CS 4610/5335: Robotics

Representation of Orientation

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The space or rotations

Special Orthogonal Group (3)

The space of all valid rotation matrices can be described as:

$$SO(3) = \{R \in R^{3\times 3} | RR^T = I, det(R) = +1\}$$

Why it's special: $det(R) = +1$ NOT $det(R) = -1$ (i.e. it uses a right handed coordinate frame)

Why it's orthogonal: the columns/rows are orthogonal

Why it's a group:

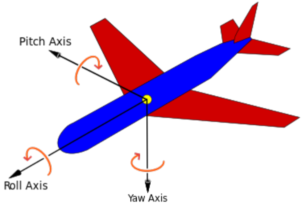
1. closed under multiplication: if $R_1, R_2 \in SO(3)$, then $R_1 R_2 \in SO(3)$.
2. has identity: $\exists I \in SO(3)$ such that $IR = R$.
3. has unique inverse
4. is associative

Representations of rotation

In addition to rotation matrices, we're going to study the following representations of rotation:

1. Euler angles
2. Axis angle coordinates (also called exponential coordinates)
3. Unit quaternions

Euler Angles



Euler angles are a 3-number representation of orientation:

$$\Gamma = \begin{pmatrix} \phi \\ \theta \\ \psi \end{pmatrix}$$

First rotate by ϕ about the x axis (yaw); then by θ about the y axis (pitch); and then by ϕ about the z axis (roll):

1. here we chose the $x - y - z$ ordering, but Euler angles can be defined for any ordering of three non-identical axes.

Euler Angles

First rotate by ϕ about the x axis (yaw); then by θ about the y axis (pitch); and then by ϕ about the z axis (roll):

$$R = R_x(\phi)R_y(\theta)R_z(\psi),$$

where:

$$R_x(\phi) = \begin{pmatrix} 1 & 0 & \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi) & \cos(\phi) \end{pmatrix}$$

$$R_y(\theta) = \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix}$$

$$R_z(\psi) = \begin{pmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Problems with Euler Angles

Euler angles do not do a good job encoding the distance between two orientations.

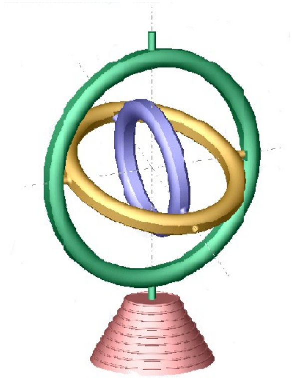
- Two orientations that have very different Euler angle representations could in fact be very similar

For example:

$$\Gamma_1 = \begin{pmatrix} 0 \\ 90 \\ 0 \end{pmatrix}, \Gamma_1 = \begin{pmatrix} 90 \\ 89 \\ 90 \end{pmatrix} \text{(in degrees)}$$

Although these two angles appear to be very far apart, in fact they are only 1 degree apart.

Problems with Euler Angles



Another problem: gimble lock

- when two axes are aligned
- at gimble lock, cannot represent angular velocities in certain directions.

Axis angle representation

Theorem: (Euler). Any orientation, $R \in SO(3)$, is equivalent to a rotation about a fixed axis, $\hat{k} \in S^2$, through an angle, $\theta \in [-\pi, \pi]$.

Axis Angle

Axis angle encodes an orientation as a unit axis, $\hat{k} = \begin{pmatrix} k_x \\ k_y \\ k_z \end{pmatrix}$, and the magnitude of the angle, θ.

- axis angle is also called "exponential coordinates"
- $\hat{k}$ and θ are often expressed as a single vector, $k = \theta \hat{k}$. (The magnitude of k now encodes the magnitude of rotation, θ.)
- Rotations of less than 180 degrees have a unique axis angle representation (unlike euler angles). A 180 deg rotation can represented two ways.

Converting to axis angle

Given rotation matrix, R, calculate $\hat{k}$ and θ:

- $\theta = \cos^{-1} \frac{Tr(R)-1}{2}$,
- $\hat{k} = \frac{1}{\sin(\theta)} \begin{pmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{pmatrix}$

where $R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$, and

$Tr(R)$ denotes the *trace* of R, $Tr(R) = r_{11} + r_{22} + r_{33}$.

Converting back to a rotation matrix

Given axis angle, $\hat{k}$ and θ, calculate the corresponding rotation matrix, R:

- $R = I + S(\hat{k})\sin(\theta) + S(\hat{k})^2(1 - \cos(\theta))$
- where $S(\hat{k})$ denotes the skew-symmetric matrix:

$$S(\hat{k}) = \begin{pmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{pmatrix}$$

Why is axis angle a nice representation?

