Differential Kinematics

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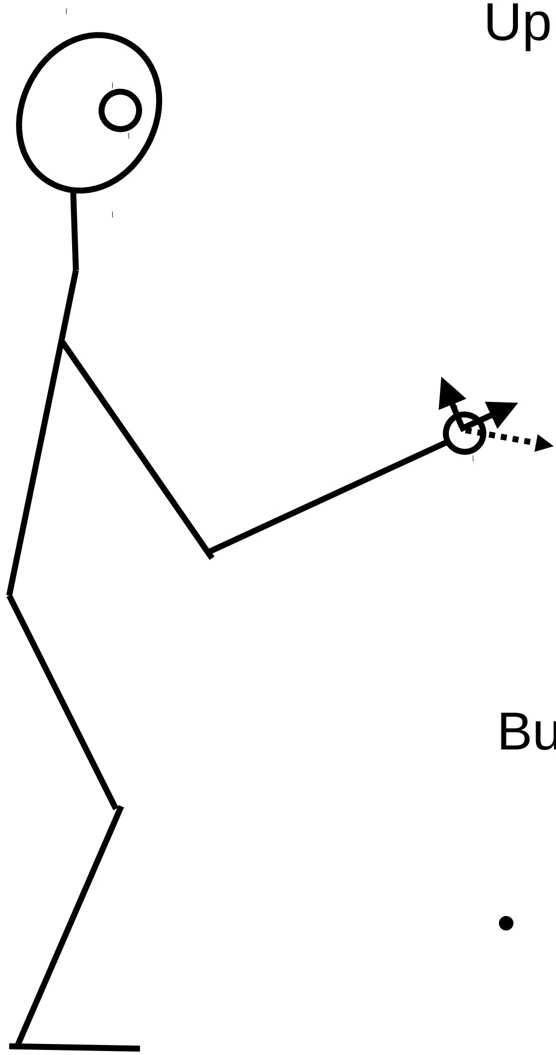
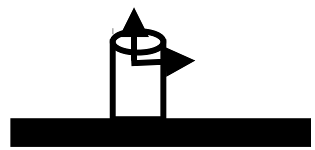


Up to this point, we have only considered the relationship of the joint angles to the Cartesian location of the end effector:

$$f(q) = x$$



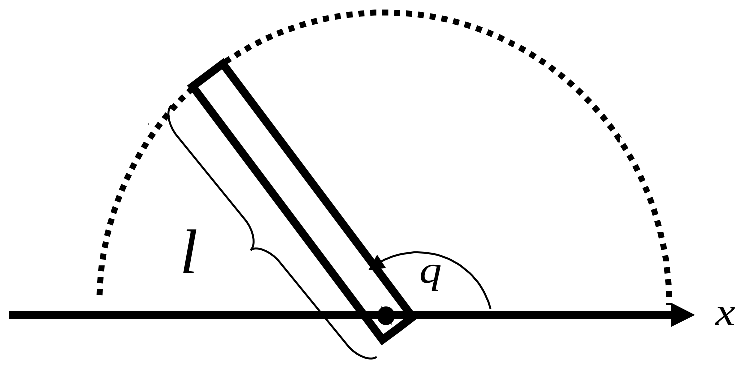
But what about the first derivative? $$\frac{\partial f(q)}{\partial q}$$

- This would tell us the velocity of the end effector as a function of joint angle velocities.

Motivating Example

Consider a one-link arm

- As the arm rotates, the end effector sweeps out an arc
- Let's assume that we are only interested in the x coordinate…



Forward kinematics: $x = l\cos(q)$

Differential kinematics: $\frac{dx}{dq} = -l\sin(q)$

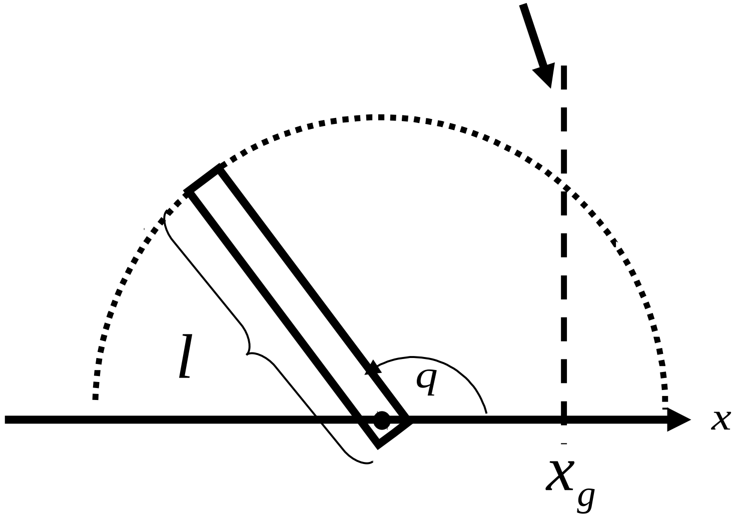
$$\delta x = -l\sin(q)\delta q$$

$$\delta q = -\frac{1}{l\sin(q)}\delta x$$

Motivating Example

Suppose you want to move the end effector above a specified point, x_g

Goal: move the end effector onto this line

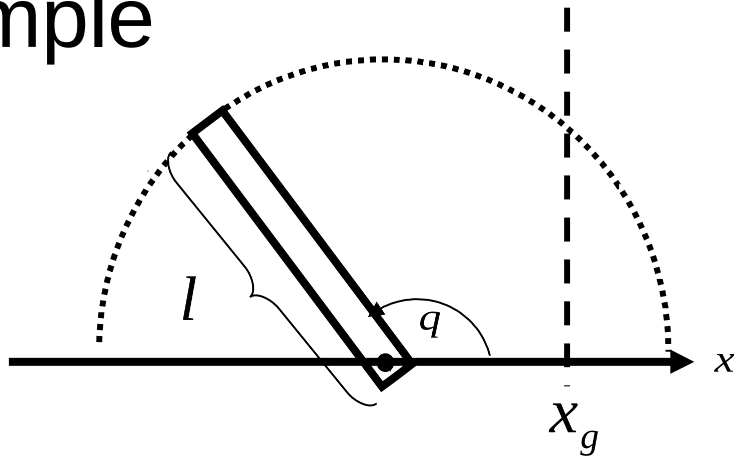


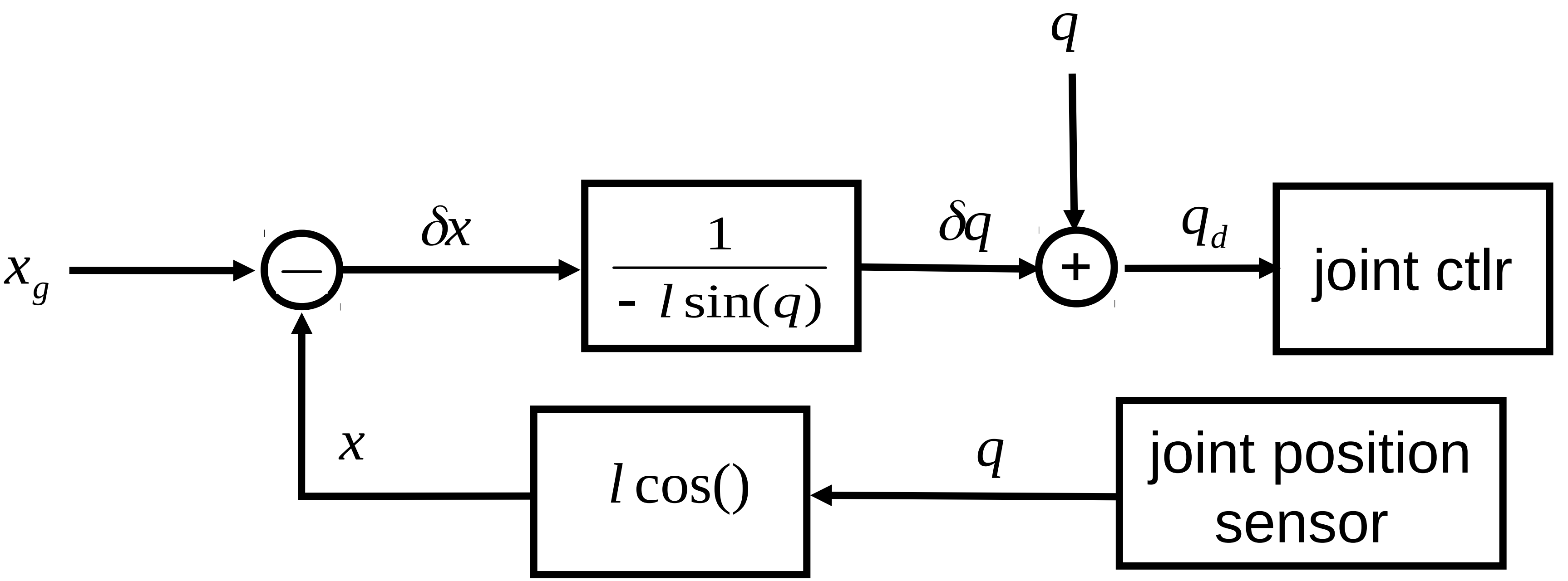
Answer #1: $q_g = \cos^{-1}\left(\frac{x_g}{l}\right)$

Answer #2:

1. $i = 0, q_0 = \text{arbitrary}$
2. $x_i = l\cos(q_i)$
3. $\delta x = \alpha\left(x_g - x_i\right)$
4. $\delta q = \frac{1}{-l\sin(q_i)}\delta x$
5. $q_{i+1} = q_i + \delta q$
6. $i{+}{+}$ goto 2.

