CS4800: Algorithms — S'18 — Jonathan Ullman

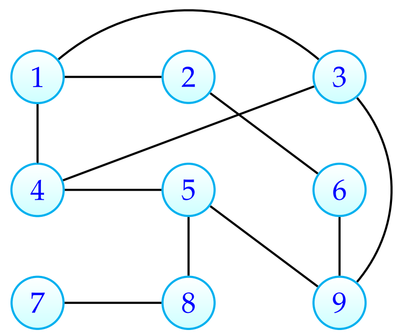
Homework 45
Due Friday February 23 at 11:59pm via Gradescope

Name:
Collaborators:

- Make sure to put your name on the first page. If you are using the LaTeX template we provided, then you can make sure it appears by filling in the `yourname` command.
- This assignment is due Friday February 23 at 11:59pm via Gradescope. No late assignments will be accepted. Make sure to submit something before the deadline.
- Solutions must be typeset in LaTeX. If you need to draw any diagrams, you may draw them by hand as long as they are embedded in the PDF. I recommend using the source file for this assignment to get started.
- I encourage you to work with your classmates on the homework problems. *If you do collaborate, you must write all solutions by yourself, in your own words.* Do not submit anything you cannot explain. Please list all your collaborators in your solution for each problem by filling in the `yourcollaborators` command.
- Finding solutions to homework problems on the web, or by asking students not enrolled in the class is strictly prohibited.

Problem 1. *Graph Representations and Exploration*

This problem is about the following graph.



(a) Draw the adjacency matrix of this graph.[1]

Solution:

(b) Draw the adjacency list of this graph.

Solution:

(c) BFS this graph starting from the node 1. Always choose the lowest-numbered node next. Draw the BFS tree and label each node with its distance from s.

Solution:

(d) Is this graph 2-colorable? Either 2-color it or explain why you can't.

[1]**Hint:** If drawing the matrix in latex, I recommend using the `tabular` or `matrix` environments.

Problem 2. *Graph Properties*

Consider an undirected graph $G = (V, E)$. The *degree* of a vertex v is the number of edges adjacent to v—that is, the number of edges of the form $(v, u) \in E$.[2]

(a) Prove that the sum of the degrees of the vertices is equal to $2m$.

Solution:

(b) Prove that there are an even number of vertices whose degree is odd.

Solution:

(c) Consider a vertex v with odd degree. Prove that there is a path connecting v with another vertex u with odd degree.

Solution:

[2]Recall the standard notation $n = |V|, m = |E|$.

Problem 3. *Fragile Connectivity*

Connectivity is a fragile property—if our graph consists of a single path

$$s - v_2 - v_3 - \cdots - v_{n-1} - t$$

then the graph is connected, but if we *remove* any one node from the graph, and every edge touching that node, then the graph is no longer connected. For example, if we remove v_3 then we would have a graph consisting of two paths

$$s - v_2 \qquad\qquad v_4 - v_5 - \cdots - v_{n-1} - t$$

In this question we will see that this is a general property of graphs with long paths.

(a) Let $G = (V, E)$ be an undirected, connected graph. Suppose there is a pair $s, t \in V$ such that the distance between s and t in G is strictly greater than $n/2$. Prove that in any such graph, there exists a node v^* such that removing v^* and all edges touching v^* from G will cause s and t to be disconnected.[3]

Solution:

(b) Modify BFS to obtain an $O(n+m)$ time algorithm that finds such a node v^*. Clearly describe your algorithm and explain why it is correct.

Solution:

[3] **Hint:** Think about what it will look like when you BFS this graph.