Probability and Decision Theory

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Some images and slides are used from:

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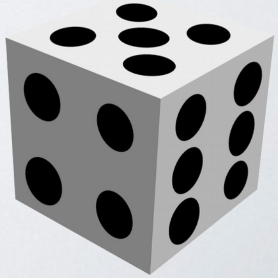
QUANTIFYING UNCERTAINTY WITH PROBABILITIES

- Generally we may use probabilities due to
- Ignorance
  - E.g., we don’t know our opponent
- Laziness
  - It is too difficult or takes too much effort to model the event in detail
- The event is inherently random



HOW DO WE INTERPRET PROBABILITIES

- Frequentist view
  - Objective: If I throw a die many times it will come up 3 one sixth of the times
- Strength of belief view (Bayesian)
  - Subjective: How strongly do I, should I, believe it will come up 3
  - These are not claims of a “probabilistic tendency” in the current situation (but might be learned from past experience)
  - Probabilities of propositions change with new evidence:
    - e.g., P (get to airport on time|no reported accidents, 5 a.m.) = 0.15

(Discrete) Random variables

What is a random variable?

Suppose that the variable *a* denotes the outcome of a role of a single six-sided die:

$$a \in \underbrace{\{1, 2, 3, 4, 5, 6\}}_{} = A$$

a is a *random variable*

this is the *domain* of *a*

Another example:

Suppose *b* denotes whether it is raining or clear outside:

$$b \in \{rain, clear\} = B$$

Probability distribution

A probability distribution associates each with a probability of occurrence, represented by a *probability mass function (pmf)*.

A probability table is one way to encode the distribution:

$$a \in \{1, 2, 3, 4, 5, 6\} = A \qquad b \in \{rain, clear\} = B$$

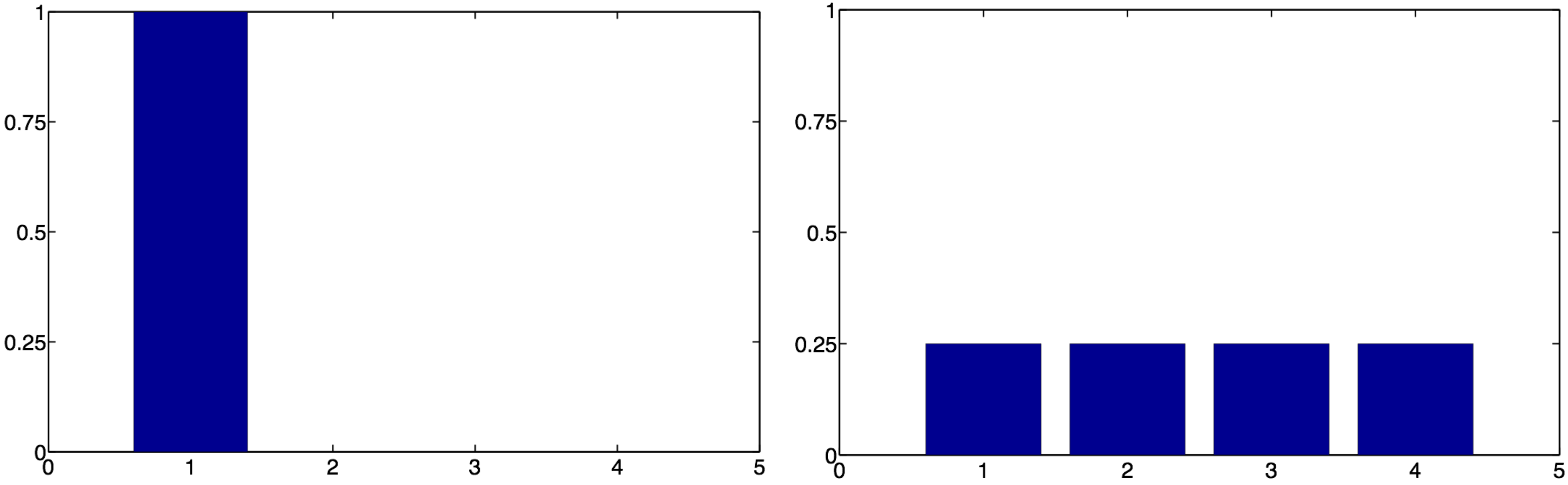
a	P(a)
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6

b	P(b)
rain	1/4
clear	3/4

<u>All probability distributions must satisfy the following:</u>

1. $\forall a \in A, a \geq 0$
2. $\sum_{a \in A} a = 1$

Example pmfs



Two pmfs over a state space of $X = \{1, 2, 3, 4\}$

Writing probabilities

a	P(a)
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6

b	P(b)
rain	1/4
clear	3/4

For example: $p(a = 2) = 1/6$

$$p(b = clear) = 3/4$$

But, sometimes we will abbreviate this as: $p(2) = 1/6$

$$p(clear) = 3/4$$

Types of random variables

Propositional or Boolean random variables

- e.g., *Cavity* (do I have a cavity?)
- *Cavity* = *true* is a proposition, also written *cavity*

