Bit-probe lower bounds for succinct data structures

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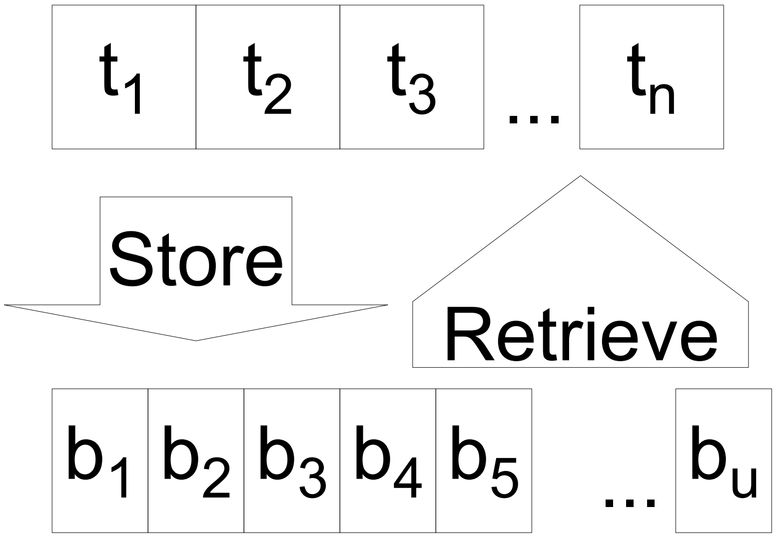
Bits vs. trits

- Store n "trits" $t_1, t_2, \ldots, t_n \in \{0,1,2\}$



In u bits $b_1, b_2, \ldots, b_u \in \{0,1\}$

- Want:

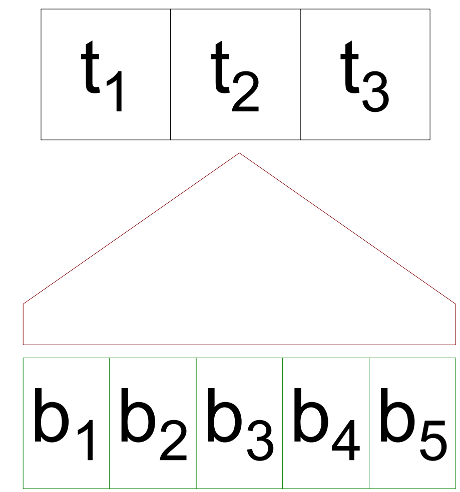
Small space u $\left(\text{optimal} = \lceil n \lg_2 3 \rceil\right)$

Fast retrieval: Get t_i by probing few bits $\left(\text{optimal} = 2\right)$

Two solutions

- Arithmetic coding:

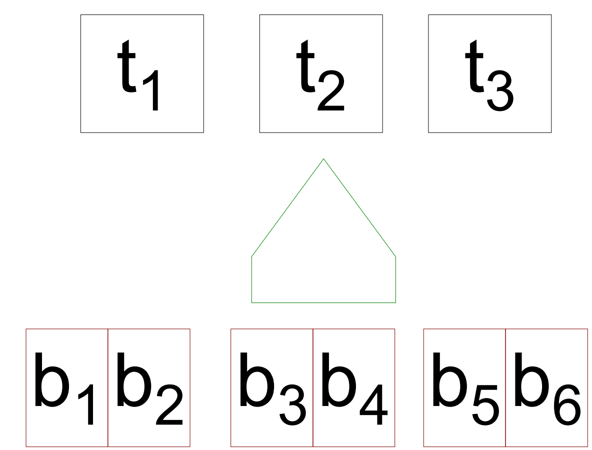
  Store bits of $(t_1, \ldots, t_n) \in \{0, 1, \ldots, 3^n - 1\}$



