The Inclusion-Exclusion Principle

1. The probability that at least one of two events happens

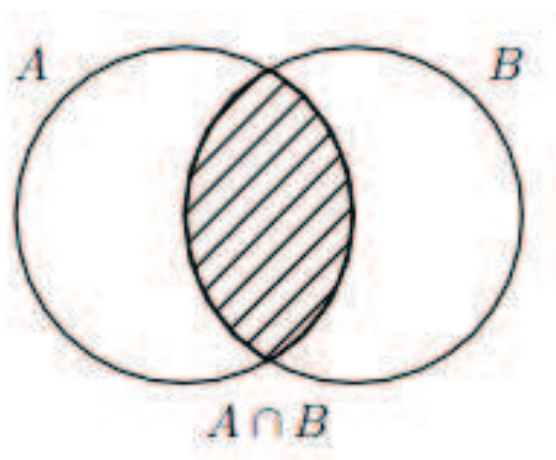
Consider a discrete sample space Ω. We define an event A to be any subset of Ω, which in set notation is written as $A \subset \Omega$. Then, Boas asserts in eq. (3.6) on p. 732 that[1]

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)\,, \tag{1}$$

for any two events $A, B \subset \Omega$. This is equivalent to the set theory result,

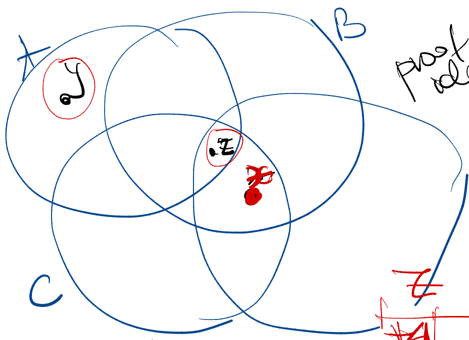
$$|A \cup B| = |A| + |B| - |A \cap B|\,, \tag{2}$$

where the notation $|A|$ means the number of elements contained in the set A, etc. In writing eq. (2), we have assumed that A and B are two finite discrete sets, so the number of elements in A and B are finite.



The proof of eq. (2) is immediate after considering the Venn diagram shown above. In particular, adding the number of elements of A and B overcounts the number of elements in $A \cup B$, since the events in $A \cap B$ have been double counted. Thus, we correct this double counting by subtracting the number of elements in $A \cap B$, which yields eq. (2). The corresponding result in probability theory is given by eq. (1).

[1]Boas uses a nonstandard notation by writing $A + B$ for $A \cup B$. The latter is standard in set theory and we shall use it in these notes. $A \cup B$ means the union of the sets A and B and is equivalent to the "inclusive or," i.e. "either A or B or both." Likewise, Boas uses a nonstandard notation by writing AB for $A \cap B$. Again, the latter is standard in set theory and we shall use it in these notes. $A \cap B$ means the intersection of the sets A and B, or equivalently "both A *and* B."



proof idea
• look at all regions (disjoint
— maybe at each element x, y, z
— make sure the total count right side
$+ - + - + \cdots = 1$
count x on right side of PIE:

x	A	B	C	D
	✓	✓	✓	✓

x		Count of x	y
$+4$	$+\sum$ "$\cap$ of 1" : $\lvert A\rvert+\lvert B\rvert+\lvert C\rvert+\lvert D\rvert$	3	$+1$
-6	$-\sum$ "$\cap$ of 2" : $-\lvert A\cap B\rvert-\lvert A\cap C\rvert-\lvert A\cap D\rvert-\lvert B\cap C\rvert-\lvert B\cap D\rvert-\lvert C\cap D\rvert$	-3	0
$+4$	$+\sum$ "$\cap$ of 3" $+\lvert A\cap B\cap C\rvert+\lvert A\cap B\cap D\rvert+\lvert A\cap C\cap D\rvert+\lvert B\cap C\cap D\rvert$	$+1$	0
-1	$-$ "$\cap$ of 4" $\lvert A\cap B\cap C\cap D\rvert$	0	0
$+1$		$+1$	✓

