APPROX ALG 2004, PROBLEM SET 1, # 5, EDGECOVER.

a) Given a matching m, find a cover C not too big.

The cover could be found using the algorithm given in d). Start from the given matching m, add at most $|V|-2|m|$ edges to cover the unmatched (hence, uncovered) vertices.

$|C| \leq |m| + (|V| - 2|m|) = |V| - |m|$

b) Given a cover C, find a matching not too small.

Observe that a set of edges is a matching iff the induced degrees upon vertices are ≤ 1. The cover C incurs a total degree of $2|C|$. There are at most $2|C| - |V|$ "redundant" edges (those incident on vertices with degree > 1). By removing at most that many edges from C, we will get a matching with size at least

$|C| - (2|C| - |V|) = |V| - |C|$.

c) From a) and b), it is straightforward to get the following identity

$|\min Cover| + |\max Matching| = |V|$.

Since MAXIMUM MATCHINING could be solved efficiently, so can the MIN EDGECOVER problem.

d) The proposed algorithm essentially first finds a *maximal* matching m, then adds on edges to take care of the uncovered vertices. According to a), $|C| \leq |V| - |m|$, combined with the result of c), we need to prove $|V| - |m| \leq 1.5(|V| - |M|)$, where M is some *maximum* matching.

First, we have $2|m| \geq |M|$. (to see this, note that $2|m|$ is the size of some vertex cover, which is lower-bounded by any matching.) It is also obvious that $|V| \geq 2|M|$.

Hence,

$$
\begin{aligned}
& 1.5(|V| - |M|) \\
= \; & |V| - |M| + 0.5(|V| - |M|) \\
\geq \; & |V| - |M| + |M| - |m| \\
= \; & |V| - |m| \\
\geq \; & |C|.
\end{aligned}
$$

This bound is tight for the proposed algorithm. Consider, eg, { o—o—o—o }. This gadget can be duplicated.