  Optimal space: $\lceil n \lg_2 3 \rceil \approx n \cdot 1.584$

  Bad retrieval: To get t_i probe all > n bits

- Two bits per trit

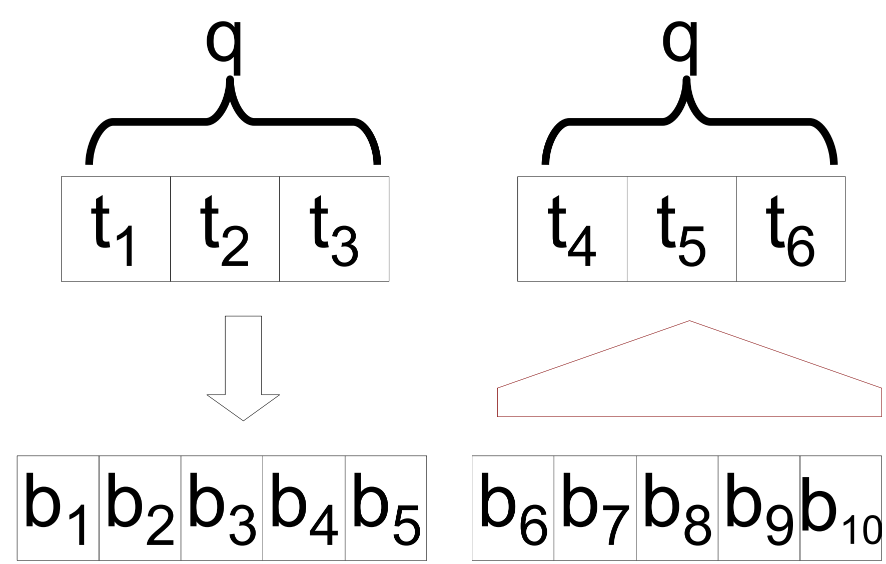


  Bad space: $n \cdot 2$

  Optimal retrieval: Probe 2 bits

Polynomial tradeoff

- Divide n trits $t_1, \ldots, t_n \in \{0,1,2\}$ in blocks of q
- Arithmetic-code each block