2. The probability that at least one of three events happens

It is straightforward to generalize the result of eq. (1) to the case of three events.[2]

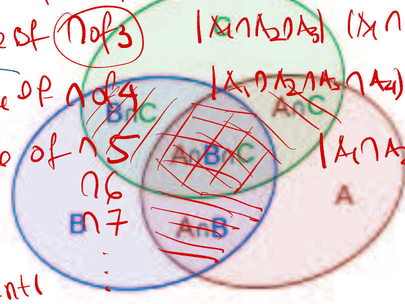
$$P(A\cup B\cup C) = P(A)+P(B)+P(C)-P(A\cap B)-P(A\cap C)-P(B\cap C)+P(A\cap B\cap C)\,, \quad (3)$$

for any three events $A, B, C \subset \Omega$. This is equivalent to the set theory result,

$$|A\cup B\cup C| = \boxed{|A|+|B|+|C|} - \boxed{|A\cap B| - |A\cap C| - |B\cap C|} + \boxed{|A\cap B\cap C|}\,. \quad (4)$$

→ 3 sets

| union of n sets | $= \sum$ sizes of 1-set $|A_1|+|A_2| \cdots + |A_n|$
$|A_1\cup A_2 \cdots A_n|$ $-\sum$ size of $\cap$ of 2 $|A_1\cap A_2|$ $|A_2\cap A_3|$ $|A_1\cap A_3| \cdots |A_{n-1}\cap A_n|$
$+\sum$ size of $\cap$ of 3 $|A_1\cap A_2\cap A_3|$ $|A_1\cap A_2\cap A_4| \cdots |A_{n-2}\cap A_{n-1}\cap A_n|$
more than 3 sets $-\sum$ size of $\cap$ of 4 $|A_1\cap A_2\cap A_3\cap A_4|$ $|A_1\cap A_2\cap A_3\cap A_5| \cdots |A_{n-3}\cap A_{n-2}\cap A_{n-1}\cap A_n|$
$+\sum$ size of $\cap$ 5 $|A_1\cap A_2\cap A_3\cap A_4\cap A_5| \cdots$
$\cap 6$
$\cap 7$
$-$
$+$
$-$
$+$



? $(-1)^{\text{power}=n+1} \sum$ size of $\cap$ of all (n) $|A_1\cap A_2\cap A_3 \cdots \cap A_{n-1}\cap A_n|$

Once again, the proof of eq. (4) is immediate after considering the Venn diagram shown above.[3] In particular, adding the number of elements of A, B and C counts elements in $A\cap B\cap C$ three times, and counts elements of $A\cap B$, $A\cap C$ and $B\cap C$ not contained in $A\cap B\cap C$ twice. Thus, $|A\cup B\cup C| = |A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|$ will include all events in A, B and C once except for the events in $A\cap B\cap C$, which were all subtracted off. Thus, to include all events in $A\cup B\cup C$ exactly once, we must add back the number of events in $A\cap B\cap C$. Thus, eq. (4) is established. The corresponding result in probability theory is given by eq. (3).

3. The Inclusion-Exclusion principle

The inclusion-exclusion principle is the generalization of eqs. (1) and (2) to n sets. Let $A_1, A_2, \ldots, A_n$ be a sequence of n events. Then,

$$P(A_1\cup A_2\cup\cdots\cup A_n) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i\cap A_j) + \sum_{i<j<k} P(A_i\cap A_j\cap A_k)$$
$$- \sum_{i<j<k<\ell} P(A_i\cap A_j\cap A_k\cap A_\ell) + \ldots + (-1)^{n+1} P(A_1\cap A_2\cap\cdots\cap A_n)\,, \quad (5)$$

where $A_1, A_2, \ldots A_n \subset \Omega$. This is equivalent to the set theory result,

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k|$$

$$- \sum_{i<j<k<\ell} |A_i \cap A_j \cap A_k \cap A_\ell| + \ldots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n| \,. \quad (6)$$

