Northeastern University
CS 7180 – Special Topics in AI (Reinforcement Learning)
Fall 2018, Robert Platt

Self Test (SOLUTION)

Name: ______________________________

Problem	Points
1. Bayes' Rule	/0
2. Probability Distributions	/0
3. Discrete Expectation	/0
4. Expectation Properties	/0
5. Matrices/Linear Equations	/0
6. Matrices	/0
Total	/0

Instructions

- This assignment will not be graded for correctness
- Use this as an opportunity to self-assess your math background and self-study as appropriate.

(0 pts.) 1. Bayes' Rule

The Weatherly app predicts rain tomorrow. In recent years, it has rained only 73 days each year. When it actually rains, the Weatherly app correctly forecasts rain 70% of the time. When it doesn't rain, the app incorrectly forecasts rain 30% of the time. What is the probability that it will rain tomorrow?

Hint: $P(H|D) = \frac{P(H)P(D|H)}{P(D)}$

$P(Rain|PredictRain) = \frac{P(Rain)P(PredictRain|Rain)}{P(PredictRain)} = \frac{P(Rain)P(PredictRain|Rain)}{P(Rain)P(PredictRain|Rain)+P(!Rain)P(PredictRain|!Rain)}$
$P(Rain) = 73/365 = 0.2$
$P(!Rain) = 1 - P(Rain) = (365 - 73)/365 = 0.8$
$P(PredictRain|Rain) = 0.7$
$P(PredictRain|!Rain) = 0.3$
$P(Rain|PredictRain) = \frac{0.2 \times 0.7}{(0.2 \times 0.7)+(0.8 \times 0.3)} \sim 0.37$

(0 pts.) 2. Probability Distributions

Given the following probability density function (PDF) of a random variable x ...

$$p(x) = \begin{cases} 4x & 0 \le x \le \frac{1}{2} \\ -4x + 4 & \frac{1}{2} \le x \le 1 \end{cases}$$

What is the equation and graph of the corresponding cumulative density function (CDF)?

The CDF is the area under the curve of the PDF – i.e. the integral.
The anti-derivative of the PDF, $P(x)$ ...

$$P(x) = \begin{cases} 2x^2 + C_1 & 0 \le x \le \frac{1}{2} \\ -2x^2 + 4x + C_2 & \frac{1}{2} \le x \le 1 \end{cases}$$

Assume the entire mass is within the range $[0, 1]$, thus $P(0) = 0$ and $P(1) = 1$ ...

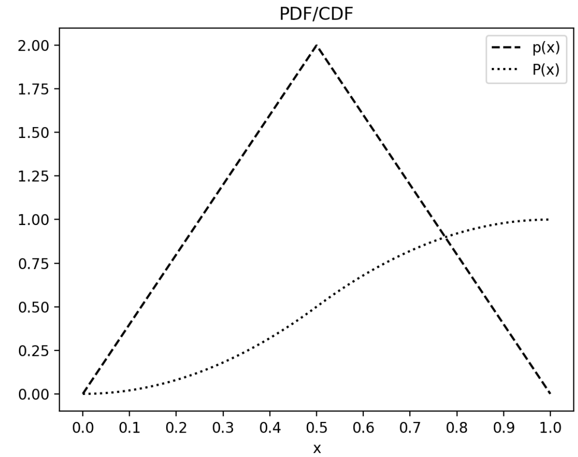
$P(0) = 0 = 2(0)^2 + C_1$
$\therefore C_1 = 0$

and ...

$P(1) = 1 = -2(1)^2 + 4(1) + C_2 = -2 + 4 + C_2 = 2 + C_2$
$\therefore C_2 = -1$

and thus ...

$$P(x) = \begin{cases} 2x^2 & 0 \le x \le \frac{1}{2} \\ -2x^2 + 4x - 1 & \frac{1}{2} \le x \le 1 \end{cases}$$



(0 pts.) 3. Discrete Expectation

Calculate the expected value of X, $E[X]$, where X is a random variable representing the outcome of a roll of a trick die. Use the sample space $x \in \{1, 2, 3, 4, 5, 6\}$ (i.e. six-sided die) and let

$$P(X = x) = \begin{cases} \frac{1}{2} & x = 1 \\ \frac{1}{10} & x \neq 1 \end{cases}$$

The expected value of a discrete random variable is ...

$$E[X] = \sum_i p(x_i) x_i$$

Thus ... $E[X] = 1/2 + 2/10 + 3/10 + 4/10 + 5/10 + 6/10 = 25/10 = 2.5$, which not a possible outcome – be aware that expectations may not be in the sample space.

