Midterm II : T 3/27 in class
(Only graph algorithms)

CS4800: Algorithms & Data
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Lecture 14:
- Minimum Spanning Trees

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Minimum Spanning Trees

Network Design

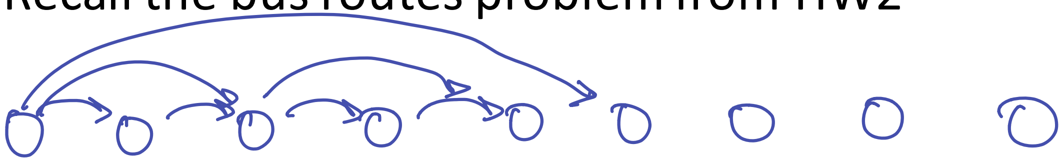
nodes in a graph

- We have a set of locations $V = \{v_1, \dots, v_n\}$
- Want to build a network to connect these locations
  - Every v_i, v_j must be connected — path from v_i to v_j
  - Must be as cheap as possible

build edges

costs for building an edge (v_i, v_j)

- Many variants
  - Build a "cheap" network that is "well connected"
  - Recall the bus routes problem from HW2



Minimum Spanning Trees (MST)

all links you could build

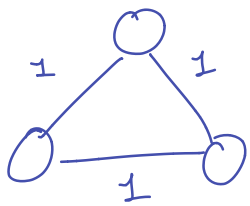
- Input: a graph $G = (V, \boxed{E}, \boxed{\{w_e\}})$ costs of building link e
  - Undirected, connected, weights may be negative
  - All edge weights are distinct (makes life much simpler)

  not without loss of generality

$G=(V,T)$ is connected

- A spanning tree is a tree T that connects all of V
  - Cost of a tree T is the sum of the edge weights
  - $T \subseteq E$ $\qquad \sum_{e \in T} w_e = \text{cost}(T)$
- Output: A spanning tree T of minimum cost