The proof of eq. (6) is an exercise in counting. Suppose a point is contained in exactly m of the sets, A_1, $A_2, \ldots A_n$, where m is a number between 1 and n. Then, the point is counted m times in $\sum_{i=1}^{n} |A_i|$, it is counted $C(m, 2)$ times in $\sum_{i<j} |A_i \cap A_j|$, it is counted $C(m, 3)$ times in $\sum_{i<j<k} |A_i \cap A_j \cap A_k|$, etc., where $C(m, k)$ is the number of combinations of m objects taken k at a time. After reaching $\sum_{i_1<i_2<\cdots i_m} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_m}|$, where the point is counted once [since $C(m, m) = 1$], one finds that the point is not counted at all in any of the terms that involve the intersection of more than m sets. The net result is that a point that is contained in exactly m of the sets will be counted S times in $|A_1 \cup A_2 \cup \cdots \cup A_n|$ given by eq. (6), where

$$S \equiv C(m, 1) - C(m, 2) + C(m, 3) - C(m, 4) + \cdots + (-1)^{m+1} C(m, m) \,, \quad (7)$$

after noting that $C(m, 1) = m$.

To compute S, we recall the binomial theorem,

$$(x + y)^m = \sum_{k=0}^{m} C(m, k) x^k y^{m-k} \,, \quad (8)$$

where

$$C(m, k) \equiv \binom{m}{k} \equiv \frac{m!}{k!(m-k)!}$$

is the number of combinations of m objects taken k at a time. Setting $x = 1$ and $y = -1$ in eq. (8) yields,

$$\sum_{k=0}^{m} (-1)^k C(m, k) = 0 \,.$$

Using $C(m, 0) = 1$, it follows that

$$1 - C(m, 1) + C(m, 2) - C(m, 3) + \ldots + (-1)^m C(m, m) = 0 \,,$$

which implies that $S = 1$ [cf. eq. (7)]. Thus, we have shown that there is no multiple counting of points in eq. (6). That is, every point contained in the union of $A_1, A_2, \ldots A_n$ is counted exactly one time. Thus, eq. (6) is established. The corresponding result in probability theory is given by eq. (5). We have therefore verified the inclusion-exclusion principle.

There are numerous applications of the inclusion-exclusion principle, both in set theory and in probability theory. In particular, it provides a powerful tool for certain types of counting problems. An example is provided in the next section of these notes.

Examples: proof by contradiction

Ex 1 Last time $\sqrt{3} \notin \mathbb{Q}$, $\sqrt{3} \neq \frac{a}{b}$; $a,b \in \mathbb{Z}$

Recap: We want $P \Rightarrow C \equiv (\neg C \Rightarrow \neg P)$
direct / contrapositive

proof by contrad.
Assume C is false
$\neg C \Rightarrow \ldots \Rightarrow \ldots \Rightarrow \ldots \Rightarrow \neg P$ CONTRADICTION
not (concl) ... not (premise) "contradicts P"
$\Rightarrow C$ is true

Ex 2 The sequence of prime numbers $p_1, p_2, p_3, p_4, p_5, p_6, \ldots$ is infinite

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, ...

$45 = 5 \cdot 9 \Rightarrow$ not prime

proof by contradiction: Assume (hypothetical) conclusion is false

$\Rightarrow$ the sequence/list of primes is finite (There are only m of them)

the only primes are $p_1, p_2, \ldots p_{m-1}, p_m$ done, no other prime

$\Longrightarrow$ construct a number bigger than p_m

$$\boxed{Z = p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot \ldots \cdot p_m + 1}$$

not have any p_i as possible factors

Product of all primes $+1$

is Z prime? $Z > p_m$, p_m is the biggest prime $\Rightarrow Z$ is $\boxed{\text{not prime}}$

$\Rightarrow Z$ is divisible by one of the primes $p_1, p_2, \ldots p_m$

$$p_i \mid (p_1 \cdot p_2 \cdot p_3 \cdot p_i \cdot \ldots \cdot p_m + 1) \Rightarrow p_i \mid 1 \quad \text{contradicts}$$

diff between multiples of p_i

$\Longrightarrow$ list of primes infinite.

