Note on Exercise 13.7

In this note, we analyze the following greedy algorithm for the set multicover algorithm: select in each step some copies of a set such that the ratio of the cost incurred to the total coverage satisfied by the copies of the set is minimized. We will argue that if the cost functions associated with each set are concave in the number of copies of the set, then this greedy algorithm has an approximation ratio strictly exceeding H_n where n is the number of elements in the universe.

Consider the following instance. The universe U consists of n elements e_1 through e_n. The requirement r_i for element e_i is 2^i. We have $n+1$ sets: $E_i = \{e_i\}$, for $1 \leq i \leq n$, and U. The cost $f(E_i, j)$ for every i and j is C; that is, the cost of including any set E_i any number of times is C. The cost $f(U, j)$ is $2C$, for all j. Clearly, all the cost functions are concave.

We now argue that the greedy algorithm selects 2^i copies of the set E_i, for $1 \leq i \leq n$, with total cost nC, while the singleton set U selected 2^n times satisfies all the coverage requirements for a total cost of $2C$. This establishes that the greedy algorithm cannot possibly yield an H_n approximation, where n is the number of elements in the universe. (It does yield an H_m approximation, where m is the maximum sum of requirements of elements in a set.)

We now consider the greedy algorithm. In the first step, the cost-effectiveness of the multiset $E_n^{2^n}$ (that is, E_n taken 2^n times) equals $C/2^n$ and is the smallest. In comparison, the cost-effectiveness of the multiset U^{2^n} is $2C/(2^{n+1}-1) > C/2^n$. So the greedy algorithm selects 2^n copies of E_n. In general, in the ith step, the multiset with the best cost-effectiveness is $E_{n-i+1}^{2^{n-i+1}}$, with a value of $C/2^{n-i+1}$. In comparison, the cost-effectiveness of the multiset $U^{2^{n-i+1}}$ is $2C/(2^{n-i+2}-1) > C/2^{n-i+1}$. Thus, the greedy algorithm will select 2^{n-i+1} copies of the set E_{n-i+1}, thus establishing the desired claim.