Discrete random variables (finite or infinite)

- e.g., *Weather* is one of ⟨*sunny, rain, cloudy, snow*⟩
- *Weather* = *rain* is a proposition
- Values must be exhaustive and mutually exclusive

Continuous random variables (bounded or unbounded)

- e.g., Temp < 22.0

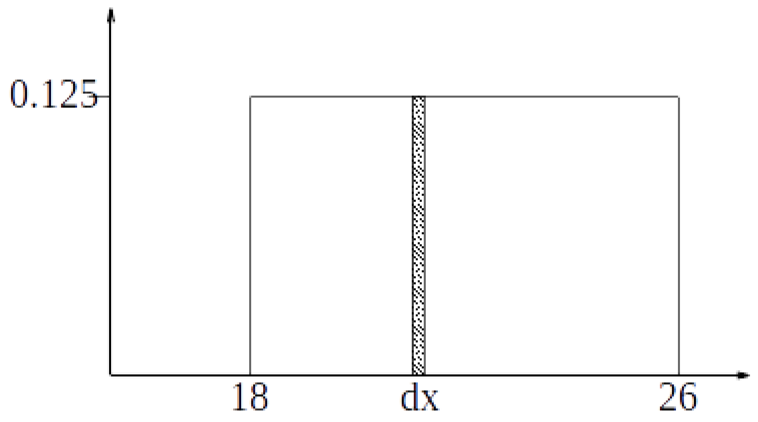
Continuous random variables

Cumulate distribution function (*cdf*), $F(q)=(X<q)$ with $P(a<X\leq b)=F(b)-F(a)$

Probability density function (*pdf*), $f(x)=\frac{d}{dx}F(x)$ with $P(a<X\leq b)=\int_a^b f(x)$

Express distribution as a parameterized function of value:

- e.g., $P(X = x) = U[18, 26](x)$ = uniform density between 18 and 26



Here *P* is a density; integrates to 1.

$P(X = 20.5) = 0.125$ really means $\lim_{dx\to 0} P(20.5 \leq X \leq 20.5 + dx)/dx = 0.125$

Joint probability distributions

Given random variables: $X_1, X_2, \ldots, X_n$

The *joint distribution* is a probability assignment to all combinations: $P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)$

or: $P(x_1, x_2, \ldots, x_n)$

Sometimes written as: $P(X_1 = x_1 \wedge X_2 = x_2 \wedge \cdots \wedge X_n = x_n)$

As with single-variate distributions, joint distributions must satisfy:

1. $P(x_1, x_2, \ldots, x_n) \geq 0$

2. $\sum_{x_1, \ldots, x_n} P(x_1, x_2, \ldots, x_n) = 1$

Prior or unconditional probabilities of propositions
e.g., P (*Cavity* = *true*) = 0.1 and P (*Weather* = *sunny*) = 0.72
correspond to belief prior to arrival of any (new) evidence

Joint probability distributions

Joint distributions are typically written in table form:

T	W	P(T,W)
Warm	snow	0.1
Warm	hail	0.3
Cold	snow	0.5
Cold	hail	0.1

An **event** is a set of particular outcomes
– e.g. it's warm and hailing...

Marginalization

Given P(T,W), calculate P(T) or P(W)...

T	W	P(T,W)
Warm	snow	0.1
Warm	hail	0.3
Cold	snow	0.4
Cold	hail	0.2

$$P(T) = \sum_{w \in W} P(T, w)$$

T	P(T)
Warm	0.4
Cold	0.6

$$P(W) = \sum_{t \in T} P(t, W)$$

W	P(W)
snow	0.5
hail	0.5

Marginalization

Given P(T,W), calculate P(T) or P(W)...

T	W	P(T,W)
Warm	snow	0.3
Warm	hail	0.2
Cold	snow	0.2
Cold	hail	0.3

$$P(T) = \sum_{w \in W} P(T, w)$$

T	P(T)
Warm	?
Cold	?

$$P(W) = \sum_{t \in T} P(t, W)$$

W	P(W)
snow	?
hail	?

Conditional Probability

At a regular checkup, assume the unconditional probability that you have a cavity is 0.2

But what if I have a toothache? Does this change the probability of a cavity?

P(cavity | toothache)

Probability of a cavity GIVEN a toothache

Conditional Probability

At a regular checkup, assume the unconditional probability that you have a cavity is 0.2

But what if I have a toothache? Does this change the probability of a cavity?