used

Th
$$\left.\begin{matrix} k \mid a \\ k \mid b \end{matrix}\right\} \Rightarrow k \mid (a-b)$$
$$\Rightarrow k \mid (a+b)$$

"if a, b are multiples of k,
$\Rightarrow$ then $a-b$ is multiple of k"
$\Rightarrow$ then $a+b$ is multiple of k"

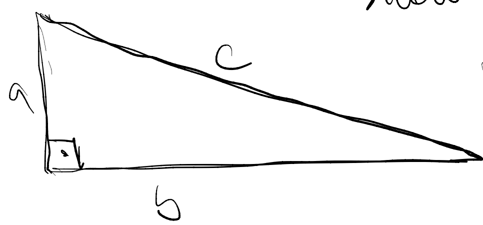
General version (N.T. holder): $(a \equiv b) \bmod k \Longleftrightarrow k \mid (a-b)$

"a and b have the same modulo/remainder with k
(not nec $a = b$)

Ex $27 \equiv 131 \bmod 13$

Ex 3 Right triangle sides a, b, long "hypotenuse"? c

show that $\boxed{c < a+b}$



proof Right $\triangle \Rightarrow$ Pythagorean th $a^2+b^2=c^2$ ✓

contradict: Assume conclusion is false

$\Rightarrow \neg(c < a+b) \Rightarrow c \geq a+b$

$(\ )^2 \Rightarrow c^2 \geq \underbrace{\underbrace{a^2+b^2}_{= c^2 \text{ (Pyth)}}+2ab}_{(a+b)^2}$

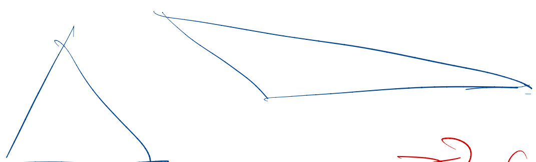
$\Rightarrow c^2 \geq c^2 + 2ab \quad \big| -c^2$

$\Rightarrow 0 \geq 2ab$ False! $(a, b > 0)$

$\Rightarrow$ conclusion is true. $c < a+b$

exercise

true for any triangle



one side < sum of the other 2

part A, Satisfiability Intro [easy]. A boolean formula is satisfiable if there exists some variable assignment that makes the formula evaluate to true. Namely, a boolean formula is satisfiable if there is some row of the truth table that comes out true. Determining whether an arbitrary boolean formula is satisfiable is called the *Satisfiability Problem*. There is no known efficient solution to this problem, in fact, an efficient solution would earn you a million dollar prize. While this is hard problem in computer science, not all instances of the problem are hard, in fact, determining satisfiability for some types of boolean formulae is easy.

i. First, let's consider why this would be hard. If you knew nothing about a given boolean formula other than that it had n variables, how large is the truth table you would need to construct? Please indicate the number of columns and rows as a function of n

ii. Now consider the following 100 variable formula.

$$x_1 \wedge (\neg x_1 \vee x_2) \wedge (\neg x_2 \vee x_3) \wedge (\neg x_3 \vee x_4) \wedge \ldots \wedge (\neg x_{99} \vee x_{100})$$

Without constructing a truth table, how many satisfying assignments does this formula have, explain your answer.

iii Now consider an arbitrary 3-DNF formula with 100 variables and 200 clauses. 3-DNF means that the formula is in disjunctive normal form and each clause has three literals. (A literal is the instantiation of the variable in the formula, so for x, $\neg x$ or x.) An example might be something like:

$$(\neg x_1 \wedge x_3 \wedge x_{10}) \vee (\neg x_3 \wedge x_{15} \wedge \neg x_{84}) \vee (x_{17} \wedge \neg x_{37} \wedge x_{48}) \vee \ldots \vee (\neg x_{87} \wedge \neg x_{95} \wedge x_{100})$$

What is the largest size truth table needed to solve this problem. What is the maximum number of such truth tables needed to determine satisfiabilty.

part B: 2CNF-SAT [hard]. The 2CNF-SAT instance is a boolean CNF formula with 2 variables in each clause, "OR" inside clauses, "AND" between clauses. There are m boolean variables $x_1, x_2, ..., x_m$) and n clauses $C_1, C_2, ..., C_n$). Every variable and its negation appears in at least one clause. Such formula is given as input in format redundantly :