Motivating Example



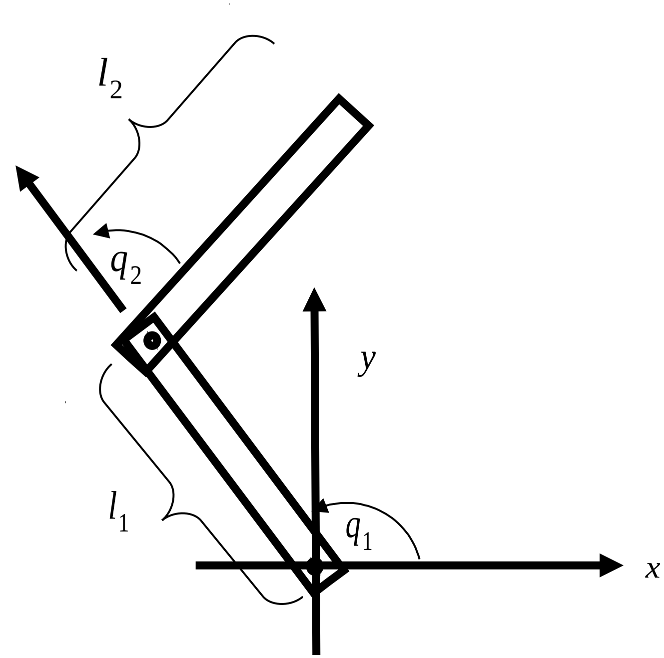


This controller moves the link asymptotically toward the goal position.

Intro to the Jacobian

$$x = \begin{bmatrix} l_1 \cos(q_1) + l_2 \cos(q_1 + q_2) \\ l_1 \sin(q_1) + l_2 \sin(q_1 + q_2) \end{bmatrix}$$

Forward kinematics of the two-link manipulator



Velocity Jacobian

$$\frac{dx}{dq} = \begin{pmatrix} -l_1 \sin(q_1) - l_2 \sin(q_1 + q_2) & -l_2 \sin(q_1 + q_2) \\ l_1 \cos(q_1) + l_2 \cos(q_1 + q_2) & -l_2 \cos(q_1 + q_2) \end{pmatrix}$$

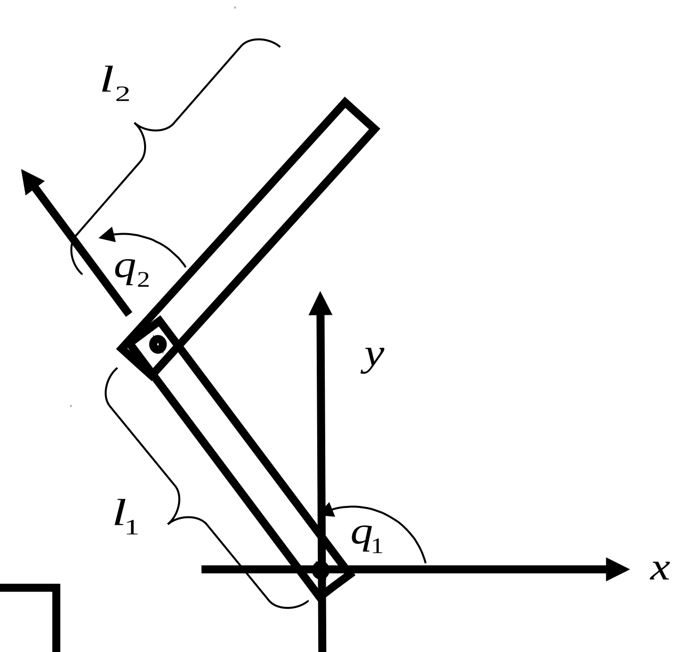
$$= J(q)$$

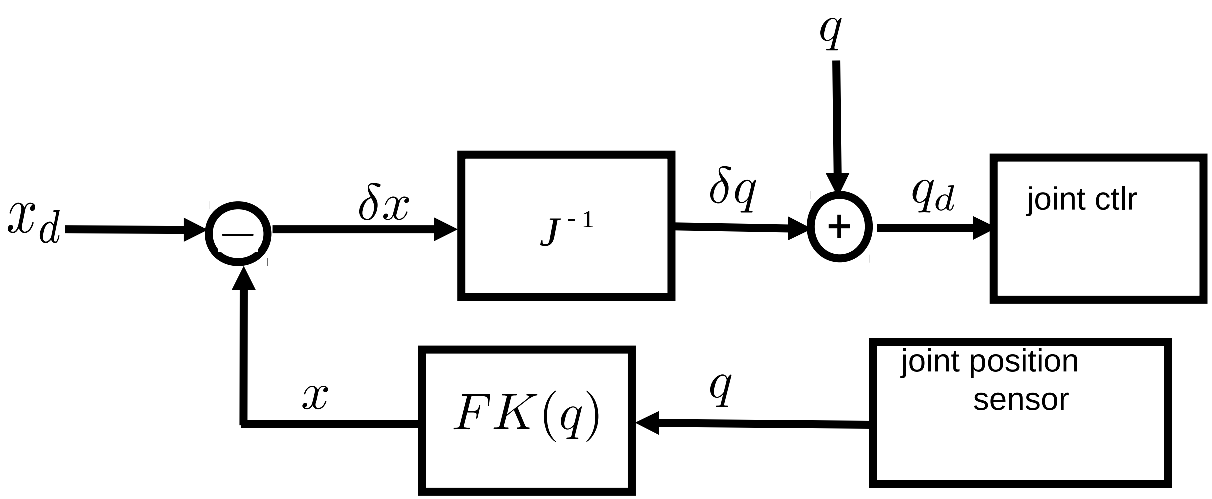
Intro to the Jacobian

$$J(q) = \begin{pmatrix} -l_1 \sin(q_1) - l_2 \sin(q_1 + q_2) & -l_2 \sin(q_1 + q_2) \\ l_1 \cos(q_1) + l_2 \cos(q_1 + q_2) & -l_2 \cos(q_1 + q_2) \end{pmatrix}$$

Chain rule: $\delta x = J \delta q$

If the Jacobian is square and full rank, then we can invert it: $\delta q = J^{-1} \delta x$





Jacobian

The Jacobian relates joint velocities with end effector *twist*:

$$\xi = J\dot{q}$$

End effector twist

Jacobian

Joint angle velocities

- First derivative of joint angles:

$$\dot{q} = \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix}$$

It turns out that you can “easily” compute the Jacobian for arbitrary manipulator structures

- This makes differential kinematics a much easier sub-problem than kinematics in general.

What is Twist?

End effector twist:

- Twist is a concatenation of linear velocity and angular velocity:

$$\xi = \begin{pmatrix} \nu \\ \omega \end{pmatrix}$$

Linear velocity

Angular velocity

- As we will show in a minute, linear and angular velocity have different units

What is Twist?

End effector twist:

- Twist is a concatenation of linear velocity and angular velocity:
- As we will show in a minute, linear and angular velocity have different units

Linear velocity

$$\xi = \begin{pmatrix} \nu \\ \omega \end{pmatrix}$$

Angular velocity

What is angular velocity?

Angular velocity is a vector that:

– points in the direction of the axis of rotation

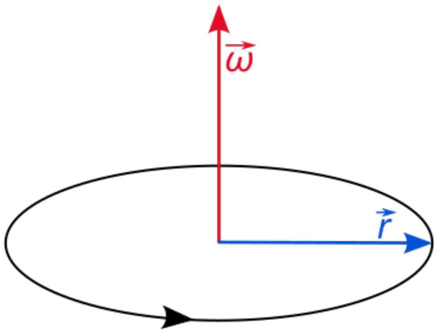
– has magnitude equal to the velocity of rotation

What is Angular Velocity?