P(cavity | toothache)

Probability of a cavity GIVEN a toothache

Definition of conditional probability: $$P(a|b) = \frac{P(a,b)}{P(b)}$$

Can be rewritten as the Product Rule: $$P(a,b) = P(a|b)P(b) = P(b|a)P(a)$$

$$P(x_n, \ldots, x_1) = P(x_n|x_{n-1}, \ldots, x_1)P(x_{n-1}, \ldots, x_1)$$

The Chain Rule

The product rule can be applied iteratively to obtain the chain rule:

$$P(x_n, \ldots, x_1) = P(x_n | x_{n-1}, \ldots, x_1) P(x_{n-1}, \ldots, x_1)$$

$$= P(x_n | x_{n-1}, \ldots, x_1) P(x_{n-1} | x_{n-2}, \ldots, x_1) \ldots P(x_1)$$

Chain rule

Infer Conditional Probabilities

	toothache		*¬ toothache*	
	catch	*¬catch*	*catch*	*¬catch*
cavity	0.108	0.012	0.072	0.008
¬cavity	0.016	0.064	0.144	0.576

We want to calculate the probability of a cavity given that we have a toothache

Recall that: $$P(a|b) = \frac{P(a,b)}{P(b)}$$

$$P(cavity \mid toothache) = \frac{P(cavity \land toothache)}{P(toothache)} = \frac{0.108 + 0.012}{0.108+0.012+0.016+0.064} = 0.6$$

$$P(\neg cavity \mid toothache) = \frac{P(\neg cavity \land toothache)}{P(toothache)} = ???$$

Infer Conditional Probabilities

P(snow|warm) = Probability that it will snow *given* that it is warm

T	W	P(T,W)
Warm	snow	0.3
Warm	hail	0.2
Cold	snow	0.2
Cold	hail	0.3

Infer Conditional Probabilities

Given P(T,W), calculate P(T|w) or P(W|t)...

T	W	P(T,W)
Warm	snow	0.3
Warm	hail	0.2
Cold	snow	0.2
Cold	hail	0.3

W	P(W|T=warm)
snow	?
hail	?

Infer Conditional Probabilities

Given P(T,W), calculate P(T|w) or P(W|t)...

T	W	P(T,W)
Warm	snow	0.3
Warm	hail	0.2
Cold	snow	0.2
Cold	hail	0.3

W	P(W|T=warm)
snow	?
hail	?

$$P(snow|warm) = \frac{P(warm, snow)}{P(warm)} = \frac{P(warm, snow)}{P(warm, hail) + P(warm, snow)}$$

Infer Conditional Probabilities

Given P(T,W), calculate P(T|w) or P(W|t)...

T	W	P(T,W)
Warm	snow	0.3
Warm	hail	0.2
Cold	snow	0.2
Cold	hail	0.3

W	P(W|T=warm)
snow	0.6
hail	?

$$P(snow|warm) = \frac{P(warm, snow)}{P(warm)} = \frac{P(warm, snow)}{P(warm, hail) + P(warm, snow)}$$

$$= \frac{0.3}{0.2 + 0.3}$$

Infer Conditional Probabilities

Given P(T,W), calculate P(T|w) or P(W|t)...

T	W	P(T,W)
Warm	snow	0.3
Warm	hail	0.2
Cold	snow	0.2
Cold	hail	0.3

W	P(W|T=warm)
snow	0.6
hail	?

$$P(snow|warm) = \frac{P(warm, snow)}{P(warm)} = \frac{P(warm, snow)}{P(warm, hail) + P(warm, snow)}$$

$$= \frac{0.3}{0.2 + 0.3}$$

Infer Conditional Probabilities

Given P(T,W), calculate P(T|w) or P(W|t)...

T	W	P(T,W)
Warm	snow	0.3
Warm	hail	0.2
Cold	snow	0.2
Cold	hail	0.3

→

W	P(W|T=warm)
snow	0.6
hail	0.4

$$P(snow|warm) = \frac{P(warm, snow)}{P(warm)} = \frac{P(warm, snow)}{P(warm, hail) + P(warm, snow)}$$

$$= \frac{0.3}{0.2 + 0.3}$$

Infer Conditional Probabilities

Given P(T,W), calculate P(T|w) or P(W|t)...

T	W	P(T,W)
Warm	snow	0.3
Warm	hail	0.2
Cold	snow	0.2
Cold	hail	0.3

W	P(W|T=warm)
snow	0.6
hail	0.4

W	P(W|T=cold)
snow	?
hail	?

Infer Conditional Probabilities

Given P(T,W), calculate P(T|w) or P(W|t)...

T	W	P(T,W)
Warm	snow	0.3
Warm	hail	0.2
Cold	snow	0.2
Cold	hail	0.3

W	P(W|T=warm)
snow	0.6
hail	0.4

W	P(W|T=cold)
snow	0.4
hail	0.6

Normalization

T	W	P(T,W)
Warm	snow	0.3
Warm	hail	0.2
Cold	snow	0.2
Cold	hail	0.3

→

W	P(W|T=warm)
snow	0.6
hail	0.4

Can we avoid explicitly computing this denominator?

$$P(snow|warm) = \frac{P(warm, snow)}{P(warm, hail) + P(warm, snow)}$$

Any ideas?

