# Counting technique: indexing / mapping

f = map

map = one-to-one function (bijection)

example

$A = \{1, 2, 3, \dots, 10\}$

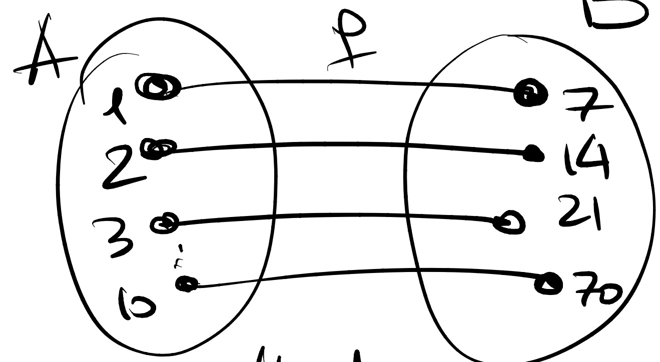
$B = \{x \in \mathbb{N};\ 2 \le x \le 72,\ x = 7k\}$ multiple of 7

$= \{7, 14, 21, \dots, 70\}$

$f: A \to B$ bijection (one-to-one)

$f(x) = 7 \cdot x$

x in A, 7x in B




called indexing if

$A = \{1, 2, 3, \dots, n\}$

$A = \{1 : n\}$

Th $\exists\, f: A \to B$ one-to-one $\Longrightarrow |A| = |B|$

$\mathbb{Z}_n$ = remainders at integer-division with $n = \{0, 1, 2, \dots, n-1\}$

$\mathbb{Z}_{10} = \{0, 1, 2, \dots 9\}$

$\mathbb{Z}_2 = \{0, 1\}$

$\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$.

$\{\text{all pairs } (x \in \mathbb{Z}_2, y \in \mathbb{Z}_5)\}$

$$\mathbb{Z}_{10} \xleftrightarrow{f} \mathbb{Z}_2 \times \mathbb{Z}_5$$

$$f(x) \longleftrightarrow (x \bmod 2, \; x \bmod 5)$$

$$x = 3 \longleftrightarrow (1, \; 3)$$

$$x = 7 \longleftrightarrow (1, \; 2)$$

$$x = 4 \longleftrightarrow (0, \; 4)$$

$X =$ set   example $X = \{a, b, c, d\}$   $a \in X$

$A \subset \mathcal{P}(X)$   $A = \{$subsets of $X$ $\underline{\text{include}}$ "$a$"$\}$   $A \cup B = \mathcal{P}(X)$

$\{a\}$ $\{a,b\}$ $\{a,b,c\}$ ...

$B \subset \mathcal{P}(X)$   $B = \{$subsets of $X$ $\underline{\text{do not}}$ include "$a$"$\}$

$\emptyset$, $\{b\}$, $\{c\}$ $\{b,c\}$ $\{b,d\}$ $\{d\}$ ...

$|A| = |B|$ ?

$$A \xleftrightarrow[\text{map}]{f} B$$

one-to-one

$$T \in A \xleftrightarrow{f \text{ one-to-one}} S \text{ in } B$$

$\boxed{T = \text{subset of } X \; \underline{a \in T}}$   $\boxed{S \text{ subset of } X \; a \notin S}$

$$f(T) = T \setminus \{a\} \; \checkmark \;\Rightarrow\; |A| = |B| = \frac{|\mathcal{P}(X)|}{2}$$

* $X$ set $a \in X$ $b \in X$ $a \neq b$. $X = \{a, b, c, d, e\}$

$A$ = set of $\{$ all subsets of $X$ contain $a$ $a \in T$ $\}$

$\{a\}$ $\{a,b\}$ $\{a,c\}$ $\{a,b,c\}$ - - -

$B = \{$ all subsets of $X$ $S$ that contain "b" $\}$

$\{b\}$ $\{b,a\}$ $\{b,c\}$ $\{b,d\}$ $\{b,c,d\}$ $\{b,a,c\}$ - - -

$A \cap B = \emptyset$ ? $\{a,b\} \in A \cap B$

$\{a,b,c\} \in A \cap B$

$A \xleftrightarrow{f \text{ one to one}} B$ ? Yes $\Rightarrow |A| = |B|$

exercise

$\mathbb{N} = \{0, 1, 2, 3, 4 \ldots\}$ infinite, countable

$\mathbb{Z} = \{\ldots -3, -2, -1, 0, +1, +2, +3 \ldots\}$

$\mathbb{N} \subset \mathbb{Z}$

$\mathbb{Z} \setminus \mathbb{N} \neq \emptyset$ forexample $-1, -2,$

$\mathbb{N} \longleftrightarrow \mathbb{Z} \longrightarrow \boxed{|\mathbb{N}| = |\mathbb{Z}|}$ Countable infinite

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\mathbb{N}$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | — |
| $\mathbb{Z}$ | 0 | +1 | −1 | +2 | −2 | +3 | −3 | +4 | −4 | |

$x=2k$ $f(2k) = -k$

$x=2k+1$ $f(2k+1) = k-1$

Technique for counting sets: product rule
(hats × pants × jackets)

$A = \{a, b, c\}$ $B = \{1, 2\}$ $C = \{\alpha, \beta, \gamma\}$

want triplet $(\underline{x \in A}, \underline{y \in B}, \underline{z \in C})$

- any combination works

$(a, 1, \beta) \neq (1, \beta, a)$

# triplets : $|A| \cdot |B| \cdot |C|$

$3 \cdot 2 \cdot 3$

is $(1, a, \alpha)$ triplet?