Angular velocity is a vector that:

- – points in the direction of the axis of rotation
- – has magnitude equal to the velocity of rotation

Symbol for angular velocity: ω



Relation between angular velocity and linear velocity: $v = \omega \times r$

We will often write it this way: $\dot{q} = \omega \times q$

Angular Velocity Derivation

$${}^{b}q = {}^{b}R_{a}\,{}^{a}q$$

$${}^{b}\dot{q} = {}^{b}\dot{R}_{a}\,{}^{a}q$$

Just differentiate all elements of the rotation matrix w.r.t. time.

$${}^{b}\dot{q} = {}^{b}\dot{R}_{a}\,{}^{b}R_{a}^{T}\,{}^{b}q$$

$$S\left({}^{b}\omega\right) = {}^{b}\dot{R}_{a}\,{}^{b}R_{a}^{T}$$

This is the matrix representation of angular velocity

$${}^{b}\dot{q} = S\left({}^{b}\omega\right){}^{b}q$$

This FO differential equation encodes how the particle rotates

Twist: Time out for skew symmetry!

$$S = -S^T$$

Def'n of skew symmetry

$$S = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

Skew symmetric matrices always look like this

If you interpret the skew symmetric matrix like this:

$$S(x) = \begin{bmatrix} 0 & -x_z & x_y \\ x_z & 0 & -x_x \\ -x_y & x_x & 0 \end{bmatrix}$$

Then this is another way of writing the cross product:

$$S(x)p = x \times p$$

Angular Velocity Derivation

Skew symmetry of $S\left({}^{b}\omega\right)$:

$$I = {}^{b}R_{a}\,{}^{b}R_{a}^{\;T}$$

$$0 = {}^{b}\dot{R}_{a}\,{}^{b}R_{a}^{\;T} + {}^{b}R_{a}\,{}^{b}\dot{R}_{a}^{\;T}$$

$${}^{b}\dot{R}_{a}\,{}^{b}R_{a}^{\;T} = -{}^{b}R_{a}\,{}^{b}\dot{R}_{a}^{\;T}$$

$$S\left({}^{b}\omega\right) = -S\left({}^{b}\omega\right)^{T}$$

$${}^{b}\dot{q} = S\left({}^{b}\omega\right){}^{b}q$$

$${}^{b}\dot{q} = {}^{b}\omega \times {}^{b}q$$

You probably already know this formula

Twist

Twist concatenates linear and angular velocity:

Linear velocity

$$\xi = \begin{bmatrix} v \\ \omega \end{bmatrix}$$

Angular velocity

Jacobian

Breakdown of the Jacobian: $v = J_v \dot{q}$

$$\omega = J_\omega \dot{q}$$

$$\xi = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix} \dot{q}$$

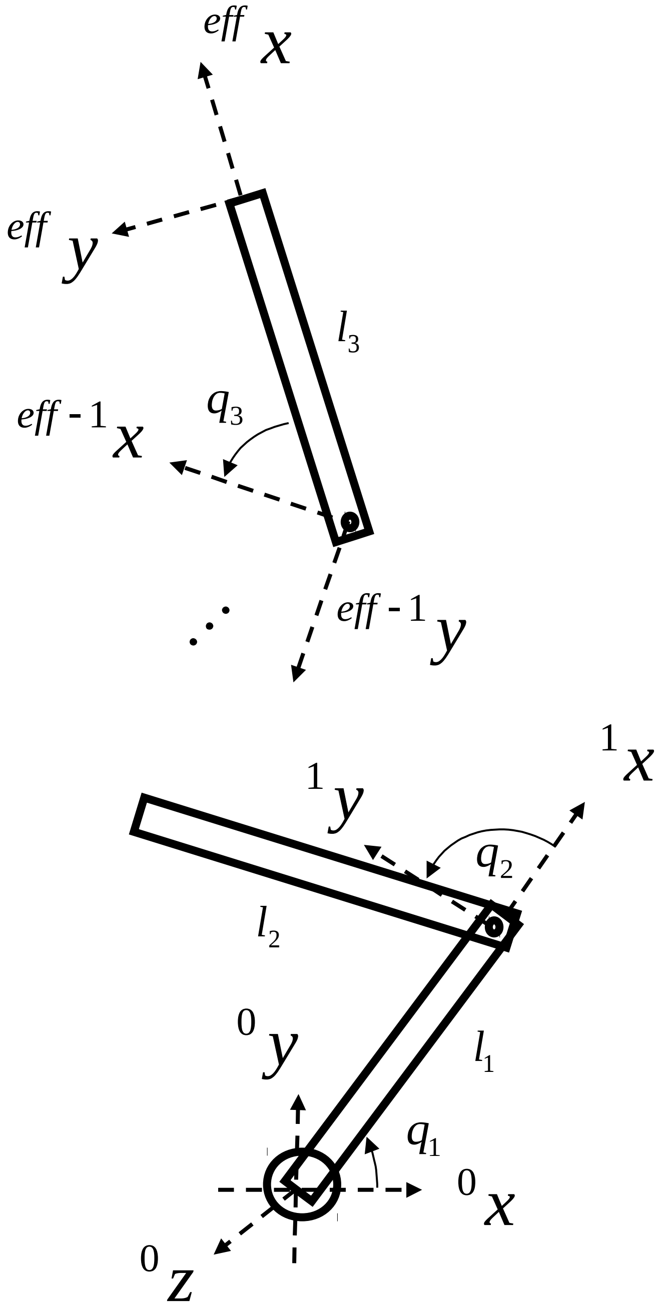
Relation to the derivative: $J_v = \frac{\partial x}{\partial q}$ **but** $J_\omega \neq \frac{\partial r_{\phi\theta\psi}}{\partial q}$

That's not an angular velocity

Calculating the Jacobian

Approach:

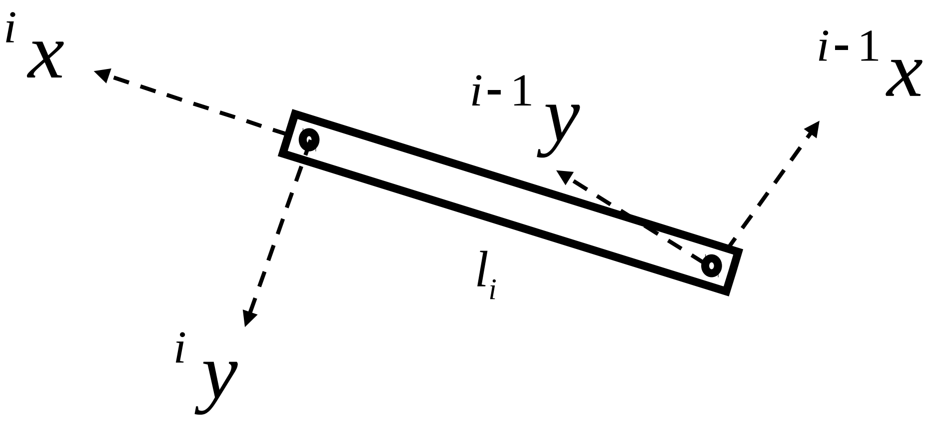
- Calculate the Jacobian one column at a time
- Each column describes the motion at the end effector due to the motion of *that joint only.*
- For each joint, *i*, pretend all the other joints are frozen, and calculate the motion at the end effector caused by *i.*



Calculating the Jacobian: Velocity

- The velocity of the end effector caused by motion at the *i-1* link:

$$ {}^{b}\dot{p}_{eff} = \underbrace{{}^{b}\omega_{i-1} \times {}^{b}p_{i-1,eff}}_{\text{Velocity at end effector due to rotation at joint } i\text{-}1} + \underbrace{{}^{b}\dot{p}_{i-1,i}}_{\text{Velocity at end effector due to change in length of link } i\text{-}1} $$