Space: $\lceil q \lg_2 3 \rceil n/q < (q \lg_2 3 + 1)\, n/q$

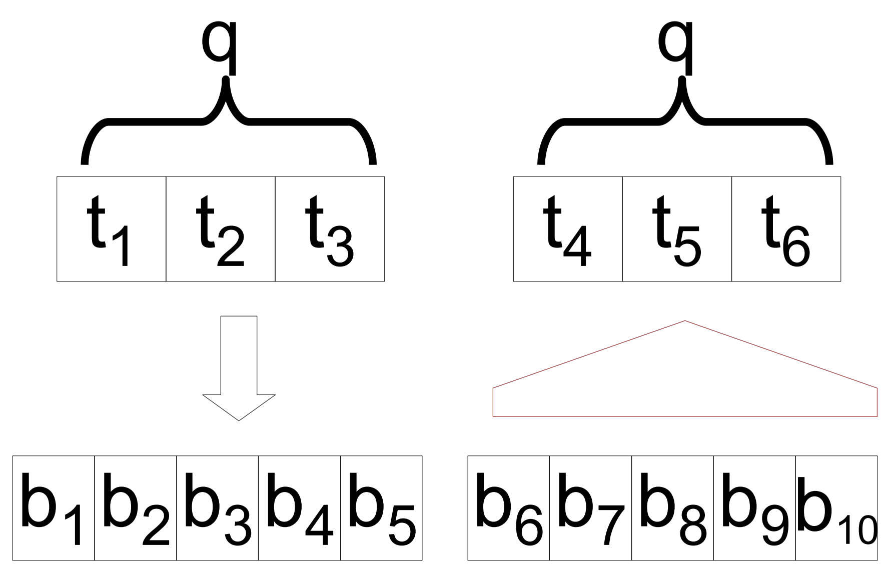
$= n \lg_2 3 + n/q$

Retrieval: Probe O(q) bits

polynomial tradeoff between redundancy, probes

Polynomial tradeoff

- Divide n trits $t_1, \ldots, t_n \in \{0,1,2\}$ in blocks of q



- Arithmetic-code each block

Space: $\lceil q \lg_2 3 \rceil n/q = (q \lg_2 3 + 1/q^{\Theta(1)})\, n/q$

$= n \lg_2 3 + n/q^{\Theta(1)}$

Retrieval: Probe $O(q)$ bits

polynomial tradeoff between redundancy, probes

Logarithmic forms

Exponential tradeoff

- Breakthrough [Pătraşcu '08, later + Thorup]

Space: $n \lg_2 3 + n/2^{\Omega(q)}$

Retrieval: Probe q bits

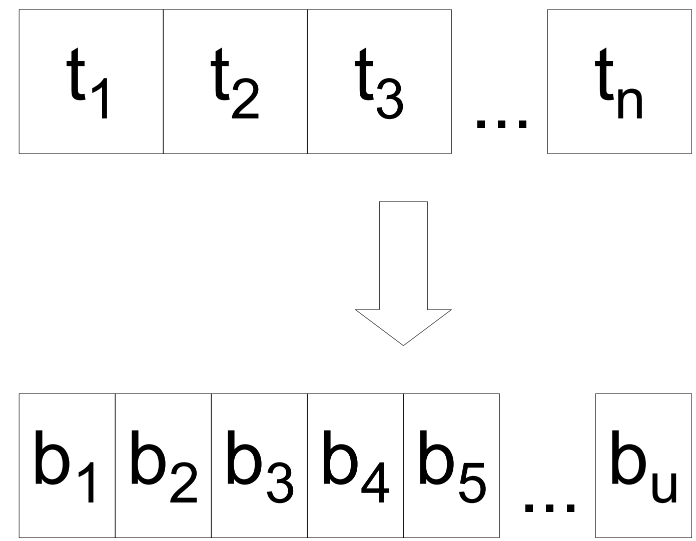
exponential tradeoff between redundancy, probes

- E.g., optimal space $\lceil n \lg_2 3 \rceil$, probe $O(\lg n)$

Our results

- Theorem[this work]:
  Store n trits $t_1, \ldots, t_n \in \{0,1,2\}$
  in u bits $b_1, \ldots, b_u \in \{0,1\}$.

  If get t_i by probing q bits
  then space $u > n \lg_2 3 + n/2^{O(q)}$.



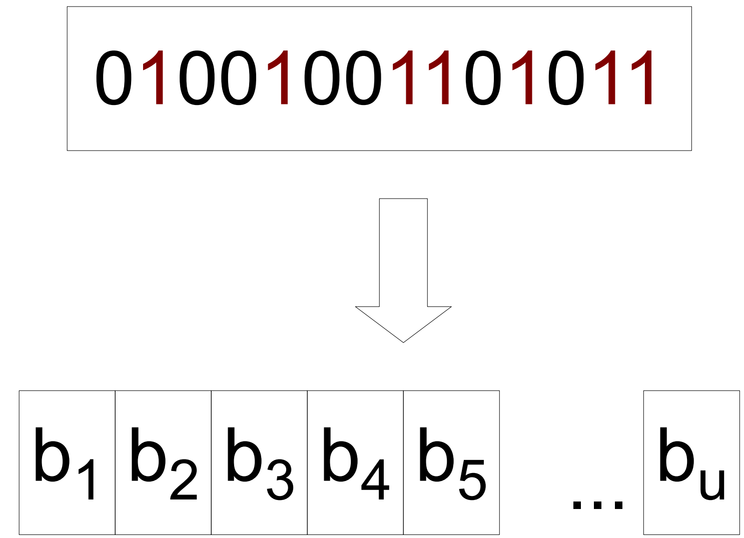
- Matches [Pătraşcu Thorup]: space $< n \lg_2 3 + n/2^{\Omega(q)}$
- Holds even for adaptive probes