Normalization

T	W	P(T,W)
Warm	snow	0.3
Warm	hail	0.2
Cold	snow	0.2
Cold	hail	0.3

$$P(W|t) = \frac{P(W,t)}{P(t)}$$

W	P(W|T=warm)
snow	0.6
hail	0.4

Two steps:

1. Copy entries

W	P(W,T=warm)
snow	0.3
hail	0.2

2. Scale them up so that entries sum to 1

W	P(W|T=warm)
snow	0.6
hail	0.4

Normalization

T	W	P(T,W)
Warm	snow	0.3
Warm	hail	0.4
Cold	snow	0.2
Cold	hail	0.1

<u>Two steps:</u>

1. Copy entries

T	P(T,W=hail)
warm	?
cold	?

2. Scale them up so that entries sum to 1

T	P(T|W=hail)
warm	?
cold	?

Normalization

T	W	P(T,W)
Warm	snow	0.3
Warm	hail	0.4
Cold	snow	0.2
Cold	hail	0.1

Two steps:

1. Copy entries

T	P(T,W=hail)
warm	0.4
cold	0.1

2. Scale them up so that entries sum to 1

T	P(T|W=hail)
warm	?
cold	?

Normalization

T	W	P(T,W)
Warm	snow	0.3
Warm	hail	0.4
Cold	snow	0.2
Cold	hail	0.1

Two steps:

1. Copy entries

T	P(T,W=hail)
warm	0.4
cold	0.1

2. Scale them up so that entries sum to 1

T	P(T|W=hail)
warm	0.8
cold	0.2

$$P(W|t) = \frac{P(W,t)}{P(t)}$$

The only purpose of this denominator is to make the distribution sum to one.
– we achieve the same thing by scaling.

A Problem...

So far we have done inference using the full joint distribution

But, this doesn't scale well with complex domains…

– e.g. 30 boolean variables requires us to specify 2^30 probabilities!

Way too many!

<u>Possible solution</u>: exploit independence!

INDEPENDENCE

- Expand our dentistry joint distribution by adding a fourth variable, weather
  - *P(Toothache, Catch, Cavity, Weather)*
- *Now have a table of 32 entries*
  - *(Assuming Weather = {cloudy, rain, sunny, snow})*
- But consider the relation these variables have:
  - *P(toothache, catch, cavity, cloudy)*
  - *Does cloudy influence dentistry?*

INDEPENDENCE

- By the product rule:
  - *P(toothache, catch, cavity, cloudy) = P(cloudy|toothache, catch, cavity) x P(toothache, catch, cavity)*
- Reasonable to assume dental facts and the weather don't influence each other, so:
  - *P(cloudy | toothache, catch, cavity) = P(cloudy)*
- *Therefore: P(toothache, catch, cavity, cloudy) = P(cloudy) x P(toothache, catch, cavity)*
- *Reducing the 32 entry table to an 8 element table and 4 element table*
  - *Calculation of joint is a product of the entries in these tables*

INDEPENDENCE

- *A* and *B* are *independent* iff $P(X, Y) = P(X)P(Y)$
  - $P(Toothache, Catch, Cavity, Weather) = P(Toothache, Catch, Cavity)P(Weather)$
- *In other words the joint distribution factors into a product of two smaller, simpler distributions*
- *Equivalently*
  - $P(X|Y) = P(X)$ *or* $P(Y|X) = P(Y)$
- *If two events are independent, knowing one has occurred does not give us any new information about the other*

INDEPENDENT?

- Formally independence is
  - *P(X|Y) = P(X) or*
  - *P(Y|X) = P(Y) or*
  - *P(X, Y) = P(X)P(Y)*

