College of Computer & Information Science
Northeastern University
CSG399: Approximation Algorithms

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Sample Solution to Part 3 of Exercise 13.4

This exposition is based on the reduction given in [1, Section 6]. Let the given covering integer program be:

$$\begin{array}{ll} \text{minimize} & \sum_{1\le i\le m} c_i x_i \\ \text{subject to} & \sum_{1\ lei \le m} a_{ij} x_i \ge b_j \ \ 1 \le j \le n \end{array}$$

The above problem can be presented as a multiset multicover problem with m multisets S_1 through S_m, and n elements e_1 through e_n; multiset S_i has a_{ij} copies of element e_j, and the requirement of element e_j is b_j. The problem with this reduction is that the size of the multiset multicover problem is polynomial in the values of a_{ij}'s rather than their sizes.

By scaling and merging, we will reduce the a_{ij} values (and, in the process, also the requirements) to a polynomial range. Without loss of generality, we assume that a_{ij} is at most b_j for all i and j.

Let $\beta_j = b_j \cdot min_i \frac{c_i}{a_{ij}}$; thus β_j is the minimum cost of covering element e_j entirely by one set alone. Let $\beta = \max_j \beta_j$. It thus follows that both the fractional as well as integral optimum are at least β and at most $n\beta$. Furthermore, any multiset that has cost more than $n\beta$ is not part of an integral optimal solution; so we can safely remove any multiset S_i with cost $c_i > n\beta$, without affecting the integral optimum. This is what we do first. Now that we have narrowed the range of the integral and fractional optimal to $[\beta, n\beta]$, we can be more liberal in our reduction. In particular, we can clump together $\alpha_i = \lceil \beta/(nc_i) \rceil$ copies of set S_i, by setting c_i to $\alpha_i c_i$ and a_{ij} to $\alpha_i a_{ij}$, to ensure that its cost is at least β/n. Note that this does not change the cost of the fractional optimum. The main reason for doing this is to be able to reduce the a_{ij} values to small numbers by scaling later. Note that any multiset that undergoes such clumping ends up having a cost of no more than $2\beta/n$; others continue to have their original cost.

If, in the process of clumping multisets, we find that an element is covered to its requirement by a multiset of cost less than $2\beta/n$, we simply include this multiset in our solution and remove the covered element. Since we do this at most n times, the total cost incurred is at most 2β which is at most a constant fraction of the fractional optimum. From now on, we can assume that we have a multiset multicover problem (or equivalently, a covering integer program) in which every set is of cost in the range $[\beta/n, n\beta]$ and no element is completely covered by any single multiset.

We now scale all the multiplicities and requirements down as follows. We set a'_{ij} to be $a_{ij} \cdot 2n^2/b_j$ and set b'_j to be $2n^2$. Let us refer to this problem as the *scaled-down* problem. By doing so, we ensure that each multiplicity and each requirement is no more than $2n^2$. Furthermore, simply scaling does not affect the fractional and integral optimal solutions. The multiplicities, however, are not integral. So we do not have a valid multiset multicover problem.

We construct the *rounded-down* problem: set a''_{ij} to be $\lfloor a'_{ij} \rfloor$, and b'_j remains as $2n^2$. Thus, the resultant multiset multicover problem is polynomial in the size of the covering integer program. All that remains to be shown is that any δ-approximate solution to the rounded-down problem can be transformed to an $O(\delta)$-approximate solution to the original covering integer program. What we will show is that the fractional optimal solution to the rounded-down problem is no more than 4 times the fractional optimal to the covering integer program.

Consider an optimal solution to the scaled-down problem with multiplicities a'_{ij}. Since the cost of this solution is at most $n\beta$ and each c_i is at least β/n, it follows that $\sum_i x_i$ is at most n^2. Consider all the multisets S_i such that $a'_{ij} < 1$. The total coverage of element e_j due to these sets is at most n^2. Therefore, the remaining sets contribute at least n^2. Since for each of these sets $a''_{ij} \geq a'_{ij} - 1 \geq a'_{ij}/2$, it follows that if we multiply each x_i by 4, element e_j will be covered to the extent of at least $2n^2$ in the rounded-down problem. This shows that the optimal fractional solution to the rounded-down problem is within a factor of 4 of the optimal solution to the scaled problem, which is the same as the optimal fractional solution to the covering integer program obtained after the preliminary processing.

We know that the multiset multicover problem has an H_ℓ integral approximation (with respect to the fractional optimal) where ℓ is the maximum sum of the multiplicities of all of the elements in a multiset. In the above reduction, the multiset multicover problem obtained has ℓ at most $2n^3$. We thus have an $O(H_n)$ approximation for the covering integer program, where the constant hidden in the big-Oh notation includes the factor 3 (from $2n^3$) and the factors given up during reduction.

References

[1] S. Rajagopalan and V. Vazirani. Primal-dual RNC approximation algorithms for set cover and covering integer programs. *SIAM Journal on Computing*, 28:525–540, 1998.