Linearizing a non-linear system

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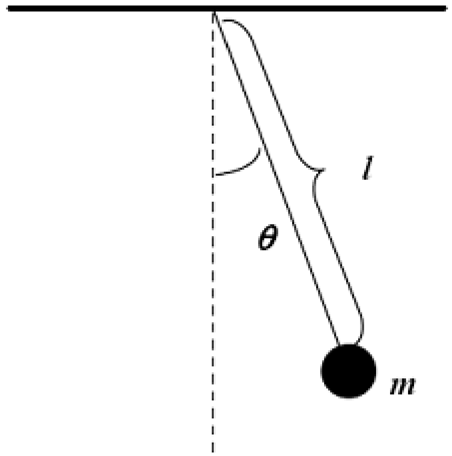
Pendulum

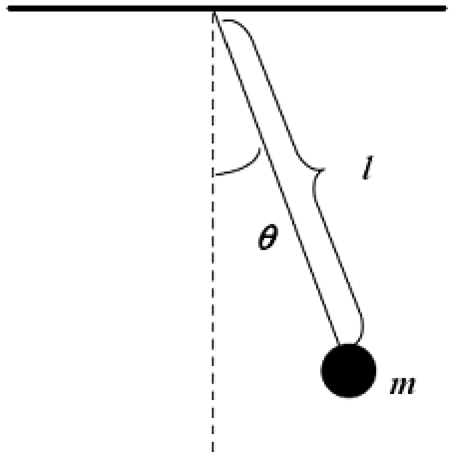


EOM for pendulum: $$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

How do we get this system in the standard form: $$x_{t+1} = Ax_t + Bu_t$$

?

Pendulum



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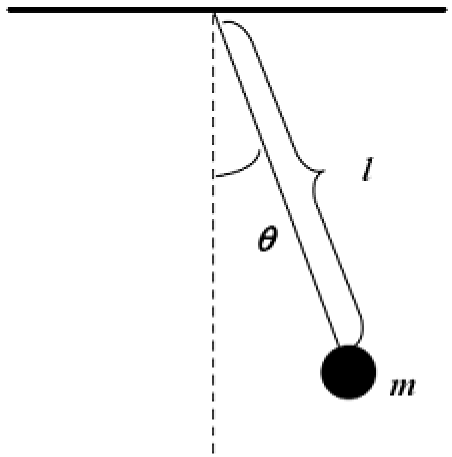
$$\theta_{t+1} = \theta_t + \dot{\theta}_t dt$$
$$\dot{\theta}_{t+1} = \dot{\theta}_t - \frac{g}{l}\sin\theta_t dt$$

Pendulum



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Linearizing a non-linear system

Idea: use first-order Taylor series expansion

$$x_{t+1} = f(x_t) + Bu_t$$ original non-linear system

Linearizing a non-linear system

<u>Idea:</u> use first-order Taylor series expansion

$$x_{t+1} = f(x_t) + Bu_t \longleftarrow \text{original non-linear system}$$

$$\text{Linearize about } x^* \longrightarrow \approx f(x^*) + \underbrace{\frac{\partial f(x^*)}{\partial x}(x_t - x^*)}_{\text{first order term}} + Bu_t$$

Linearizing a non-linear system

<u>Idea:</u> use first

We just linearized the system about x^*

$$x_{t+1} = f(x_t) + Bu_t$$ original non-linear system

Linearize about x^*

$$\approx f(x^*) + \underbrace{\frac{\partial f(x^*)}{\partial x}(x_t - x^*)}_{\text{first order term}} + Bu_t$$

Linearizing a non-linear system

$$x_{t+1} \approx f(x^*) + \frac{\partial f(x^*)}{\partial x}(x_t - x^*) + Bu_t$$

Suppose that x^* is a fixed point (or a steady state) of the system...

Then: $f(x^*) = x^*$

Linearizing a non-linear system

$$x_{t+1} \approx f(x^*) + \frac{\partial f(x^*)}{\partial x}(x_t - x^*) + Bu_t$$

Suppose that x^* is a fixed point (or a steady state) of the system...