Velocity at end effector due to change in length of link *i-1*

Velocity at end effector due to rotation at joint *i-1*

Calculating the Jacobian: Velocity

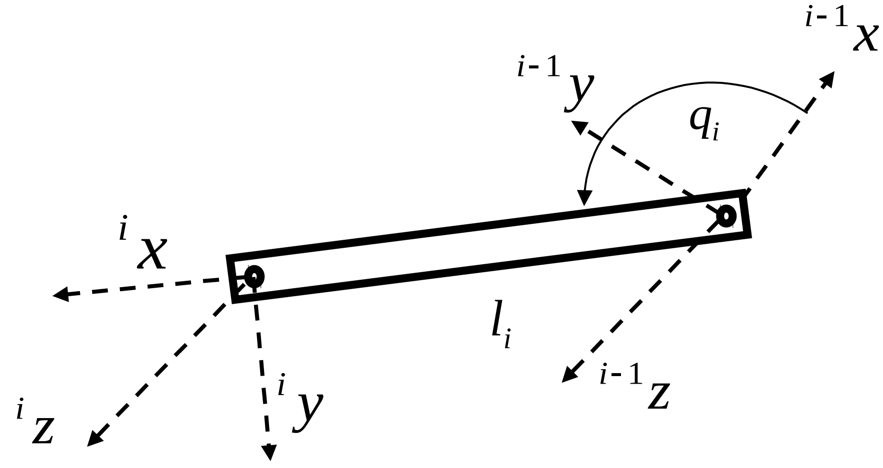
Rotational DOF

- Rotates about ${}^{i-1}z$

$$J_{v_i} = {}^{b}z_{i-1} \times {}^{b}p_{i-1,eff}$$

$$J_{v_i} = {}^{b}z_{i-1} \times \left({}^{b}p_{eff} - {}^{b}p_{i-1} \right)$$

Vector from i-1 to the end effector

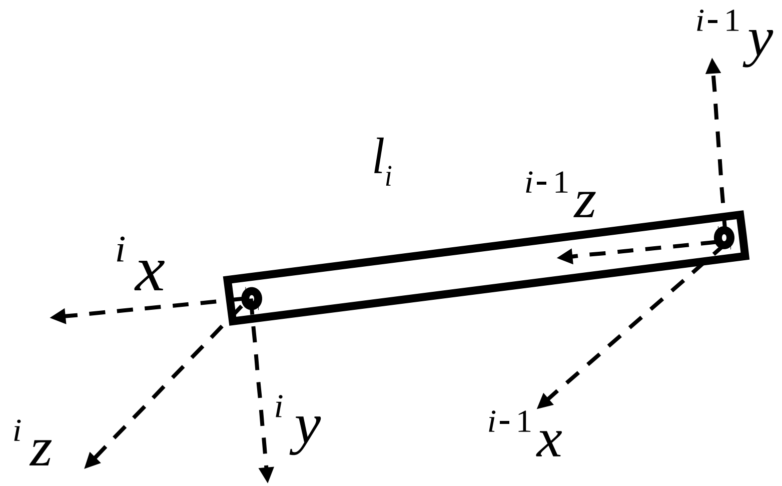


Rotation about ${}^{i-1}z$

Prismatic DOF

- Translates along ${}^{i-1}z$

$$J_{v_i} = {}^{b}z_{i-1}$$



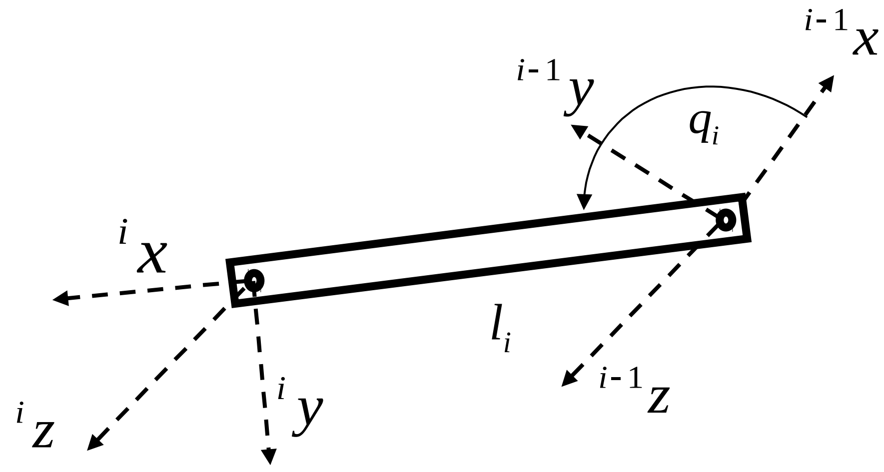
Extension/contraction along ${}^{i-1}z$

Calculating the Jacobian: Velocity

Rotational DOF

- Rotates about ${}^{i-1}z$

$$J_{\omega_i} = {}^{b}z_{i-1,i}$$

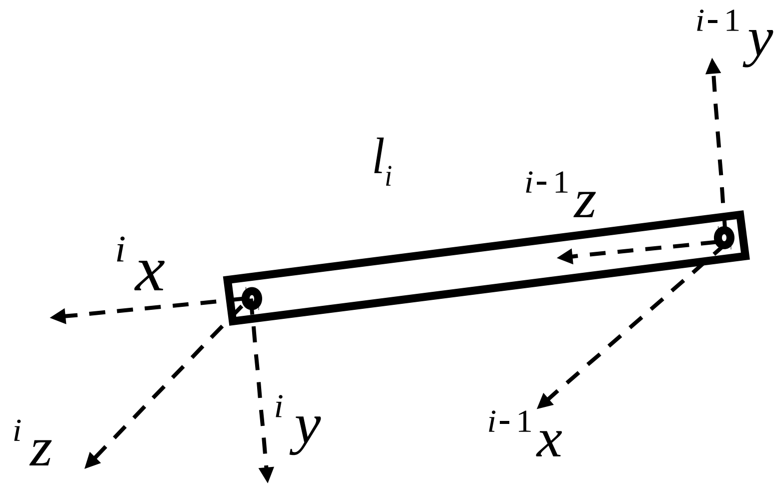


Rotation about ${}^{i-1}z$

Prismatic DOF

- Translates along ${}^{i-1}z$

$$J_{\omega_i} = 0$$



Extension/contraction along ${}^{i-1}z$

Calculating the Jacobian: putting it together

$$J_v = \begin{bmatrix} J_{v_1} & \cdots & J_{v_n} \end{bmatrix}$$

Where

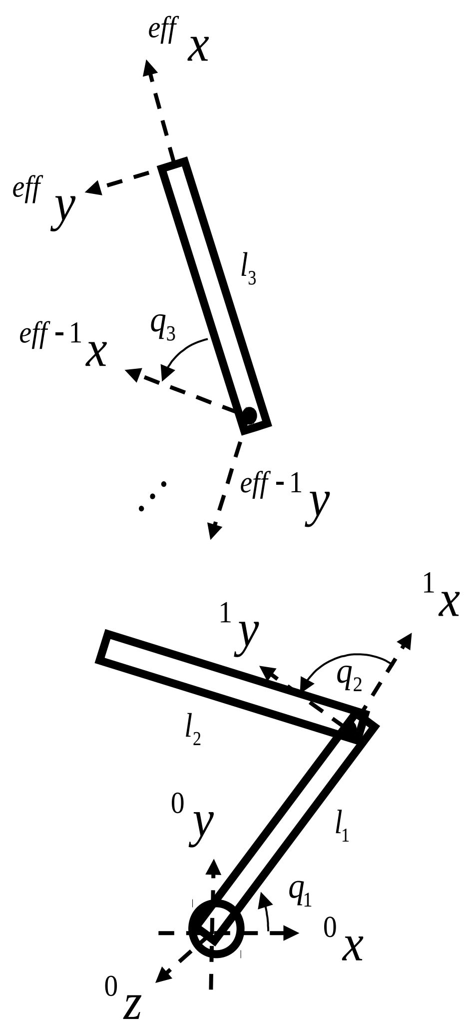
- rotational $J_{v_i} = {}^b z_{i-1} \times \left({}^b p_{eff} - {}^b p_{i-1} \right)$
- prismatic $J_{v_i} = {}^b z_{i-1}$

$$J_\omega = \begin{bmatrix} J_{\omega_1} & \cdots & J_{\omega_n} \end{bmatrix}$$

Where

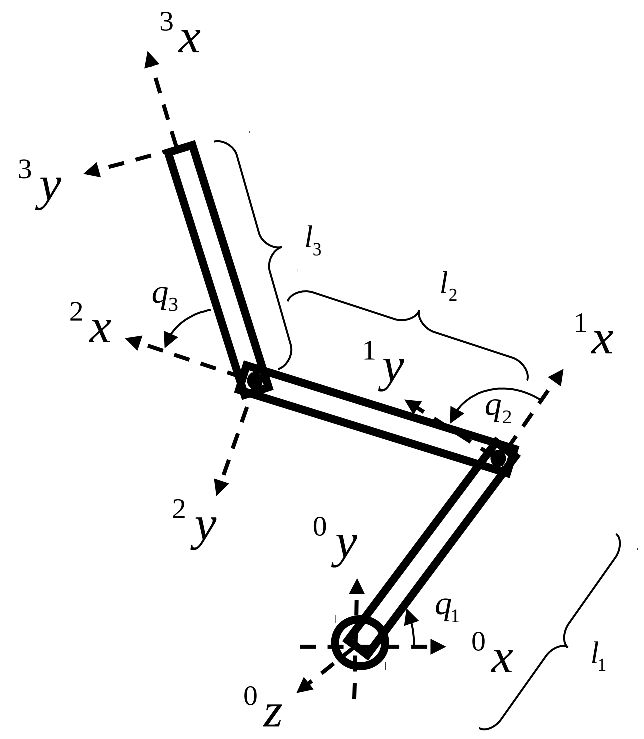
- rotational $J_{\omega_i} = {}^b z_{i-1}$
- prismatic $J_{\omega_i} = 0$

$$J = \begin{bmatrix} J_{v_1} & \cdots & J_{v_n} \\ J_{\omega_1} & \cdots & J_{\omega_n} \end{bmatrix}$$