NO

triplet = sequence

$A \times B \times C = \{(a, 1, \alpha), (a, 1, \beta), (a, 1, \gamma),$
$(a, 2, \alpha), (a, 2, \beta), (a, 2, \gamma),$
$\vdots$
$(c, 2, \alpha), (c, 2, \beta), (c, 2, \gamma)\}$

$n = 25$ students
$k = 3$ classrooms $\Rightarrow \exists$ one classroom with $\geq \lceil \frac{25}{3} \rceil = 9$

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10 people $x_1, x_2 \ldots x_{10}$ salaries avg 80,000 / sum of salaries is 800,000

$$\frac{x_1 + x_2 + \ldots + x_{10}}{10} = 80{,}000$$

$\Rightarrow$ at least one $\underline{x_i \geq 80{,}000}$
($\exists i$)

# Pigeon Hole principle

non-math version

- $n$ items placed in $n-1$ boxes (spots) $\Rightarrow$ $\exists$ at least one box with 2 items or more
- $n$ items placed on $k$ boxes $\Rightarrow$ $\exists$ at least one box with $\lceil \frac{n}{k} \rceil$ items

math-version

$x_1, x_2, x_3 \cdots x_n \in \mathbb{R}$ $\quad \mu = \dfrac{x_1 + x_2 \cdots + x_n}{n} = E[x]$

- at least one of them $x_i \geq \mu$ $\rightarrow$ prove by contradiction
- at least one of them $x_j \leq \mu$

