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CS G399: Algorithmic Power Tools I	Scribe: Zhifeng Sun

Lecture Outline:

- Multicommodity flow and Sparset cut

1 Multicommodity flow

Demands multicommodity flow: Given graph $G = (V, E)$, edge capacity function $C : E \to \mathbb{Z}^+$. There are $k \geq 1$ commodities, each with its own source s_i, sink t_i, and demand $dem(i)$. The objective is to maximize f such that we can send $f \cdot dem(i)$ units of commodity i from s_i to t_i for each i **simultaneously**, without violating the capacity constraint of any edge.

Sum-flow multicommodity flow: Given graph $G = (V, E)$, edge capacity function $C : E \to \mathbb{Z}^+$. There are $k \geq 1$ commodities, each with its own source s_i, sink t_i. The objective is to maximize the sum of the flow sent from s_i to t_i, over all i, without violating the capacity constraint for any edge.

2 Two examples where Max-Flow is not equal to Min-Cut

It is well known that Max-Flow is equal to Min-Cut in Single Commodity Flow problem. But this is not true for Multicommodity Flow when the number of commodities is greater than 2.

First we give the definition of Min-Cut in multicommodity flow problem.

Definition 1. For any cut $\langle S, \bar{S} \rangle$ of the graph, let $C(S, \bar{S}) = \sum_{e \in \langle S, \bar{S} \rangle} C(e)$ which is the total capacities across this cut, and $D(S, \bar{S}) = \sum_{\{i | s_i \in S \wedge t_i \in \bar{S} \text{ or } s_i \in \bar{S} \wedge t_i \in S\}} dem(i)$ which is the total demand across this cut. Define the Min-Cut as $\eta = \min_{S \subseteq V} \dfrac{C(S, \bar{S})}{D(S, \bar{S})}$. We refer to $\dfrac{C(S, \bar{S})}{D(S, \bar{S})}$ as the *ratio* of cut $(S, \bar{S})$.

Let f^* be the optimal value for demands multicommodity flow. It is clear that $f^* \leq \eta$. The first example (Figure 1, taken from Jon Kleinberg's lecture notes) shows f^* could be strictly smaller than η in multicommodity flow problem.

In the graph, there are 4 flow pairs, each with a demand of 1, and the shortest path between each pair is 2 hops. So the total capacity consumed when we send f^* flow for each commodity is $8f^*$. And there are only 6 edges in the graph. So we have $f^* \leq 3/4$.

The second example gives an even worse ratio between Max-Flow and Min-Cut, where $f^* \leq O\left(\frac{\eta}{\log n}\right)$. This example makes use of Uniform Multicommodity Flow and 3-regular expander graph.

3-regular expanders: 3-regular expander graph has the following properties: