$

check
solution

$$n \log_2 n + n \stackrel{?}{=} 2 \cdot \left\{ \frac{n}{2} \log_2 \frac{n}{2} + \frac{n}{2} \right\} + n$$

$$= n \log_2 \frac{n}{2} + n + n$$

$$= n \cdot (\log_2 n - \log_2 2) + n + n$$

$$= n \cdot (\log_2 n - 1) + n + n$$

$$= n \log_2 n - \cancel{n} + \cancel{n} + n$$

$$= n \log_2 n + n \quad \checkmark$$

Claim: $\forall k \geq 1$, the iterative pattern

$$T(n) = k \cdot n + 2^k T(n/2^k) \quad \text{holds.}$$

$$T(n) = n + 2 T(n/2)$$

Proof (by weak induction):

B.C. $k=1$ $T(n) = 1 \cdot n + 2^1 T(n/2^1)$
 $= n + 2 \cdot T(n/2)$

(get back original recurrence)

I.S. Show that if true for $k-1$, then true for k

$$\begin{aligned} T(n) &= (k-1) \cdot n + 2^{k-1} \cdot T(n/2^{k-1}) \\ &= (k-1)n + 2^{k-1} \left[\frac{n}{2^{k-1}} + 2 T\left(\frac{n/2^{k-1}}{2}\right) \right] \\ &= (k-1) \cdot n + n + 2^k T(n/2^k) \\ &= k \cdot n + 2^k T(n/2^k) \quad \checkmark \end{aligned}$$

E.g. $T(n) = 2 \cdot T(n/2) + n^2$; $T(1) = 1$

$$T(\boxed{n}) = n^2 + 2 \cdot T(n/2)$$

$$T(n) = n^2 + 2 T(n/2)$$

$$= n^2 + 2 \left[(n/2)^2 + 2 \cdot T(n/2) \right]$$

$$= n^2 + \frac{n^2}{2} + 2^2 \cdot T(n/2^2)$$

$$= n^2 + \frac{n^2}{2} + 2^2 \cdot \left[(n/2^2)^2 + 2 \cdot T(n/2^2) \right]$$

$$= n^2 + \frac{n^2}{2} + \frac{n^2}{2^2} + 2^3 \cdot T(n/2^3)$$

$$= n^2 + \frac{n^2}{2} + \frac{n^2}{2^2} + 2^3 \left[(n/2^3)^2 + 2 \cdot T(n/2^3) \right]$$

$$= n^2 + \frac{n^2}{2} + \frac{n^2}{2^2} + \frac{n^2}{2^3} + 2^4 \cdot T(n/2^4)$$

$$= n^2 \left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} \right) + 2^5 \cdot T(n/2^5)$$

$$\vdots$$

$$k \text{ times} \Rightarrow n^2 \left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^{k-1}} \right) + 2^k T(n/2^k)$$

$$= n^2 \sum_{i=0}^{k-1} \frac{1}{2^i} + 2^k T(n/2^k)$$

$$\frac{n}{2^k} = 1 \Leftrightarrow k = \log_2 n$$

$$T(n) = n^2 \sum_{i=0}^{k-1} \frac{1}{2} i + 2^k T(n/2^k)$$

$$\sum_{i=0}^k r^i = \frac{1-r^{k+1}}{1-r}$$

$$k = \log_2 n$$

$$= n^2 \sum_{i=0}^{\log_2 n - 1} \frac{1}{2} i + 2^{\log_2 n} T(n/2^{\log_2 n})$$

$$= n^2 \sum_{i=0}^{\log_2 n - 1} \left(\frac{1}{2}\right)^i + n \cdot T(1)$$

$$= n^2 \cdot \sum_{i=0}^{\log_2 n - 1} \left(\frac{1}{2}\right)^i + n$$

$$= n^2 \frac{1 - \left(\frac{1}{2}\right)^{\log_2 n}}{1 - \frac{1}{2}} + n$$

$$= n^2 \cdot \frac{1 - \frac{1}{2^{\log_2 n}}}{1/2} + n$$

$$= n^2 \frac{1 - 1/n}{1/2} + n = n^2 (2 - 2/n) + n$$

$$= 2n^2 - 2n + n = \boxed{2n^2 - n}$$

check $T(n) = 2T(n/2) + n^2$

$$2n^2 - n \stackrel{?}{=} 2 \left[2 \cdot \left(\frac{n}{2}\right)^2 - \left(\frac{n}{2}\right) \right] + n^2$$