Suppose you are currently at rotation R_1 and you want to rotate into R_2. What axis should you rotate about?

Answer:

- Calculate desired delta rotation, $\delta R = R_1^T R_2$.
- Convert δR into axis angle, $\hat{k}$ and θ
- Rotate θ about $\hat{k}$
- or ... start rotating with some angular velocity, $\omega = \alpha \hat{k}$, where α is the speed of rotation.

But, axis angle has its problems too...

Suppose you have two orientations encoded in axis angle,

$$k_1 = \begin{pmatrix} \pi/2 \\ 0 \\ 0 \end{pmatrix}, \text{ and } k_2 = \begin{pmatrix} 0 \\ \pi/2 \\ 0 \end{pmatrix}.$$

Calculate the delta rotation:

- This is **not** the right answer: $k_1 - k_2 = \begin{pmatrix} \pi/2 \\ -\pi/2 \\ 0 \end{pmatrix}$.
- according to that, the magnitude of $k_1 - k_2$ is $\|k_1 - k_2\| = \frac{\pi}{\sqrt{2}} = 127.27$ degrees

But, axis angle has its problems too...

$$k_1 = \begin{pmatrix} \pi/2 \\ 0 \\ 0 \end{pmatrix} \qquad k_2 = \begin{pmatrix} 0 \\ \pi/2 \\ 0 \end{pmatrix}$$

$$^1R_2 = {^BR_1}^T \, {^BR_2}$$

$$^bR_1 = R_x(\pi/2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\pi/2) & -\sin(\pi/2) \\ 0 & \sin(\pi/2) & \cos(\pi/2) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$^bR_2 = R_y(\pi/2) = \begin{pmatrix} \cos(\pi/2) & 0 & \sin(\pi/2) \\ 0 & 1 & 0 \\ -\sin(\pi/2) & 0 & \cos(\pi/2) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$^1R_2 = {^BR_1}^T \, {^BR_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\theta = \cos^{-1}\left(\frac{trace(R)-1}{2}\right) = \cos^{-1}\left(-\frac{1}{2}\right) = \tfrac{2}{3}\pi \qquad \hat{k} = \frac{1}{2\sin\theta}\begin{pmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \qquad k = \frac{\frac{2}{3}\pi}{\sqrt{3}}\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

But, $\|k_1 - k_2\| = \frac{\pi}{\sqrt{2}} \neq \frac{2\pi}{3}$!

What's going on here?

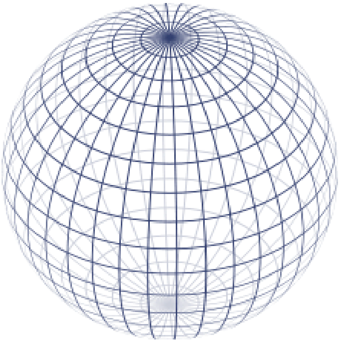
- If k_1 and k_2 are both close to the origin, then Euclidean distance is a good approximation for the magnitude of the angle between them.
- But, the approximation gets worse (overestimates angle) as you get further away from the origin...
- This effect is kind of like creating a world map using the Mercator projection ...

What's going on here?

- ...the problem is that $SO(3)$ is *not* a Euclidean space...

What's going on here?

- so what it it?



Quaternions

Generalization of complex numbers: $Q = q_0 + iq_1 + jq_2 + kq_3$

$$Q = (q_0, q)$$

Essentially a 4-dimensional quantity

Properties of complex dimensions:

$$ii = jj = kk = ijk = -1 \qquad jk = -kj = i$$

$$ij = -ji = k \qquad ki = -ik = j$$

Multiplication:

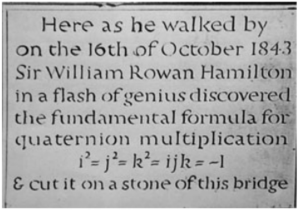
$$QP = (q_0 + iq_1 + jq_2 + kq_3)(p_0 + ip_1 + jp_2 + kp_3)$$

$$QP = (p_0 q_0 - p \cdot q,\ p_0 q + q_0 p + p \times q)$$

Complex conjugate: $Q^* = (q_0, q)^* = (q_0, -q)$

Quaternions

Invented by Hamilton in 1843:


Here as he walked by
on the 16th of October 1843
Sir William Rowan Hamilton
in a flash of genius discovered
the fundamental formula for
quaternion multiplication
$i^2 = j^2 = k^2 = ijk = -1$
& cut it on a stone of this bridge

Along the royal canal in Dublin...