$P_1(T,W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$P(T)$

T	P
hot	0.5
cold	0.5

$P(W)$

W	P
sun	0.6
rain	0.4

$P_2(T,W)$

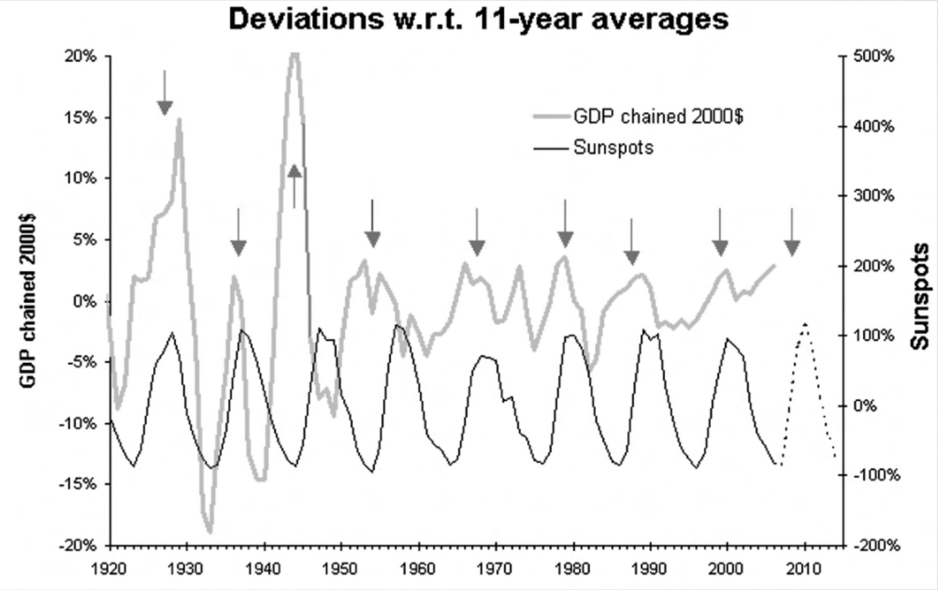
T	W	P
hot	sun	0.3
hot	rain	0.2
cold	sun	0.3
cold	rain	0.2

INDEPENDENCE

- Formally independence is (equivalently)
  - *In terms of conditional distributions*
    - $P(X|Y) = P(X)$ or $P(Y|X) = P(Y)$
  - *In terms of joint distributions*
    - $P(X, Y) = P(X)P(Y)$
- Independence can greatly reduce the probabilities that need to be specified

INDEPENDENCE

- However full independence is rare
  - Just as separate subgraphs were rare in CSPs
- In empirical data especially, it's rare not to see some interaction (real or not)
  - Sunspot activity and GDP,
  - Sunspot activity and stock market
  - Sunspot activity and …
- We need to consider a more limited form of independence



CONDITIONAL INDEPENDENCE

We will consider a more limited form of independence
Conditional Independence:

$$P(X,Y|Z) = P(X|Z)P(Y|Z)$$

P(Toothache, Cavity, Catch) has 8 − 1 = 7 independent entries

If I have a cavity, the probability that the probe catches in it doesn't depend on whether I have a toothache:

$$P(catch|toothache, cavity) = P(catch|cavity)$$

The same independence holds if I haven't got a cavity:

$$P(catch|toothache, \neg cavity) = P(catch|\neg cavity)$$

Catch is conditionally independent of Toothache given Cavity:

$$P(Catch|Toothache, Cavity) = P(Catch|Cavity)$$

Equivalent statements:

$$P(Toothache|Catch, Cavity)=P(Toothache|Cavity)$$

$$P(Toothache, Catch|Cavity)=P(Toothache|Cavity)P(Catch|Cavity)$$

CONDITIONAL INDEPENDENCE

- Write out full joint distribution using chain rule:
- P(*Toothache, Catch, Cavity*)
  = P(*Toothache*|*Catch, Cavity*)P(*Catch, Cavity*)
  = P(*Toothache*|*Catch, Cavity*)P(*Catch*|*Cavity*)P(*Cavity*)
  = P(*Toothache*|*Cavity*)P(*Catch*|*Cavity*)P(*Cavity*)
- 2 + 2 + 1 = 5 independent numbers
- In most cases, the use of conditional independence reduces the size of the representation of the joint distribution from exponential in *n* to linear in *n*
- Conditional independence is our most basic and robust form of knowledge about uncertain environments

Bayes Rule

$$P(a|b) = \frac{P(b|a)P(a)}{P(b)}$$

Thomas Bayes (1701 – 1761):
– English statistician, philosopher and Presbyterian minister
– formulated a specific case of the formula above
– his work later published/generalized by Richard Price

Bayes Rule

$$P(a|b) = \frac{P(b|a)P(a)}{P(b)}$$

It's easy to derive from the product rule:

$$P(a,b) = P(b|a)P(a) = \underbrace{P(a|b)}P(b)$$

Solve for this

Using Bayes Rule

$$P(a|b) = \frac{P(b|a)P(a)}{P(b)}$$

$$P(cause|effect) = \frac{P(effect|cause)P(cause)}{P(effect)}$$

Using Bayes Rule

$$P(a|b) = \frac{P(b|a)P(a)}{P(b)}$$

$$\Downarrow$$

$$P(cause|effect) = \frac{P(effect|cause)P(cause)}{P(effect)}$$

But harder to estimate this

It's often easier to estimate this

Bayes Rule Example

$$P(cause|effect) = \frac{P(effect|cause)P(cause)}{P(effect)}$$

Suppose you have a stiff neck...

Suppose there is a 70% chance of meningitis if you have a stiff neck:

stiff neck → s meningitis → m

$$P(s|m) = 0.7$$

What are the chances that you have meningitis?

Bayes Rule Example

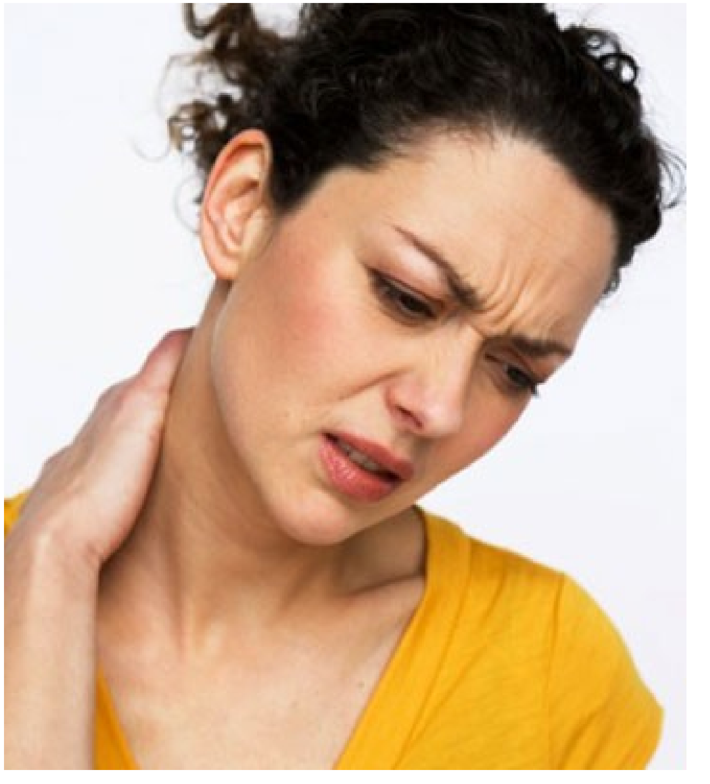
$$P(cause|effect) = \frac{P(effect|cause)P(cause)}{P(effect)}$$

Suppose you have a stiff neck...

Suppose there is a 70% chance of meningitis
if you have a stiff neck:

stiff neck → s, meningitis → m

$$P(s|m) = 0.7$$



What are the chances that you have meningitis?

We need a little more information...

Bayes Rule Example

$$P(cause|effect) = \frac{P(effect|cause)P(cause)}{P(effect)}$$

$$P(s|m) = 0.7$$

$$P(s) = 0.01$$

Prior probability of stiff neck

$$P(m) = \frac{1}{50000}$$

Prior probability of meningitis

$$P(m|s) = \frac{P(s|m)P(m)}{P(s)} = \frac{0.7 \times \frac{1}{50000}}{0.01} = 0.0014$$

Bayes Rule Example

Given:

W	P(W)
snow	0.8
hail	0.2

T	W	P(T|W)
Warm	snow	0.3
Warm	hail	0.4
Cold	snow	0.7
Cold	hail	0.6

Calculate P(W|warm):

$$P(W|warm) = \frac{P(warm|W)P(W)}{P(warm)}$$

Bayes Rule Example

Given:

W	P(W)
snow	0.8
hail	0.2

T	W	P(T|W)
Warm	snow	0.3
Warm	hail	0.4
Cold	snow	0.7
Cold	hail	0.6

Calculate P(W|warm):

$$P(W|warm) = \frac{P(warm|W)P(W)}{P(warm)}$$

$$P(hail|warm) = \frac{0.4 \times 0.2}{P(warm)} = \frac{0.08}{P(warm)}$$

=0.25

$$P(snow|warm) = \frac{0.3 \times 0.8}{P(warm)} = \frac{0.24}{P(warm)}$$

normalize

=0.75

SO FAR: IMPORTANT FORMULAS

- *Basic axioms*
  - $P(a) \geq 0$
  - $\sum_{a \in X} P(a) = 1$
  - $P(\neg a) = 1 - P(a)$
  - *(holds for joint distributions as well)*
- *Inclusion-exclusion principle*
  - $P(a \vee b) = P(a) + P(b) - P(a \wedge b)$
- Marginal probability
  - $P(Y) = \sum_{z \in Z} P(Y,z)$

- X and Y are independent IF
  - $P(X, Y) = P(X)P(Y)$
  - $P(X|Y) = P(X)$ *or* $P(Y|X) = P(Y)$ *or*
- *X, Y are conditional independent given Z* IF
  - $\mathbf{P}(X,Y|Z) = \mathbf{P}(X|Z)\, \mathbf{P}(Y|Z)$
  - $\mathbf{P}(X|Y,Z) = \mathbf{P}(X|Z)$
  - $\mathbf{P}(Y|X,Z) = \mathbf{P}(Y|Z)$

- Definition of conditional probability:
  - $P(a \mid b) = \dfrac{P(a \wedge b)}{P(b)}$
  - $P(X \mid Y) = \dfrac{P(X, Y)}{P(Y)}$
- Product Rule:
  - $P(a \wedge b) = P(a|b)\, P(b)$ or $P(a \wedge b) = P(b|a)\, P(a)$
  - $P(X_n, \ldots, X_1) = P(X_n|X_{n-1}, \ldots, X_1)\, P(X_{n-1}, \ldots, X_1)$
- Chain Rule:
  - $P(X_n, \ldots, X_1) = P(X_n|X_{n-1}, \ldots, X_1)\, P(X_{n-1}|X_{n-2}, \ldots, X_1) \ldots P(X_1)$
- Bayes Rule
  - $P(Y|X) = \dfrac{P(X|Y)\, P(Y)}{P(X)}$
  - $P(Y|X, e) = \dfrac{P(X|Y, e)\, P(Y \mid e)}{P(X \mid e)}$

MAKING DECISIONS UNDER UNCERTAINTY

- Going to the airport: A_X means leaving X minutes before the flight
- Suppose I believe the following:
- P(A_{25} gets me there on time|...) = 0.04
- P(A_{90} gets me there on time|...) = 0.70
- P(A_{120} gets me there on time|...) = 0.95
- P (A_{1440} gets me there on time|...) = 0.9999
- Which action to choose?

MAKING DECISIONS UNDER UNCERTAINTY

- Suppose I believe the following:
- $P(A_{25}$ gets me there on time|...$) = 0.04$
- $P(A_{90}$ gets me there on time|...$) = 0.70$
- $P(A_{120}$ gets me there on time|...$) = 0.95$
- $P(A_{1440}$ gets me there on time|...$) = 0.9999$
- Which action to choose?
- Depends on my preferences for missing flight vs. airport cuisine, etc.
- Utility theory is used to represent and infer preferences
- Decision theory = utility theory + probability theory

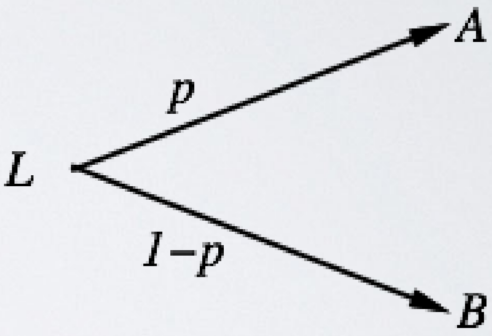
MAKING DECISIONS UNDER UNCERTAINTY

- Rational decision making requires reasoning about one's *uncertainty* and *objectives*
- Previous section focused on uncertainty
- This section will discuss how to make rational decisions based on a *probabilistic model* and *utility function*
- Focus will be on single step decisions, later we will consider sequential decision problems

PREFERENCES

- An agent chooses among prizes (A, B, etc.) and lotteries, i.e., situations with uncertain prizes

- Lottery $L=[p,A;\ (1-p),B]$



- Notation:

  - $A \succ B$ A preferred to B

  - $A \sim B$ indifference between A and B

  - $A \succsim B$ B not preferred to A

RATIONAL PREFERENCES

- Idea: preferences of a rational agent (*not* a human!) must obey constraints
- Rational preferences ⇒ behavior describable as maximization of expected utility
- The Axioms of Rationality:

Orderability

$$(A \succ B) \vee (B \succ A) \vee (A \sim B)$$

Transitivity

$$(A \succ B) \wedge (B \succ C) \Rightarrow (A \succ C)$$

Continuity

$$A \succ B \succ C \Rightarrow \exists p\ [p, A;\ 1-p, C] \sim B$$

Substitutability

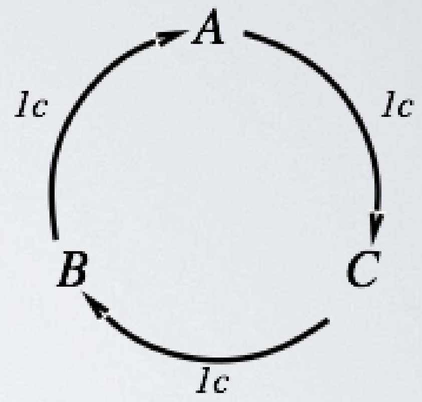
$$A \sim B \Rightarrow [p, A;\ 1-p, C] \sim [p, B; 1-p, C]$$

Monotonicity

$$A \succ B \Rightarrow$$
$$(p \geq q \Leftrightarrow [p, A;\ 1-p, B] \succeq [q, A;\ 1-q, B])$$

RATIONAL PREFERENCES

- Violating the constraints leads to self-evident irrationality
- For example: an agent with intransitive preferences can be induced to give away all its money



- If $B \succ C$, then an agent who has C would pay (say) 1 cent to get B
- If $A \succ B$, then an agent who has B would pay (say) 1 cent to get A
- If $C \succ A$, then an agent who has A would pay (say) 1 cent to get C

REMINDER: EXPECTATION

- The expected value of a function of a random variable is the average, weighted by the probability distribution over outcomes

- Example: How long to get to the airport?

