From Parametricity to Conservation Laws, via Noether's Theorem

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2014-04-07
\[ TE = mgh + \frac{1}{2}mv^2 \]
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\[ TE = PE + KE \]
distance = $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
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Distance is **invariant** under translation and change of coordinate representation.
weights of about 2-3kg

swivel chair

initial angular velocity of about one revolution every couple of seconds

final angular velocity of up to two or three revolutions per second
Noether's Theorem
Noether's Theorem

(1915) "Any differentiable symmetry of the action of a physical system has a corresponding conservation law"
Quick Example
\[ L(t, z_1, z_2, \dot{z}_1, \dot{z}_2) = \frac{1}{2} m (\dot{z}_1^2 + \dot{z}_2^2) - \frac{1}{2} k (z_1 - z_2)^2 \]
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\[ \forall d \in \mathbb{R}^2 \]
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\[ \forall d \in \mathbb{R}^2 \quad L(t, z_1, z_2, \dot{z}_1, \dot{z}_2) = L(t, z_1 + d, z_2 + d, \dot{z}_1, \dot{z}_2) \]
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(Noether's Theorem)

\[ \frac{d}{dt} m (\dot{x}_1 + \dot{x}_2) = 0 \]
Conservation of Momentum

\[ L(t, z_1, z_2, \dot{z}_1, \dot{z}_2) = \frac{1}{2} m (\dot{z}_1^2 + \dot{z}_2^2) - \frac{1}{2} k (z_1 - z_2)^2 \]

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(Noether's Theorem)

\[ \frac{d}{dt} m (\dot{x}_1 + \dot{x}_2) = 0 \]
If the action

$$S[q; a; b] = \int_a^b L(t, q, \dot{q}) \, dt$$

is invariant under $\Phi_\epsilon$ and $\Psi_\epsilon$, then

$$\frac{d}{dt} \left( \sum_{i=1}^n \frac{\partial L}{\partial \dot{q}_i} \psi_i + \left( L - \sum_{i=1}^n \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right) \phi \right) = 0$$

where $\phi = \frac{\partial \Phi}{\partial \epsilon} \bigg|_{\epsilon=0}$ and $\psi = \frac{\partial \psi}{\partial \epsilon} \bigg|_{\epsilon=0}$
Pretty cool, right?
\( \tau \)

\[(\lambda x : \text{unit} . \ 42 : \text{int})\]

\[
\lambda f : (\text{int} \to \text{int}) \ \text{ref} . \ 
\lambda n : \text{int} . \\
\  f := (\lambda acc : \text{int} \ \text{ref} . \ 
\  \lambda m : \text{int} . \\
\  \  \text{case} \ (n = m : \text{bool}) \ \text{of} \\
\  \  \ (\text{acc} := (\text{mul} \ \!\!\!\!\!\!\!\!\!\!\text{acc} \ m); \ \text{acc}) : \text{int} \\
\  \  \  (!f (acc := (\text{mul} \ \!\!\!\!\!\!\!\!\!\!\text{acc} \ m); \ \text{acc}) (m+1)) : \text{int} \\
\  \  \ ) \ (\text{ref} \ 1) \ 1) \ (\text{ref} \ \lambda x : \text{int} . x)\]
Atkey (2014)
Atkey (2014)

Define a type system for Lagrangian Mechanics.
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Define a type system for Lagrangian Mechanics.

Derive conservation laws as "free theorems" by parametricity.
Lagrangian: \[ L(t, z_1, z_2, \dot{z}_1, \dot{z}_2) = \frac{1}{2} m (\dot{z}_1^2 + \dot{z}_2^2) - \frac{1}{2} k (z_1 - z_2)^2 \]
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Type: \( \forall y : T(1) \).

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Reference:

\( \forall y : T(1) \rightarrow \text{for all translations } y \text{ in one-dimensional space} \)
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\( C^\infty(\_ , \_ ) \quad \rightarrow \) type for smooth functions between spaces
Lagrangian: $L(t, z_1, z_2, \dot{z}_1, \dot{z}_2) = \frac{1}{2} m (\dot{z}_1^2 + \dot{z}_2^2) - \frac{1}{2} k (z_1 - z_2)^2$

Type: $\forall y : T(1)$.