Unit Quaternions as a representation of rotation

Let's consider the set of unit quaternions: $Q^2 = q_0{}^2 + q_1{}^2 + q_2{}^2 + q_3{}^2 = 1$

This is a four-dimensional hypersphere, *i.e.* the 3-sphere S^3

The identity quaternion is: $Q = (1,0)$

Since: $QQ^* = (q_0, q)(q_0, -q) = (q_0 q_0 - q^2, q_0 q - q_0 q + q \times q) = (1,0)$

Therefore, the inverse of a unit quaternion is: $Q^* = Q^{-1}$

Unit Quaternions as a representation of rotation

Associate a rotation with a unit quaternion as follows:

Given a unit axis, $\hat{k}$, and an angle, θ: ← (just like axis angle)

The associated quaternion is: $Q_{\hat{k},\theta} = \left(\cos\left(\frac{\theta}{2}\right), \hat{k}\sin\left(\frac{\theta}{2}\right) \right)$

Therefore, Q represents the same rotation as $-Q$

Let ${}^iP = \left(0, {}^ip\right)$ be the quaternion associated with the vector ${}^ip$

You can rotate ${}^aP$ from frame *a* to *b*: ${}^bP = Q_{ba}\,{}^aP Q_{ba}^*$

Composition: $Q_{ca} = Q_{cb}Q_{ba}$

Inversion: $Q_{cb} = Q_{ca}Q_{ba}^{-1}$

Quaternions example 1

Rotate $\quad {}^aP = \left(0, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right)$ by $\quad Q = \left(\frac{1}{\sqrt{2}}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}\right)$

$$
\begin{aligned}
{}^bP = Q\,{}^aPQ^* &= \left(\frac{1}{\sqrt{2}}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}\right)\left(0, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right)\left(\frac{1}{\sqrt{2}}, \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}\right) \\
&= \left(\frac{1}{\sqrt{2}}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}\right)\left(0, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}\right) \\
&= \left(0, \begin{pmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{pmatrix}\right) = \left(0, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}\right)
\end{aligned}
$$

Quaternions example 2

Find the difference between these two axis angle rotations:

$$k_1 = \begin{pmatrix} \pi/2 \\ 0 \\ 0 \end{pmatrix} \qquad k_2 = \begin{pmatrix} 0 \\ \pi/2 \\ 0 \end{pmatrix}$$

$$\sin\left(\tfrac{\pi}{4}\right) = \cos\left(\tfrac{\pi}{4}\right) = \tfrac{1}{\sqrt{2}} \qquad Q_{cb} = \left(\tfrac{1}{\sqrt{2}}, \begin{pmatrix} 0 \\ \tfrac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \right) \quad Q_{ba} = \left(\tfrac{1}{\sqrt{2}}, \begin{pmatrix} \tfrac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix} \right)$$

$$QP = \left(p_0 q_0 - p \cdot q,\ p_0 q + q_0 p + p \times q \right)$$

$$Q_{cb} = Q_{ca} Q_{ba}^{-1} = \left(\tfrac{1}{\sqrt{2}}, \begin{pmatrix} 0 \\ \tfrac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \right) \left(\tfrac{1}{\sqrt{2}}, \begin{pmatrix} -\tfrac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix} \right)$$

$$= \left(\tfrac{1}{2}, \tfrac{1}{\sqrt{2}} \begin{pmatrix} -\tfrac{1}{\sqrt{2}} \\ \tfrac{1}{\sqrt{2}} \\ -\tfrac{1}{\sqrt{2}} \end{pmatrix} \right) = \left(\tfrac{1}{2}, \begin{pmatrix} -\tfrac{1}{2} \\ \tfrac{1}{2} \\ -\tfrac{1}{2} \end{pmatrix} \right)$$

$$\theta_{cb} = \cos^{-1}\left(\tfrac{1}{2}\right) = \tfrac{2}{3}\pi$$

$$k_{cb} = \begin{pmatrix} -\tfrac{1}{\sqrt{3}} \\ \tfrac{1}{\sqrt{3}} \\ -\tfrac{1}{\sqrt{3}} \end{pmatrix}$$

Interpolation using quaterions

Suppose you're given two rotations, R_1 and R_2

How do you calculate intermediate rotations?

$$R_i = \alpha R_1 + (1-\alpha) R_2 \longleftarrow$$

This does not even result in a rotation matrix

Do quaternions help?

$$Q_i = \frac{\alpha Q_1 + (1-\alpha) Q_2}{\left|\alpha Q_1 + (1-\alpha) Q_2\right|}$$

Suprisingly, this actually works

- Finds a geodesic

This method normalizes automatically (SLERP):

$$Q_i = \frac{Q_1 \sin(1-\alpha)\Omega + Q_2 \sin \alpha\Omega}{\sin \Omega}$$