Example 1: calculating the Jacobian

From before:



$$^{0}T_1 = \begin{pmatrix} c_{q_1} & -s_{q_1} & 0 & l_1 c_{q_1} \\ s_{q_1} & c_{q_1} & 0 & l_1 s_{q_1} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad ^{1}T_2 = \begin{pmatrix} c_{q_2} & -s_{q_2} & 0 & l_2 c_{q_2} \\ s_{q_2} & c_{q_2} & 0 & l_2 s_{q_2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$^{2}T_3 = \begin{pmatrix} c_{q_3} & -s_{q_3} & 0 & l_3 c_{q_3} \\ s_{q_3} & c_{q_3} & 0 & l_3 s_{q_3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$J_\omega = \begin{bmatrix} {}^0\hat{z}_0 & {}^0\hat{z}_1 & {}^0\hat{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

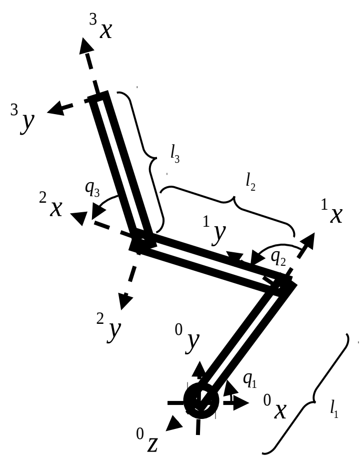
Example 1: calculating the Jacobian

$$J_{v_1} = {}^0\hat{z}_0 \text{¿} \left({}^0o_3 - {}^0o_0\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \left(\begin{bmatrix} l_1 c_1 + l_2 c_{12} + l_3 c_{123} \\ l_1 s_1 + l_2 s_{12} + l_3 s_{123} \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} - l_3 s_{123} \\ l_1 c_1 + l_2 c_{12} + l_3 c_{123} \\ 0 \end{bmatrix}$$

$$J_{v_2} = {}^0\hat{z}_1 \text{¿} \left({}^0o_3 - {}^0o_1\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \left(\begin{bmatrix} l_1 c_1 + l_2 c_{12} + l_3 c_{123} \\ l_1 s_1 + l_2 s_{12} + l_3 s_{123} \\ 0 \end{bmatrix} - \begin{bmatrix} l_1 c_1 \\ l_1 s_1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} -l_2 s_{12} - l_3 s_{123} \\ l_2 c_{12} + l_3 c_{123} \\ 0 \end{bmatrix}$$

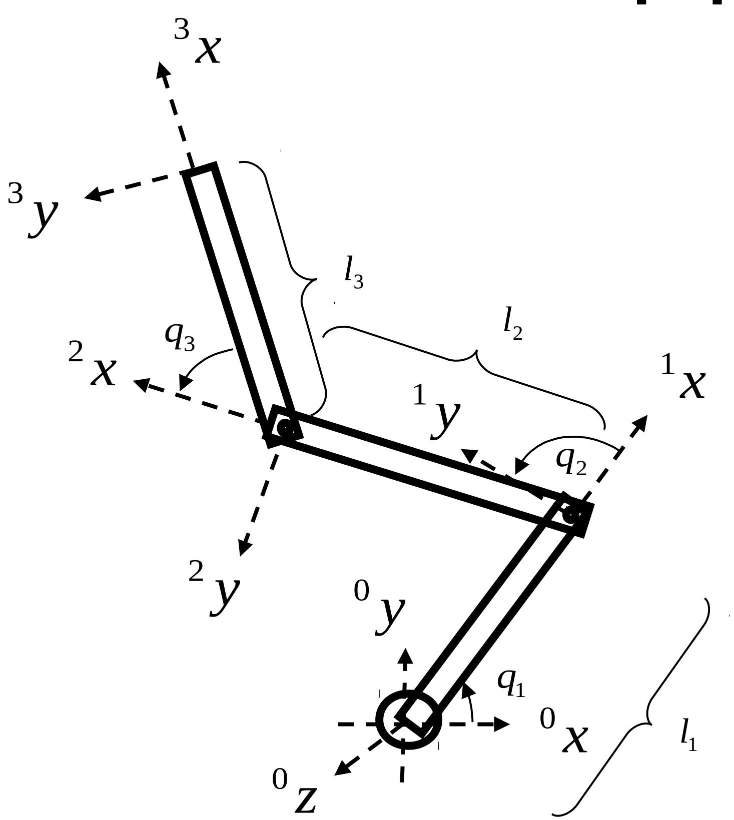
$$J_{v_3} = {}^0\hat{z}_2 \text{¿} \left({}^0o_3 - {}^0o_2\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \left(\begin{bmatrix} l_1 c_1 + l_2 c_{12} + l_3 c_{123} \\ l_1 s_1 + l_2 s_{12} + l_3 s_{123} \\ 0 \end{bmatrix} - \begin{bmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12} \\ 0 \end{bmatrix} \right) = \begin{bmatrix} -l_3 s_{123} \\ l_3 c_{123} \\ 0 \end{bmatrix}$$

$$J_v = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} - l_3 s_{123} & -l_2 s_{12} - l_3 s_{123} & -l_3 s_{123} \\ l_1 c_1 + l_2 c_{12} + l_3 c_{123} & l_2 c_{12} + l_3 c_{123} & l_3 c_{123} \\ 0 & 0 & 0 \end{bmatrix}$$

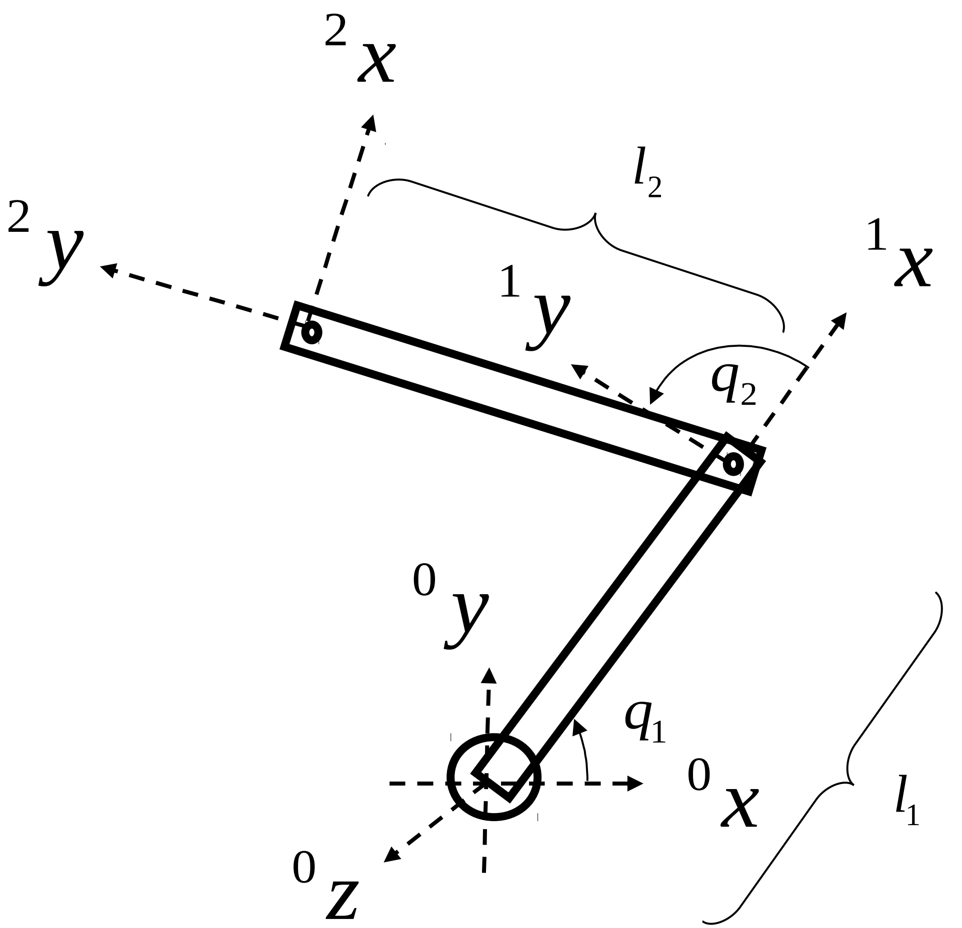


Example 1: calculating the Jacobian

$$J=\begin{bmatrix} J_v \\ J_\omega \end{bmatrix}=\begin{bmatrix} -l_1 s_1 - l_2 s_{12} - l_3 s_{123} & -l_2 s_{12} - l_3 s_{123} & -l_3 s_{123} \\ l_1 c_1 + l_2 c_{12} + l_3 c_{123} & l_2 c_{12} + l_3 c_{123} & l_3 c_{123} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$