Outline

- Bits vs. trits
- Bits vs. sets
- Proof

Bits vs. sets

- Store $S \subseteq \{1, 2, \ldots, n\}$ of size $|S| = k$



In u bits $b_1, \ldots, b_u \in \{0,1\}$

- Want:

Small space u $\left(\text{optimal} = \lceil \lg_2 (n \text{ choose } k) \rceil\right)$

Answer "$i \in S$?" by probing few bits $(\text{optimal} = 1)$

Previous results

- Store S $\subseteq$ {1, 2, … , n}, |S| = k in bits, answer "i $\in$ S?"

- [Minsky Papert '69] Average-case study

- [Buhrman Miltersen Radhakrishnan Venkatesh; Pagh '00]

  Space O(optimal), probe O(lg(n/k))

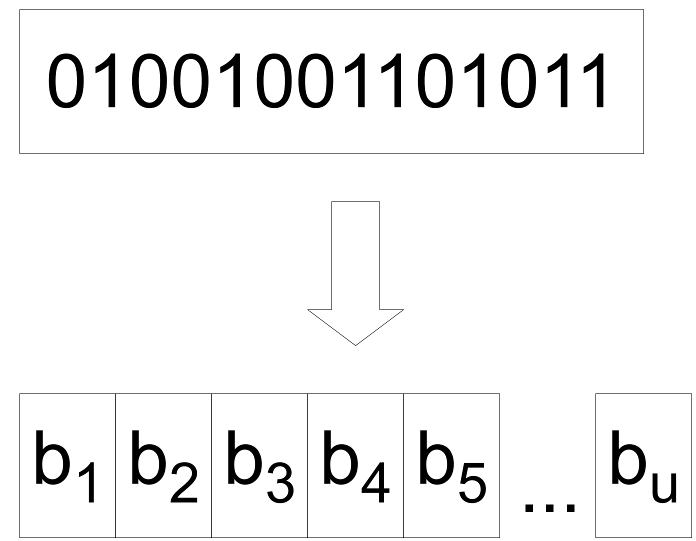
  Lower bounds for $k < n^{1-\varepsilon}$

- No lower bound was known for $k = \Omega(n)$

Our results

- Theorem[this work]:
  Store $S \subseteq \{1, 2, \ldots, n\}$, $|S| = n/3$
  in u bits $b_1, \ldots, b_u \in \{0,1\}$

  If answer "$i \in S$?" probing q bits
  then space $u >$ optimal $+ n/2^{O(q)}$.



- First lower bound for $|S| = \Omega(n)$

- Holds even for adaptive probes

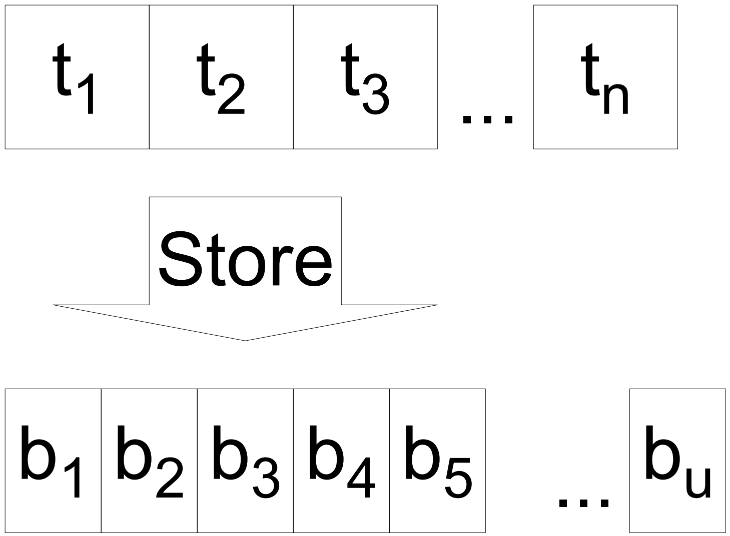
Outline

- Bits vs. trits
- Bits vs. sets
- Proof

Recall our results

- Theorem:
  Store n trits $t_1, \ldots, t_n \in \{0,1,2\}$
  in u bits $b_1, \ldots, b_u \in \{0,1\}$.