- for each variable there is a list of clauses containing it
- for each clause there there are 2 variables

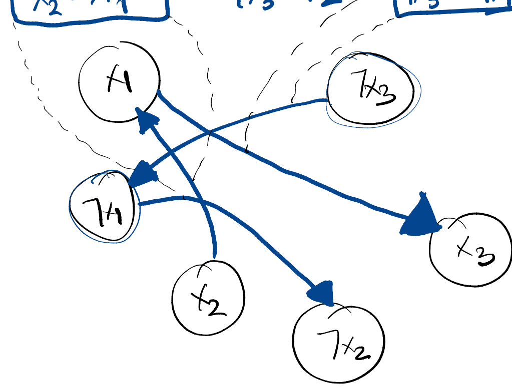
• how to think about 2CNF formulas
— every clause can be written as 2 implications (redundant)

For example the formula $\boxed{x_1 \vee \neg x_2} \wedge \boxed{x_2 \vee x_3} \wedge \boxed{\neg x_1 \vee x_3} \wedge \boxed{\neg x_2 \vee \neg x_3}$ will be given as:

$\boxed{x_1 \Rightarrow \neg x_2}$ $\wedge$ $\neg x_2 \Rightarrow x_3$ $\wedge$ $\boxed{x_1 \Rightarrow x_3}$ $\wedge$ $x_2 \Rightarrow \neg x_3$

$\boxed{x_2 \Rightarrow x_1}$ $\neg x_3 \Rightarrow x_2$ $\boxed{\neg x_3 \Rightarrow \neg x_1}$ $x_3 \Rightarrow \neg x_2$

$m = 3, n = 4$
$x_1 : C_1$
$\neg x_1 : C_3$
$x_2 : C_2$
$\neg x_2 : C_1, C_4$
$x_3 : C_2, C_3$
$\neg x_3 : C_4$
$C_1 : x_1, \neg x_2$
$C_2 : x_2, x_2$
$C_3 : \neg x_1, x_3$
$C_4 : \neg x_2, \neg x_3$



• draw these implication (not nec, but useful)
• trial and error start literal = T
for example try $x_2 = T \Rightarrow x_1 = T \Rightarrow x_3 = T$..
see if it works out.

Your task is to design a strategy that determines, for a given formula, the boolean assignments for the variables such that all clauses are satisfied, thus the formula is true (if more such assignments are possible, you only need to output one). If no such assignment is possible, output "FALSE".

As established inpart A, there are 2^m possible assignments for the variable set. So if one were to build the truth table and "brute force" search all rows/assignments until one works, it would take exponential time — not good! Instead: do trial and error, but in a smart way that only tries at most $2 * m^2$ boolean assignments.

Your strategy can be pseudocode, or you can informally describe a procedure with bullets and English statements. You can write in your procedure statements like

* $x = x_1$

2CNF
$(x_2 \vee \neg x_3) \wedge (x_3 \vee x_1) \wedge (x_2 \vee \neg x_1) \wedge (x_3 \vee \neg x_2)$

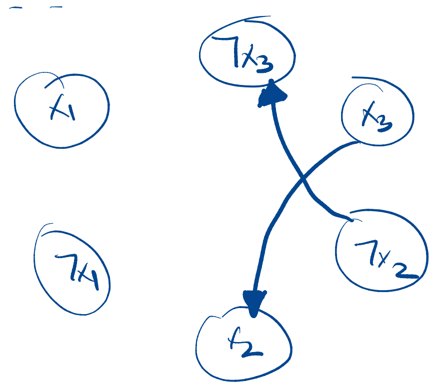
* foreach C containing variable x {
- - - -
}

$\boxed{x_3 \Rightarrow x_2}$
$\neg x_2 \Rightarrow \neg x_3$

* C= next clause, or C = next clause containing x

* loop C through all clauses that contain x or $\neg x$

* for each $x \in C$ {
- - - -
}



* y = the other variable in clause C, other than x or $\neg x$