Think-pair-share



Calculate the end effector Jacobian
with respect to the base frame

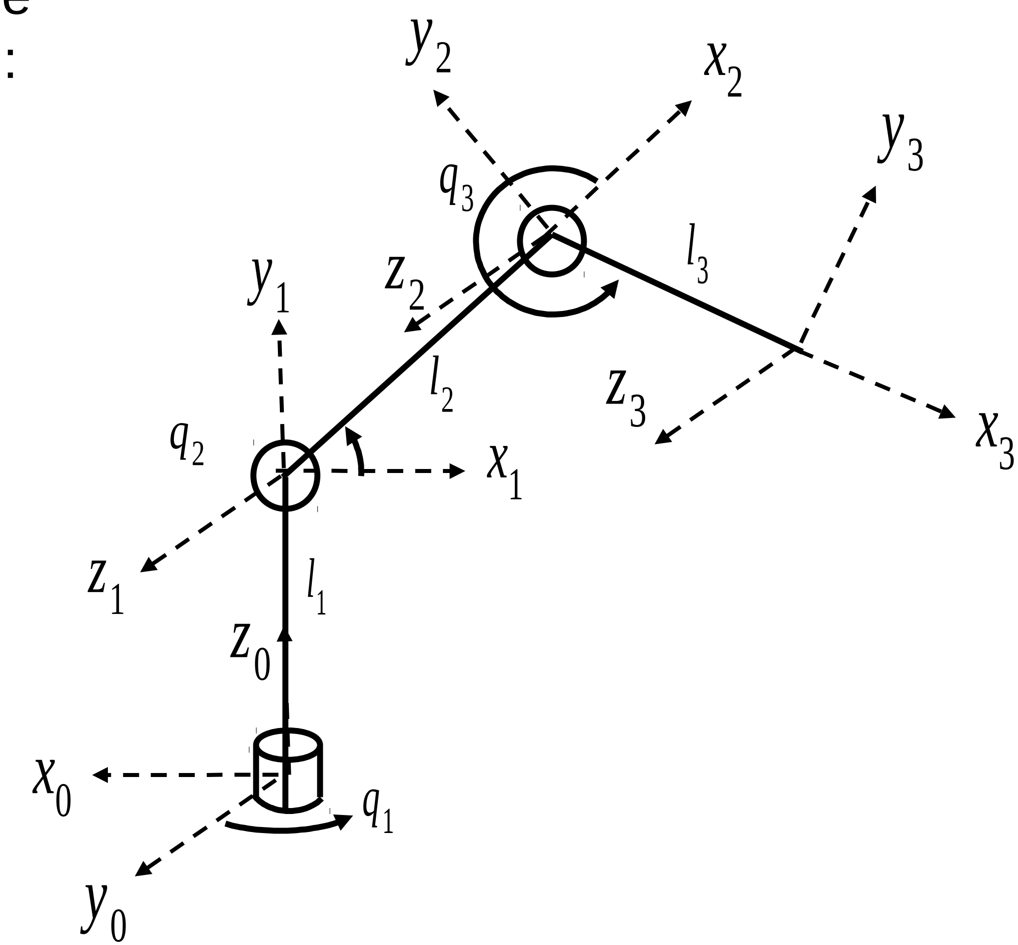
Example 2: calculating the Jacobian

The kinematics of this arm are described by the following:

$$^{0}T_{1}=\begin{pmatrix} -c_1 & 0 & -s_1 & 0 \\ -s_1 & 0 & c_1 & 0 \\ 0 & 1 & 0 & l_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$^{1}T_{2}=\begin{pmatrix} c_2 & -s_2 & 0 & l_2 c_2 \\ s_2 & c_2 & 0 & l_2 s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$^{2}T_{3}=\begin{pmatrix} c_3 & -s_3 & 0 & l_3 c_3 \\ s_3 & c_3 & 0 & l_3 s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



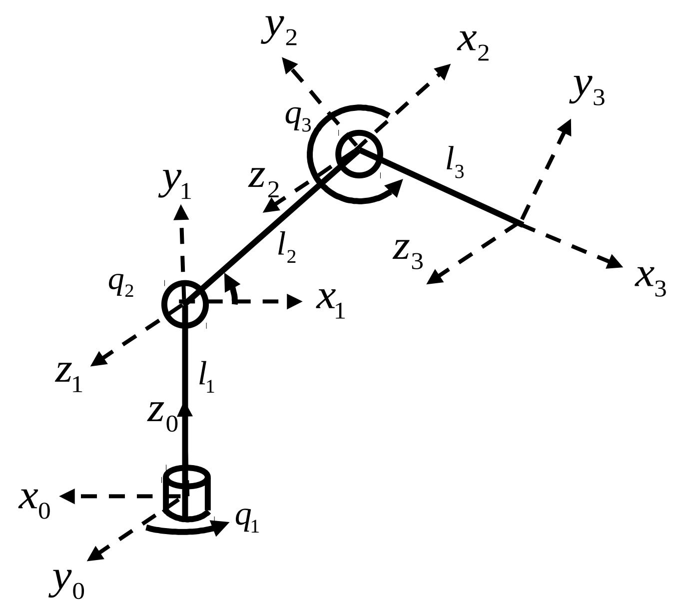
Example 2: calculating the Jacobian

$$^{b}p_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad ^{b}z_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$^{b}p_1 = \begin{pmatrix} 0 \\ 0 \\ l_1 \end{pmatrix} \qquad ^{b}z_1 = \begin{pmatrix} -s_1 \\ c_1 \\ 0 \end{pmatrix}$$

$$^{b}p_2 = \begin{pmatrix} -l_2 c_1 c_2 \\ -l_2 s_1 c_2 \\ l_2 s_2 + l_1 \end{pmatrix} \qquad ^{b}z_2 = \begin{pmatrix} -s_1 \\ c_1 \\ 0 \end{pmatrix}$$

$$^{b}p_3 = \begin{pmatrix} -c_1 \left(l_2 c_2 + l_3 c_{23} \right) \\ -s_1 \left(l_2 c_2 + l_3 c_{23} \right) \\ l_2 s_2 + l_3 s_{23} + l_1 \end{pmatrix}$$



$$J_{v_1} = {}^{b}z_0 \times \left({}^{b}p_3 - {}^{b}p_0 \right)$$

$$J_{v_2} = {}^{b}z_1 \times \left({}^{b}p_3 - {}^{b}p_1 \right)$$

$$J_{v_3} = {}^{b}z_2 \times \left({}^{b}p_3 - {}^{b}p_2 \right)$$

Example 2: calculating the Jacobian

$$J_{v_1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} -c_1\left(l_2 c_2 + l_3 c_{23}\right) \\ -s_1\left(l_2 c_2 + l_3 c_{23}\right) \\ l_2 s_2 + l_3 s_{23} + l_1 \end{pmatrix} = \begin{pmatrix} s_1\left(l_2 c_2 + l_3 c_{23}\right) \\ -c_1\left(l_2 c_2 + l_3 c_{23}\right) \\ 0 \end{pmatrix}$$