Then: $f(x^*) = x^*$

$$x_{t+1} - f(x^*) \approx \frac{\partial f(x^*)}{\partial x}(x_t - x^*) + Bu_t$$

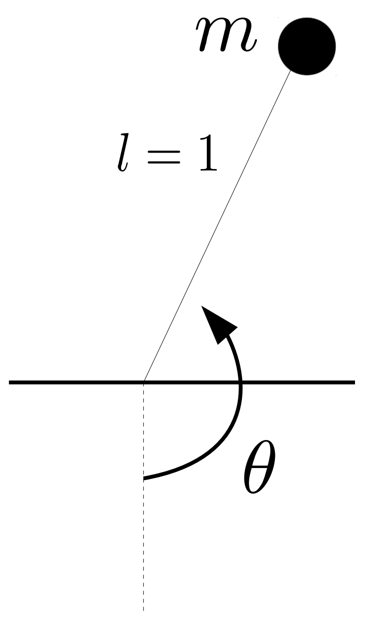
$$x_{t+1} - x^* \approx \frac{\partial f(x^*)}{\partial x}(x_t - x^*) + Bu_t$$

$$\bar{x}_{t+1} \approx \frac{\partial f(x^*)}{\partial x}\bar{x}_t + Bu_t$$

where $\bar{x}_t = x_t - x^*$

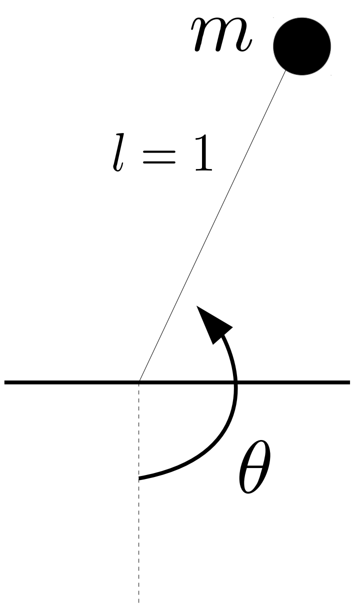
Change of coordinates

Example: inverted pendulum



$$\theta_{t+1} = \theta_t + \dot{\theta}_t dt$$
$$\dot{\theta}_{t+1} = \dot{\theta}_t - g\sin\theta_t dt$$

Example: inverted pendulum

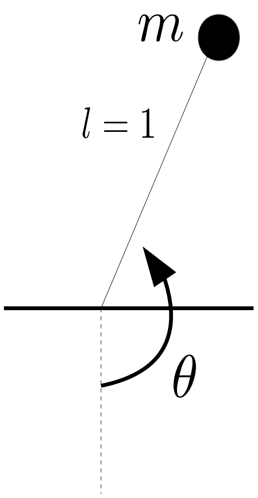


$$\theta_{t+1} = \theta_t + \dot{\theta}_t dt$$
$$\dot{\theta}_{t+1} = \dot{\theta}_t - g \sin \theta_t dt$$

$$\frac{\partial f(x^*)}{\partial x} = \begin{pmatrix} 1 & dt \\ -g \cos \theta_t dt & 1 \end{pmatrix}$$

Linearize about: $x^* = \begin{pmatrix} \pi \\ 0 \end{pmatrix} \longrightarrow \frac{\partial f(x^*)}{\partial x} = \begin{pmatrix} 1 & dt \\ g dt & 1 \end{pmatrix}$

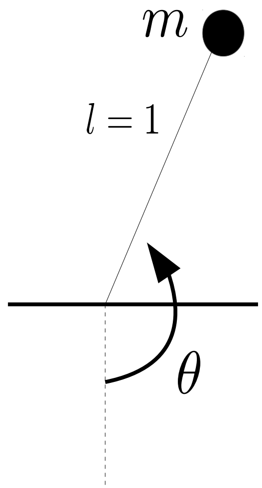
Example: inverted pendulum



$$\bar{x}_{t+1} \approx A\bar{x}_t + Bu_t \quad \text{where} \quad A = \begin{pmatrix} 1 & dt \\ gdt & 1 \end{pmatrix}$$

$$\bar{x}_t = x_t - \begin{pmatrix} \pi \\ 0 \end{pmatrix}$$

Example: inverted pendulum



Another way to think about this is:

$$\theta_{t+1} = \theta_t + \dot{\theta}_t dt$$
$$\dot{\theta}_{t+1} = \dot{\theta}_t - g \sin \theta_t dt$$

$$\theta_{t+1} = \theta_t + \dot{\theta}_t dt$$
$$\dot{\theta}_{t+1} \approx \dot{\theta}_t + (\theta_t - \pi) g dt$$

$$\sin(\theta) \approx -(\theta - \pi) \ \text{ near } \theta = \pi$$