(0 pts.) 4. Expectation Properties

Use the properties of expectation to show that we can rewrite the variance of a random variable X ...

$Var[X] = E[(X - \mu)^2]$

as ...

$Var[X] = E[X^2] - (E[X])^2$

$Var[X] = E[(X - \mu)^2]$
$Var[X] = E[X^2 - 2\mu X + \mu^2]$
$Var[X] = E[X^2] - 2\mu E[X] + \mu^2$
$Var[X] = E[X^2] - 2\mu^2 + \mu^2$
$Var[X] = E[X^2] - \mu^2$
$Var[X] = E[X^2] - (E[X])^2$

(0 pts.) 5. Matrices/Linear Equations

Consider the following system of equations .....

$2x_1 + x_2 + x_3 = 3$
$4x_1 + 2x_3 = 10$
$2x_1 + 2x_2 = -2$

a. Write the system as a matrix equation of the form $Ax = b$.

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 4 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad b = \begin{pmatrix} 3 \\ 10 \\ -2 \end{pmatrix}$$

b. Write the solution of the system as a column s and verify by matrix multiplication that $As = b$.

$As = b$
$s = A^{-1}b$

Gauss-Jordan: $A^{-1}\ldots$

$$\begin{pmatrix} 2 & 1 & 1 \\ 4 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$R_2 - 2R_1$

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & -2 & 0 \\ 2 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$R_2 + R_3$

$$\begin{pmatrix} 2 & 1 & 1 \\ 2 & 0 & 0 \\ 2 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$R_3 - R_2$

$$\begin{pmatrix} 2 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 1 \\ 2 & -1 & 0 \end{pmatrix}$$

$R_1 - R_2$

$$\begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 3 & -1 & -1 \\ -2 & 1 & 1 \\ 2 & -1 & 0 \end{pmatrix}$$

$R_1 - \frac{1}{2}R_3$

$$\begin{pmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 2 & -\frac{1}{2} & -1 \\ -2 & 1 & 1 \\ 2 & -1 & 0 \end{pmatrix}$$

$R_2 \rightleftarrows R_3$
$R_1 \rightleftarrows R_3$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2 & 1 & 1 \\ 2 & -1 & 0 \\ 2 & -\frac{1}{2} & -1 \end{pmatrix}$$

$\frac{1}{2}R_1$
$\frac{1}{2}R_2$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & \frac{1}{2} & \frac{1}{2} \\ 1 & -\frac{1}{2} & 0 \\ 2 & -\frac{1}{2} & -1 \end{pmatrix}$$

$s = A^{-1}b$

$$s = \begin{pmatrix} -1 & \frac{1}{2} & \frac{1}{2} \\ 1 & -\frac{1}{2} & 0 \\ 2 & -\frac{1}{2} & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 10 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

$As = b$

$$\begin{pmatrix} 2 & 1 & 1 \\ 4 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2-2+3 \\ 4+6 \\ 2-4 \end{pmatrix} = \begin{pmatrix} 3 \\ 10 \\ -2 \end{pmatrix} = b$$

c. Write b as a linear combination of the columns in A.

$b = A_{*1} - 2A_{*2} + 3A_{*3}$

(0 pts.) 6. Matrices

Consider the following matrix . . .

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix}$$

a. What is the determinant, $det(A)$ or $|A|$, of the matrix?

Given . . .

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$det(A) = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$
$det(A) = a(ei - fh) - b(di - fh) + c(dh - eg)$
$det(A) = 1(16 - 9) - 2(4 - 3) + 3(3 - 4)2$
$det(A) = 1(7) - 2(1) + 3(-1)$
$det(A) = 7 - 2 - 3 = 2$

b. Is the matrix invertible?

Yes: $|A| \neq 0$

c. What is the rank of the matrix?

Rows/columns are linearly independent, so rank = # rows/columns (i.e. full rank): $rank(A) = 3$