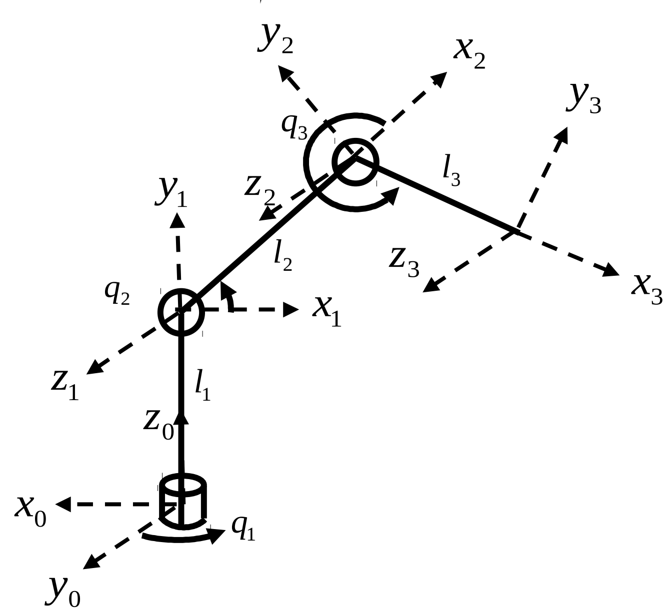
$$J_{\omega_1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$J_{v_2} = \begin{pmatrix} -s_1 \\ c_1 \\ 0 \end{pmatrix} \text{¿} \begin{pmatrix} -c_1\left(l_2 c_2 + l_3 c_{23}\right) \\ -s_1\left(l_2 c_2 + l_3 c_{23}\right) \\ l_2 s_2 + l_3 s_{23} \end{pmatrix} = \begin{pmatrix} c_1\left(l_2 c_2 + l_3 c_{23}\right) \\ s_1\left(l_2 c_2 + l_3 c_{23}\right) \\ l_2 c_2 + l_3 c_{23} \end{pmatrix}$$

$$J_{\omega_2} = \begin{pmatrix} -s_1 \\ c_1 \\ 0 \end{pmatrix}$$

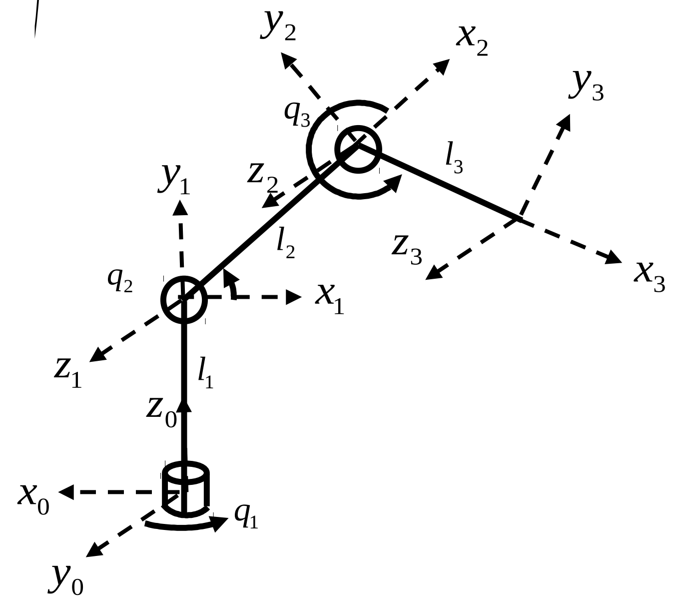
$$J_{v_3} = \begin{pmatrix} -s_1 \\ c_1 \\ 0 \end{pmatrix} \text{¿} \begin{pmatrix} -c_1 l_3 c_{23} \\ -s_1 l_3 c_{23} \\ l_3 s_{23} \end{pmatrix} = \begin{pmatrix} l_3 c_1 s_{23} \\ l_3 s_1 s_{23} \\ l_3 c_{23} \end{pmatrix}$$

$$J_{\omega_3} = \begin{pmatrix} -s_1 \\ c_1 \\ 0 \end{pmatrix}$$

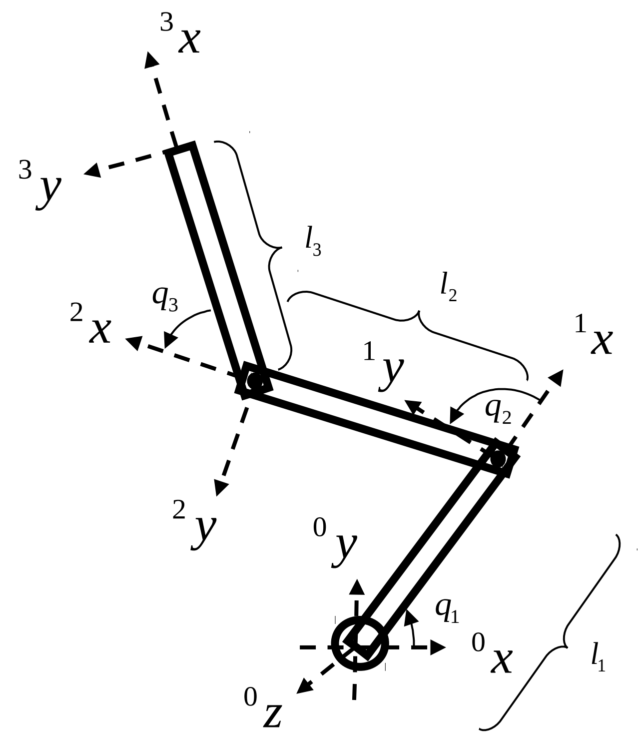


Example 2: calculating the Jacobian

$$J=\begin{pmatrix} s_1\left(l_2c_2+l_3c_{23}\right) & c_1\left(l_2c_2+l_3c_{23}\right) & l_3c_1s_{23} \\ -c_1\left(l_2c_2+l_3c_{23}\right) & s_1\left(l_2c_2+l_3c_{23}\right) & l_3c_1s_{23} \\ 0 & l_2c_2+l_3c_{23} & l_3c_{23} \\ 0 & -s_1 & -s_1 \\ 0 & c_1 & c_1 \\ 1 & 0 & 0 \end{pmatrix}$$



Expressing the Jacobian in Different Reference Frames



In the preceeding, the Jacobian has been expressed in the base frame

- It can be expressed in other reference frames using rotation matrices

Velocity is transformed from one reference frame to another using:

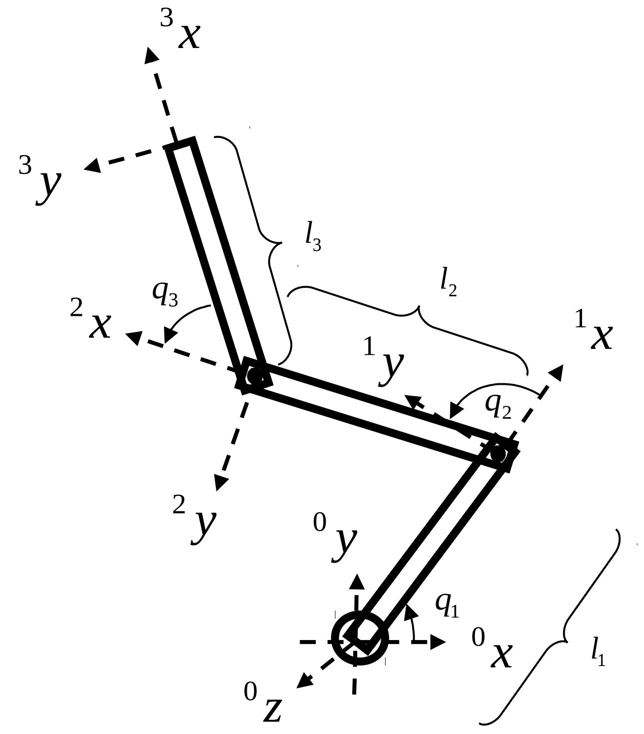
$${}^{k}p = {}^{k}R_b \, {}^{b}p$$

$${}^{k}\dot{p} = {}^{k}R_b \, {}^{b}\dot{p}$$

Therefore, the velocity Jacobian can be transformed using:

$${}^{k}J_v = {}^{k}R_b \, {}^{b}J_v$$

Expressing the Jacobian in Different Reference Frames



First, let's express angular velocity in a different reference frame:

$$^b\dot{p} = S\left(^b\omega\right)^b p$$ ← Def'n of angular velocity

$$^kR_b\, ^b\dot{p} = {^kR_b} S\left(^b\omega\right)^b p$$

$$^k\dot{p} = {^kR_b} S\left(^b\omega\right){^kR_b}^T\, {^kp}$$

$$^k\dot{p} = S\left(^kR_b\, ^b\omega\right)^k p$$

$$^k\omega = {^kR_b}\, ^b\omega$$ ← Angular velocity can also be rotated by a rotation matrix