$$= n^2 - n + n^2$$

$$= 2n^2 - n \quad \checkmark$$

claim: $\forall k \geq 1$, the iterative pattern

$$T(n) = n^2 + 2 \cdot T(n/2)$$

$$T(n) = n^2 \cdot \sum_{i=0}^{k-1} \frac{1}{2^i} + 2^k T(n/2^k) \text{ holds.}$$

Pf by induction: B.C. $k=1$ $T(n) = n^2 \cdot \sum_{i=0}^{1-1} \frac{1}{2^i} + 2^1 \cdot T(n/2^1)$

$$= n^2 \cdot \frac{1}{2^0} + 2 T(n/2)$$

$$= n^2 + 2 T(n/2) \quad \checkmark$$

I.S. Show that if true for $k-1$,
then true for k

$$T(n) = n^2 \sum_{i=0}^{k-2} \frac{1}{2^i} + 2^{k-1} T\left(\frac{n}{2^{k-1}}\right)$$

$$= n^2 \sum_{i=0}^{k-2} \frac{1}{2^i} + 2^{k-1} \left(\left(\frac{n}{2^{k-1}}\right)^2 + 2 T\left(\frac{n/2^{k-1}}{2}\right) \right)$$

$$= n^2 \sum_{i=0}^{k-2} \frac{1}{2^i} + \frac{n^2}{2^{k-1}} + 2^k T(n/2^k)$$

$$= n^2 \sum_{i=0}^{k-1} \frac{1}{2^i} + 2^k T(n/2^k) \quad \checkmark$$

$$T(n) = 4T(n/2) + n$$

$$T(\frac{n}{2}) = \frac{n}{2} + 4T(\frac{n}{4})$$

$$T(n) = n + 4T(n/2)$$

$$= n + 4 \left(\frac{n}{2} + 4T(n/4) \right)$$

$$= n + \frac{4}{2}n + 4^2 T(n/4)$$

$$= n + \frac{4}{2}n + 4^2 \left(\frac{n}{2^2} + 4T(n/8) \right)$$

$$= n + \frac{4}{2}n + \frac{4^2}{2^2}n + 4^3 T(n/8)$$

$$= n + \frac{4}{2}n + \left(\frac{4}{2}\right)^2 n + \left(\frac{4}{2}\right)^3 n + 4^4 T(n/16)$$

$$= n + \left(\frac{4}{2}\right)n + \left(\frac{4}{2}\right)^2 n + \dots + \left(\frac{4}{2}\right)^{k-1} n + 4^k T(n/2^k)$$

$$= n (1 + 2 + 2^2 + 2^3 + \dots + 2^{k-1}) + 4^k T(n/2^k)$$

$$= n \cdot \sum_{i=0}^{k-1} 2^i + 4^k T(n/2^k)$$

$$\frac{n}{2^k} = 1 \Leftrightarrow k = \log_2 n$$

$$T(n) = n \cdot \sum_{i=0}^{k-1} 2^i + 4^k T(n/2^k)$$

$$k = \log_2 n$$

$$= n \cdot \sum_{i=0}^{\log_2 n - 1} 2^i + 4^{\log_2 n} T(n/2^{\log_2 n})$$

$$= n \cdot \frac{1-2^{\log_2 n}}{1-2} + n^{\log_2 4} \cdot T(1)$$

$$= n \cdot \frac{1-n}{-1} + n^2 \cdot 1$$

$$= n(n-1) + n^2$$

$$= n^2 - n + n^2$$

$$= \boxed{2n^2 - n}$$

Facts

$$(1) \sum_{i=0}^k r^i = \frac{1-r^{k+1}}{1-r}$$

$$(2) a^{\log_b c} = c^{\log_b a}$$

- to show correct,
take $\log_b$
of both sides.

To finish proof, prove
the pattern correct
by induction.