Probability:	0.25		0.50		0.25		
	x	+	x	+	x	⇨	35 min
Time:	20 min		30 min		60 min		

MAXIMIZING EXPECTED UTILITY (MEU)

- Theorem (Ramsey, 1931; von Neumann and Morgenstern, 1944): Given preferences satisfying the constraints there exists a real-valued function U such that
  - $U(A) \geq U(B) \Leftrightarrow A \gtrsim B$
  - $U(A) > U(B) \Leftrightarrow A \succ B$
  - $U(A) = U(B) \Leftrightarrow A \sim B$
  - $U([p_1, s_1; \cdots; p_n, s_n]) = \sum_i p_i U(s_i)$
- *MEU principle:* Choose the action that maximizes expected utility

PREFERENCES LEAD TO UTILITIES

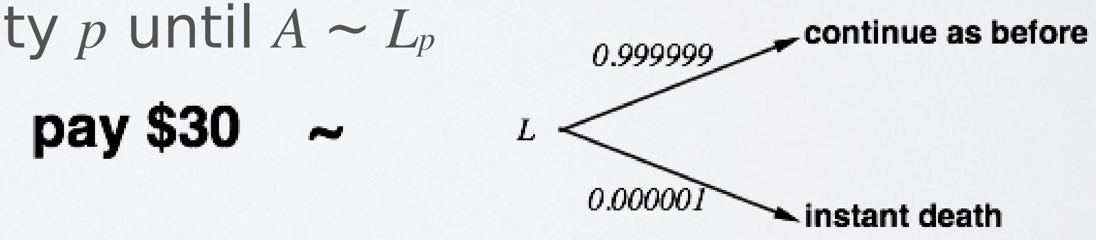
- Note: an agent can be entirely rational (consistent with MEU) without ever representing or manipulating utilities and probabilities
  - E.g., a lookup table for perfect tic-tac-toe
- Although a utility function must exist, it is not unique
  - If $U'(S)=aU(S)+b$ and a and b are constants with $a>0$, then preferences of U' are the same as U
  - E.g., temperatures in Celcius, Fahrenheit, Kelvin

MAXIMIZING EXPECTED UTILITY

- Agent has made some (imperfect) observation o of the state of the world
- If the agent executes action a, the probability the state of the world becomes s' is given by $P(s' \mid o, a)$
- Preferences on outcomes is encoded using utility function $U(s)$
- Expected utility:
- Principal of maximum expected utility says that a rational agent should choose the action that maximizes expected utility $a^* = argmax_a EU(a|o)$

UTILITIES: PREFERENCE ELICITATION

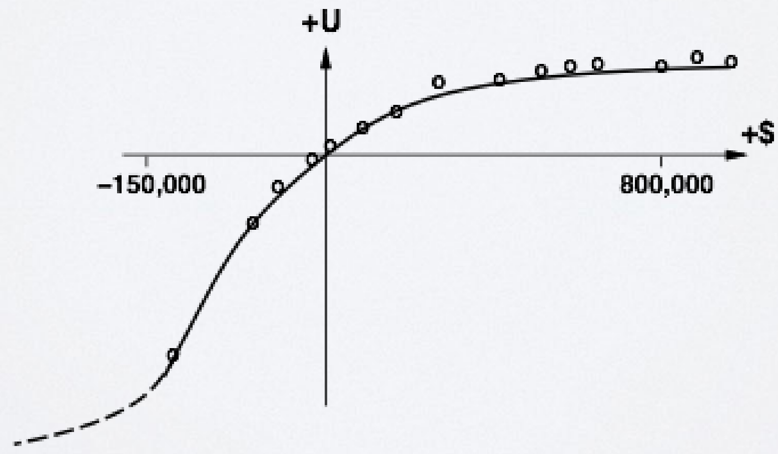
- When building a decision-making or decision-support system, it is often helpful to infer the utility function from a human
- Utilities map states to real numbers. Which numbers?
- Standard approach to assessment of human utilities: compare a given state A to a standard lottery L_p that has
  - "best possible prize" $u_\top$ with probability p
  - "worst possible catastrophe" $u_\perp$ with probability $(1 - p)$
- Adjust lottery probability p until $A \sim L_p$



- Alternatively, set best possible utility to 1 and worst possible to 0

MONEY

- Money does not behave as a utility function
- Given a lottery L with expected monetary value $EMV(L)$, usually $U(L) < U(EMV(L))$, i.e., people are risk-averse
- Utility curve: for what probability p am I indifferent between a prize x and a lottery $[p,\$M; (1-p),\$0]$ for large M?
- Typical empirical data, extrapolated with risk-prone behavior (utility of money is proportional to the logarithm of the amount):



SUMMARY

- Probability is a rigorous formalism for uncertain knowledge
- Joint probability distribution specifies probability of every atomic event
- Queries can be answered by summing over atomic events
- For nontrivial domains, we must find a way to reduce the joint size
- Independence and conditional independence provide the tools
- Next time: sequential decision making!