  If get t_i by probing q bits
  then space $u > n \lg_2 3 + n/2^{O(q)}$.



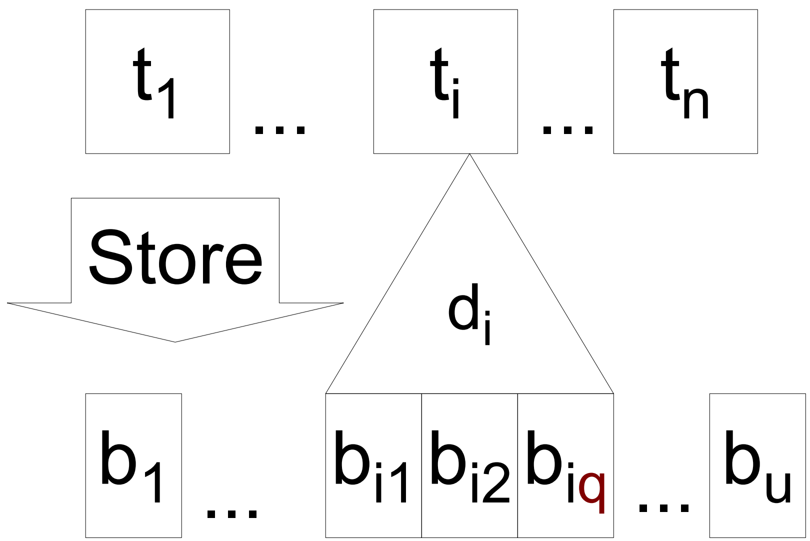
- For now, assume non-adaptive probes:
  $t_i = d_i (b_{i1}, b_{i2}, \ldots, b_{iq})$

Proof idea

- $t_i = d_i\ (b_{i1}, b_{i2}, \ldots, b_{iq})$

- Uniform $(t_1, \ldots, t_n) \in \{0,1,2\}^n$

  Let $(b_1, \ldots, b_u) := \text{Store}(t_1, \ldots, t_n)$



- Space $u \approx$ optimal $\Rightarrow (b_1, \ldots, b_u) \in \{0,1\}^u \approx$ uniform $\Rightarrow$