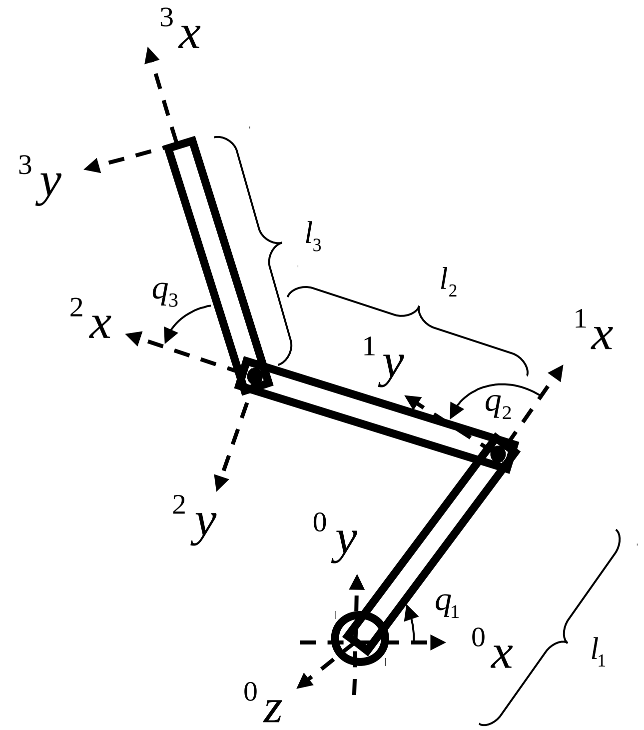
Therefore, the angular velocity Jacobian can be transformed using:

$$^kJ_\omega = {^kR_b}\, ^bJ_\omega$$

Expressing the Jacobian in Different Reference Frames

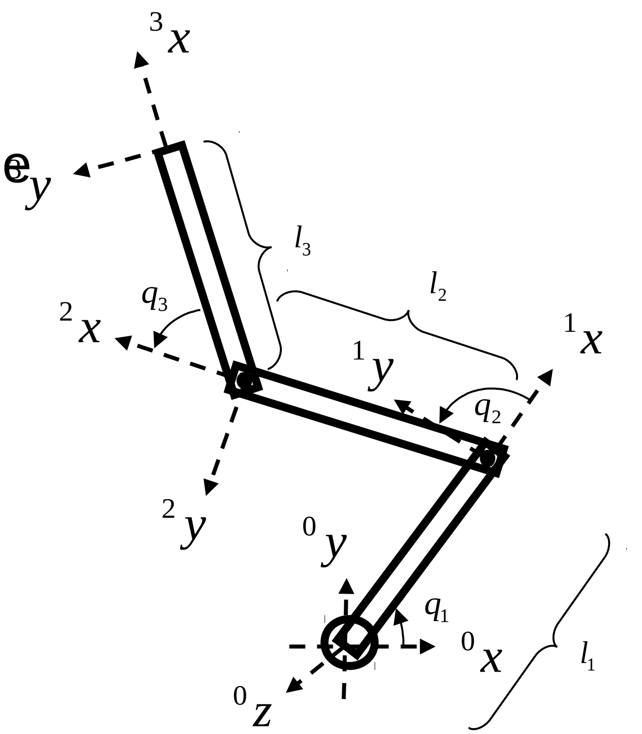
Therefore, the full Jacobian is rotated:

$$ {}^{k}J = \begin{pmatrix} {}^{k}R_b & 0 \\ 0 & {}^{k}R_b \end{pmatrix} {}^{b}J $$



Different Jacobian Reference Frames: Example

Express the Jacobian for the three-link arm in the reference frame of the end effector:

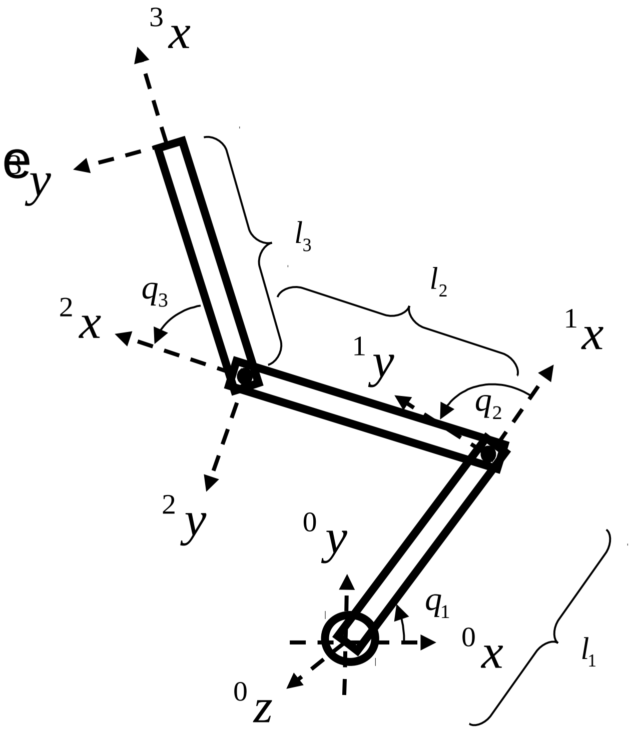


$$
{}^0R_3 = \begin{pmatrix} c_{123} & -s_{123} & 0 \\ s_{123} & c_{123} & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

$$
J = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} - l_3 s_{123} & -l_2 s_{12} - l_3 s_{123} & -l_3 s_{123} \\ l_1 c_1 + l_2 c_{12} + l_3 c_{123} & l_2 c_{12} + l_3 c_{123} & l_3 c_{123} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}
$$

Different Jacobian Reference Frames: Example

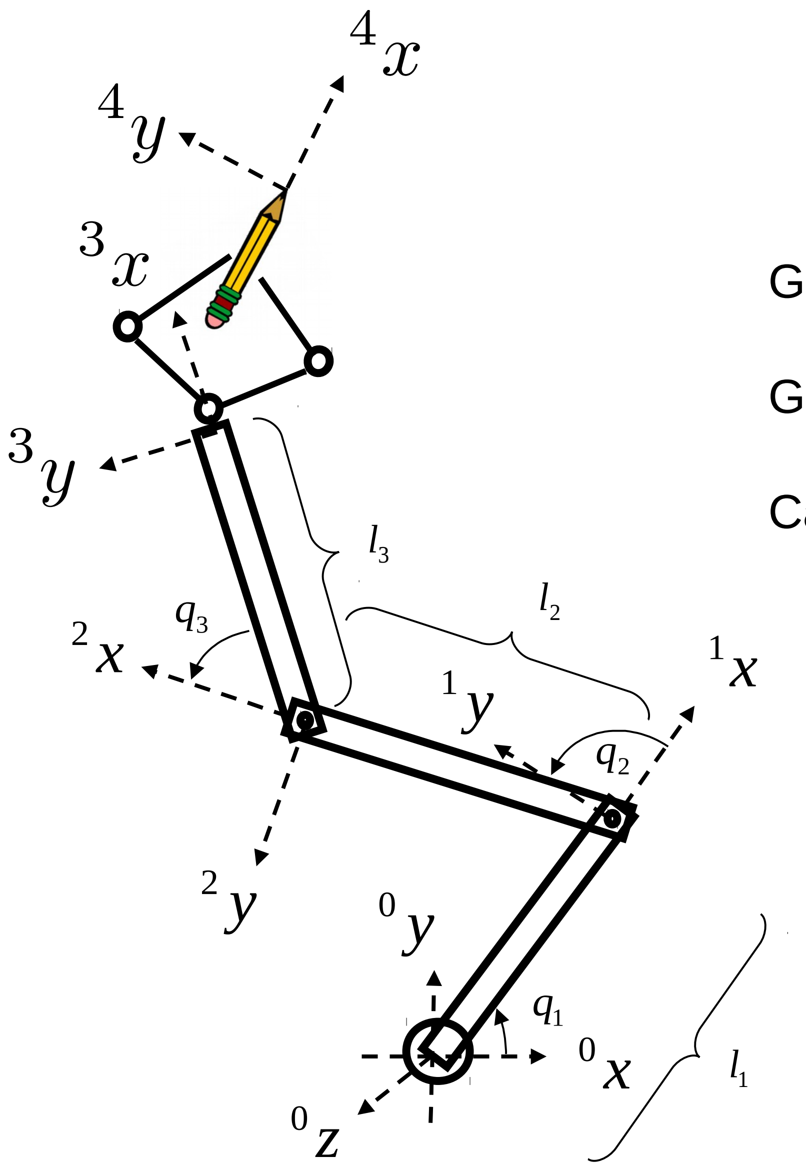
Express the Jacobian for the three-link arm in the reference frame of the end effector:



$$^{0}R_{3} = \begin{pmatrix} c_{123} & -s_{123} & 0 \\ s_{123} & c_{123} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$^{3}J = \begin{pmatrix} c_{123} & s_{123} & 0 & 0 & 0 & 0 \\ -s_{123} & c_{123} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{123} & s_{123} & 0 \\ 0 & 0 & 0 & -s_{123} & c_{123} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} -l_1 s_1 - l_2 s_{12} - l_3 s_{123} & -l_2 s_{12} - l_3 s_{123} & -l_3 s_{123} \\ l_1 c_1 + l_2 c_{12} + l_3 c_{123} & l_2 c_{12} + l_3 c_{123} & l_3 c_{123} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Think-pair-share



Given ^{0}J (in base frame)

Given: $^{3}R_0$ and $^{4}R_3$

Calculate: ^{4}J