$C^\infty(\mathbb{R}^1(1, 0) \times \mathbb{R}^1(1, y) \times \mathbb{R}^1(1, y) \times \mathbb{R}^1(1, 0) \times \mathbb{R}^1(1, 0), \_\_)$

Reference:

$\forall y : T(1) \rightarrow$ for all translations $y$ in one-dimensional space

$C^\infty(\_, \_\_) \rightarrow$ type for smooth functions between spaces
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\[ C^\infty(\mathbb{R}^1 1, 0 \times \mathbb{R}^1 1, y \times \mathbb{R}^1 1, y \times \mathbb{R}^1 1, 0 \times \mathbb{R}^1 1, 0, \_ \) \]

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\[C^\infty(\mathbb{R}^n \langle g, t_1 \rangle \times \mathbb{R}^n \langle g, t_2 \rangle, \mathbb{R}^n \langle g, t_1 - t_2 \rangle)\]

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\[T(n) \rightarrow \text{translations in } n\text{-dimensional space}\]
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\( \text{T}(n) \rightarrow \) translations in \( n \)-dimensional space

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Why $GL(n)$?
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**Theorem (Noether).** Let $L(x, u, D^1_u, \ldots, D^n_u)$, be a Lagrangian for $A \subseteq \mathbb{R}^n$, let $\varphi \in Aut(A)$ be a symmetry of $A$ such that

$$\varphi(L) + LD^i(\xi) = D^i(B^i) \quad B^i \in A$$

Then the Euler-Lagrange equations admit a conservation law $\forall i. D^i(C^i) = 0$. 
Why $GL(n)$?

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"Give me a Lagrangian and a group action satisfying these constraints, I'll give you a conservation law."
Why $\text{GL}(n)$?

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**Key point:** we need an automorphism (i.e. symmetry) to start with
What does this mean?
Reynolds:
types are
relations
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Wadler:
relations are free theorems
Reynolds: types are relations

Wadler: relations are free theorems
Reynolds: types are relations

Wadler: relations are free theorems

Atkey: free theorems are symmetries
Atkey gives us a geometric interpretation of types
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We'll argue: Atkey subsumes Reynolds + Wadler
Kinds are reflexive graphs
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Reynolds: types are sets, parametricity comes from the relations between them.
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Basic relation between Reynolds' types is the subset relation (⊆).
Kinds are reflexive graphs

- Reynolds: types are sets, parametricity comes from the relations between them.

- Basic relation between Reynolds' types is the subset relation ($\subseteq$).

- Form a graph where the objects are types and the edges order types by $\subseteq$. 
Example: bool
Example: bool
Example: `bool`

- True
- False
Example: nat
Example: nat

0
Example: nat

0 1 2 3
Example: cartesian space \((\mathbb{R}^1 \ldots \mathbb{R}^n)\)
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Example: cartesian space \((\mathbb{R}^1 \ldots \mathbb{R}^n)\)
Example: cartesian space \((\mathbb{R}^1 \ldots \mathbb{R}^n)\)

Each arrow represents a family of diffeomorphisms
Example: cartesian space

$\mathbb{R}^2$
Example: cartesian space
Example: cartesian space

\[ \text{id} \]

\[ \mathbb{R}^2 \]

\[ g_1 \]
Example: cartesian space
Example: cartesian space
Example: cartesian space

\[ g_i \in Aut(\mathbb{R}^2) \]
Free Theorems
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Give me a function
Free Theorems

Give me a function

\[ g : \forall X. X \text{ list} \rightarrow X \text{ list} \]
Free Theorems

Give me a function

\[ g : \forall X. X \text{ list} \to X \text{ list} \]

Then for every function
Free Theorems

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\[ (\text{map } f) \circ g_X = g_X \circ (\text{map } f) \]
Free Theorems

\[(\text{map } f) \circ g_x = g_x \circ (\text{map } f)\]
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\(X \text{ list}\)
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\[X \text{ list} \]

\[\downarrow \quad g_x \]

\[X \text{ list} \quad \text{map } f \quad X' \text{ list} \]
Free Theorems

\[(map \ f) \circ g_x = g_x \circ (map \ f)\]
Free Theorems

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\[
\begin{array}{ccc}
X \text{ list} & \xrightarrow{\text{map } f} & X' \text{ list} \\
\downarrow g_x & & \downarrow g_{x'} \\
X \text{ list} & \xrightarrow{\text{map } f} & X' \text{ list}
\end{array}
\]
Atkey: Main Points

- Extend System Fω with type system encoding geometric invariances.
- Interpret kinds as reflexive graphs, types as reflexive graph morphisms.
- Connect free theorems of Wadler/Reynolds with Noether's theorem via symmetries of these reflexive graphs.
Atkey: Takeaways

- Types as geometries is a powerful new way of manipulating our "syntactic discipline".
- Visual intuition, connections to group theory.
- Physics is only one potential application!
The End