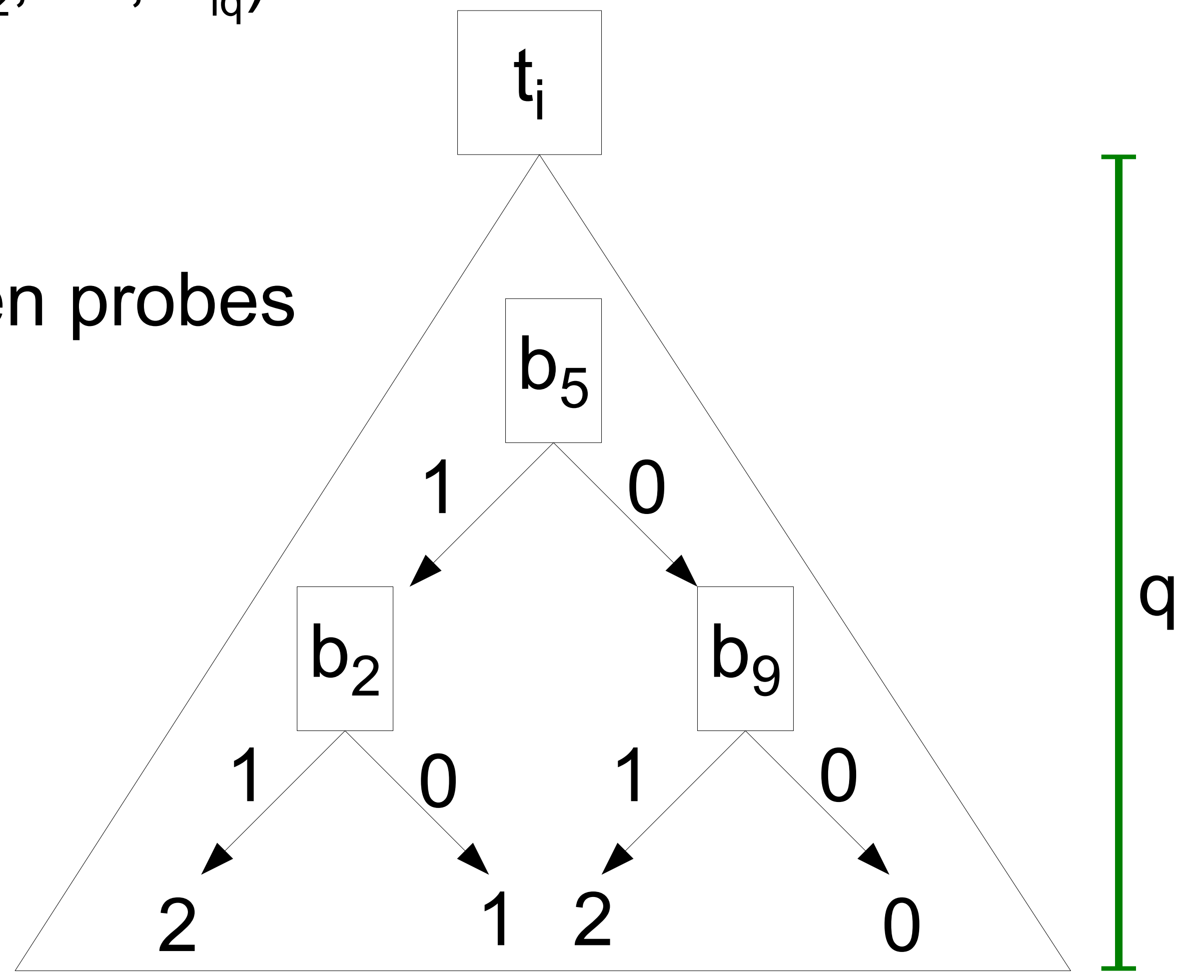
$$1/3 = \Pr[\, t_i = 2 \,] = \Pr[\, d_i\ (b_{i1}, \ldots, b_{iq}) = 2 \,] \approx A / 2^q \neq 1/3$$

Contradiction, so space u >> optimal Q.e.d.

Handling adaptivity

- So far $t_i = d_i\,(b_{i1}, b_{i2}, \ldots, b_{iq})$

- In general,
  q adaptively chosen probes
  = decision tree
    2^q bits
    depth q



$$1/3 = \Pr[\, t_i = 2 \,] = \Pr[\, d_i\,(b_{i1}, \ldots, b_{i2^q}) = 2 \,] \approx A / 2^q \neq 1/3$$

Remarks on proof

- Use ideas from lower bounds for locally decodable codes [Shaltiel V.]

- New approach to data structures lower bounds

Conclusion

- Thm: Store n trits $t_1, \ldots, t_n \in \{0,1,2\}$.

  Get t_i by probing q bits $\Rightarrow$ space > optimal + $n/2^{O(q)}$

  Matches [Pătraşcu Thorup]: space < optimal + $n/2^{\Omega(q)}$

- Thm: Store $S \subseteq \{1, 2, \ldots, n\}$, $|S| = n/3$.

  Answer "$i \in S$?" probing q bits $\Rightarrow$ space > optimal + $n/2^{O(q)}$

  First lower bound for $|S| = \Omega(n)$

- New approach to lower bounds for basic data structures