assume $x_i < \mu \ \forall i$

$\sum x_i < n \cdot \mu$

$\sum x_i < \sum x_i$ ! contradiction

$\Rightarrow \exists i \ x_i \geq \mu$

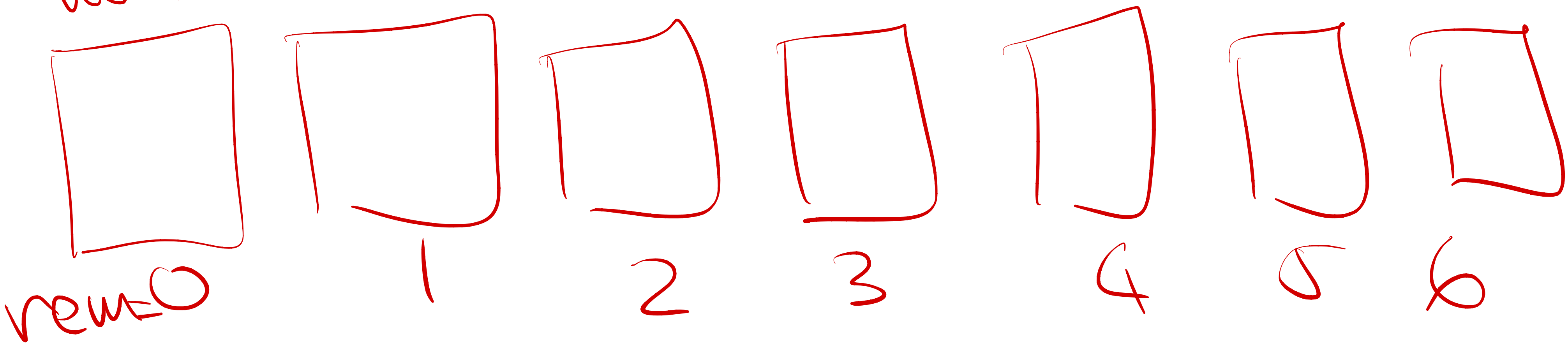
# PHP 1

100 integers $\Longrightarrow$ ($\exists$) select 15 of them

any diff of 2 = multiple of 7

$$a-b=7k \iff 7 \mid a-b \iff \boxed{a \equiv b \bmod 7}$$

mod 7 $\Longrightarrow$ 7 boxes

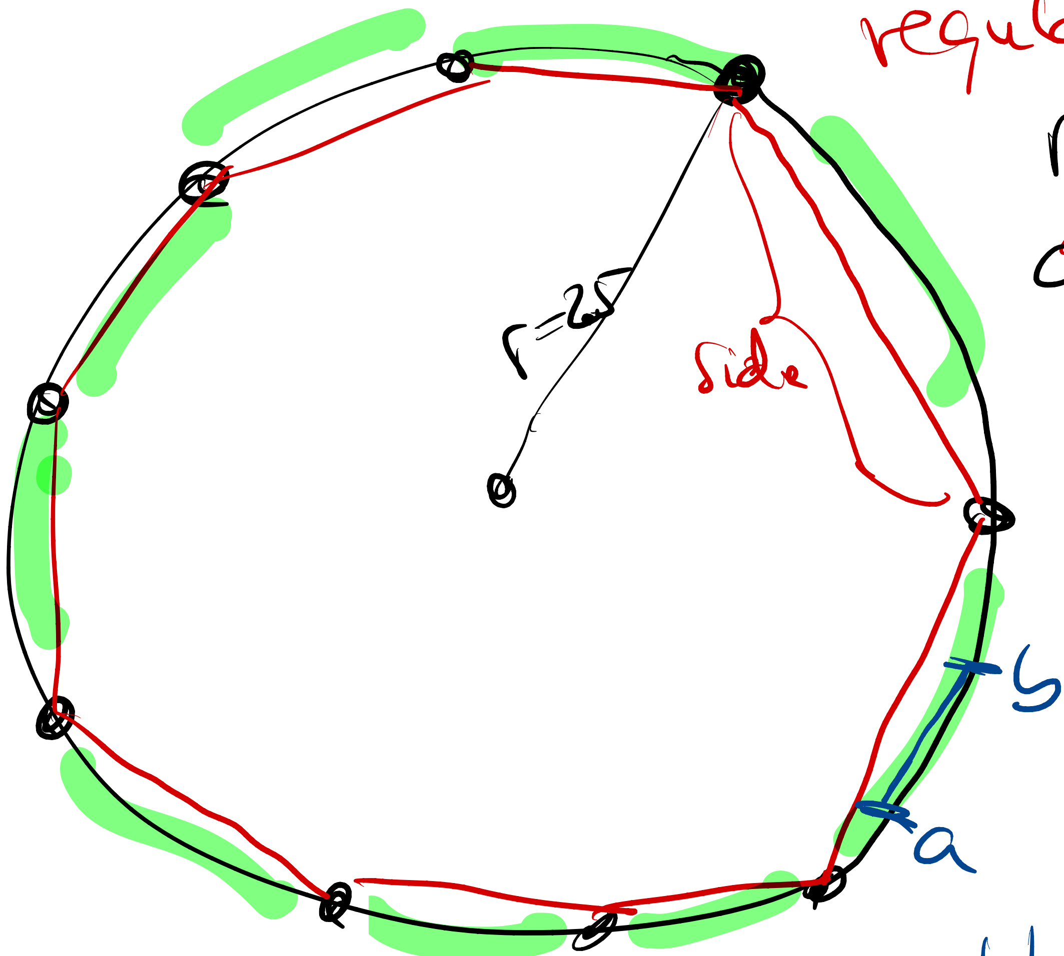



100 integers, 7 boxes $\Big\}$ $\xRightarrow{\text{PHP}}$ $\exists$ box with $\geq$ 1 box has at least $\left\lceil \frac{100}{7} \right\rceil = 15$

same rem at 15 $\Longrightarrow$ $|a-b|$ = multiple of 7.

10 point on a circle of diameter $= 2r = 5$

PHP2 $\Rightarrow \exists$ 2 of them at $\mathrm{dist}(a,b) < 2$.
a, b




regular 9-gon (equal sides)

$r = 2.5 \Rightarrow$ side $\simeq 1.71$?

9-gon splits circle into 9 regions (green)

10 points on circle

$\Downarrow$ PHP

2 of them same region

geom: $|ab| <$ side 9-gon $\simeq 1.71$

PB6 50 cats + 50 dogs in 9 rooms. What is min

A guaranteed to be in a room?

$$\left\lceil \frac{100}{9} \right\rceil = 12$$

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B per room:
no more than 6 cats ; at least 2 dogs

What is the maximum # animals in a room

R = max animals (room with) = 6 cats + max dogs

$\Rightarrow$ all other 8 rooms (except R) minimize # dogs

$8 \times 2 = 16$ dogs $\Longrightarrow$ R has $50 - 16 = 34$ dogs

$+ 6$ cats

40.

General pigeonhole principle.

ceiling

$p$ pigeons $h$ holes $\Rightarrow \left\lceil \frac{p}{h} \right\rceil$ pigeons in some hole.

<u>Example</u> 250 students

26 first letter of last name

$10 = \left\lceil \frac{250}{26} \right\rceil$ students with same first letter of last name.

Example 250 students 2letter initials

No guarantee two have same initials.

Example : cabinet with 10 black socks and 20 white socks. How many to guarantee matching pair?

Ans : 3

Example In any group of $n$ people there will be two with the exact same number of friends.

(Alt: any graph has two nodes with same degree).

Ans $n$ - pigeons, $n$ - holes, number of friends can be $0..n-2$ or $1...n-1$ since cannot have extrovert with $n-1$ friends and hermit with 0 friends simultaneously

So $n \leq n-1$. Thus two people

with exact same number of friends