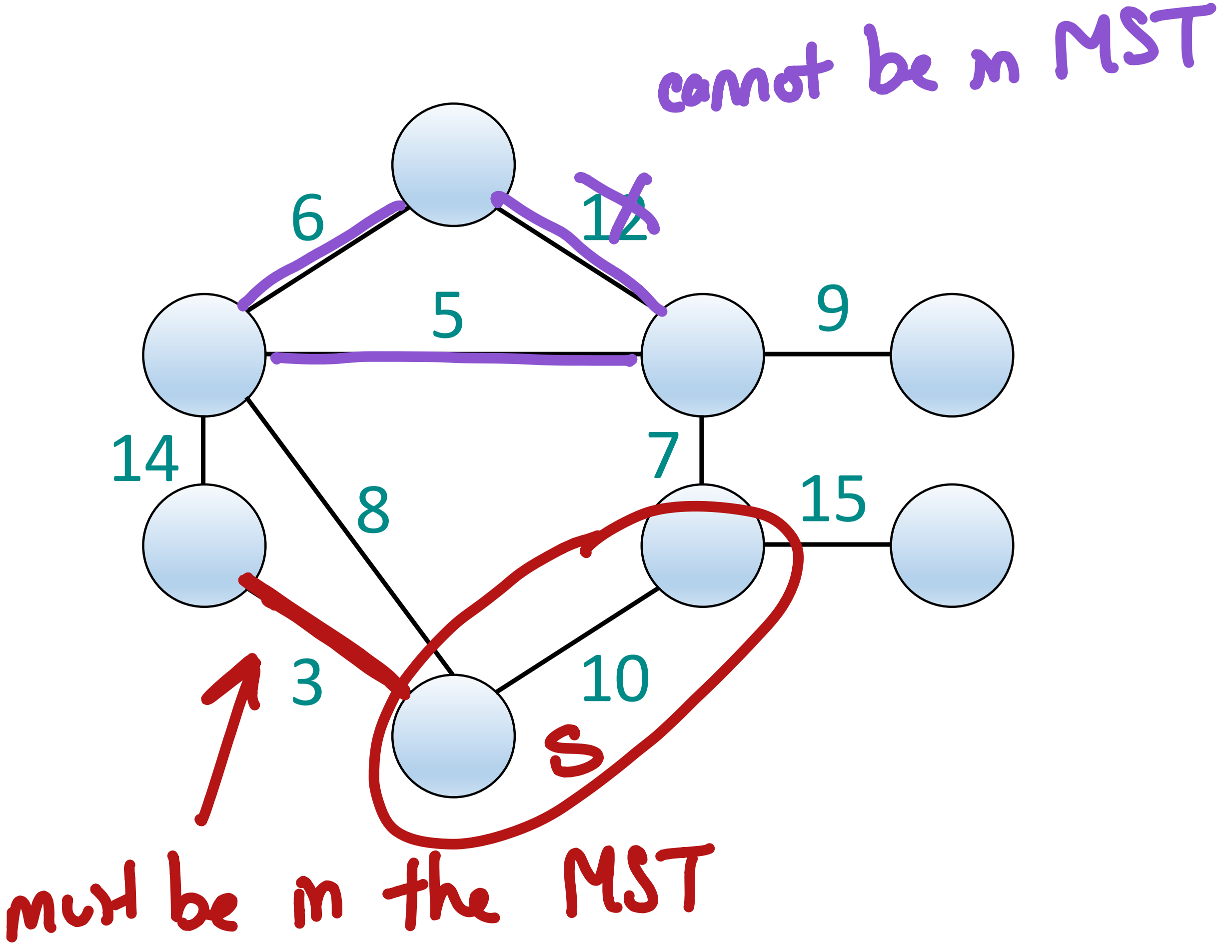
notation: T^* is the MST



MST is not unique in general

If all w_e are distinct then MST is unique

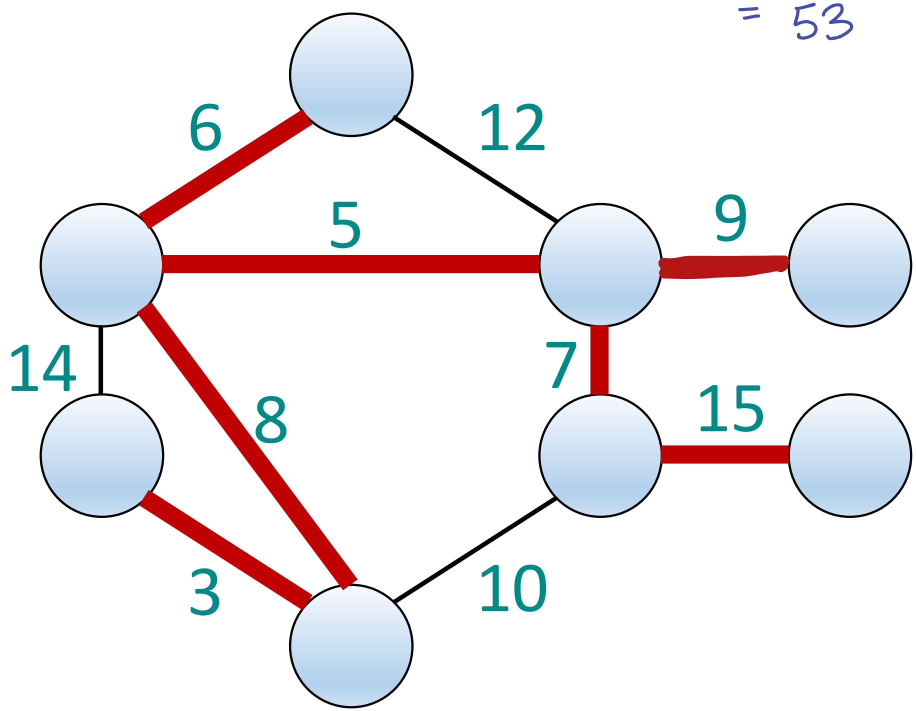
Minimum Spanning Trees (MST)



Minimum Spanning Trees (MST)

$\text{Cost}(T^*) = 6 + 5 + 7 + 9 + 15 + 8 + 3$

$= 53$

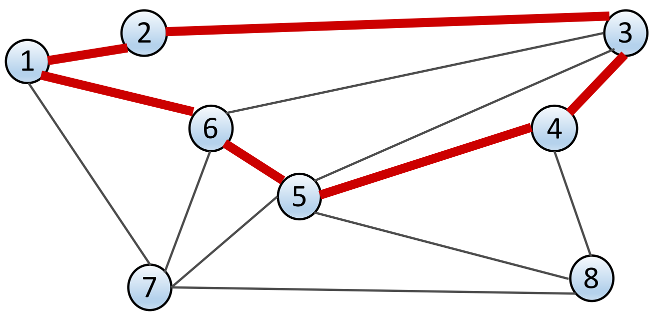


MST Algorithms

- There are at least four reasonable MST algorithms
  - Borůvka's Algorithm: start with $T = \emptyset$, in each round add cheapest edge out of each connected component
  - Prim's Algorithm: start with some s, at each step add cheapest edge that grows the connected component
  - Kruskal's Algorithm: start with $T = \emptyset$, consider edges in ascending order, adding edges unless they create a cycle
  - Reverse-Kruskal: start with $T = E$, consider edges in descending order, deleting edges unless it disconnects

Cycles and Cuts

- Cycle: a set of edges $(v_1, v_2), (v_2, v_3), \ldots, (v_k, v_1)$

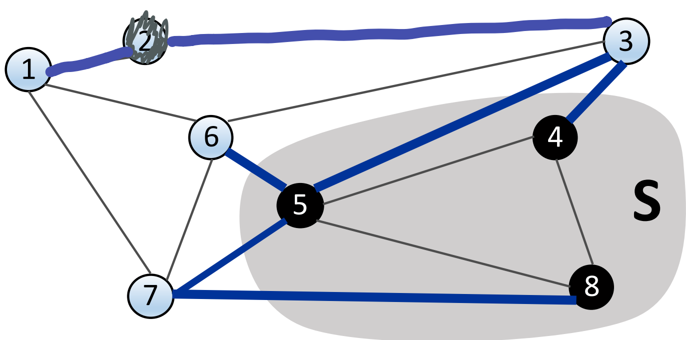


Cycle C = (1,2),(2,3),(3,4),(4,5),(5,6),(6,1)

Cutset(S) :

$\{e = (u,v) : u \in S \; v \notin S\}$

- Cut: a subset of nodes $S \subseteq V$

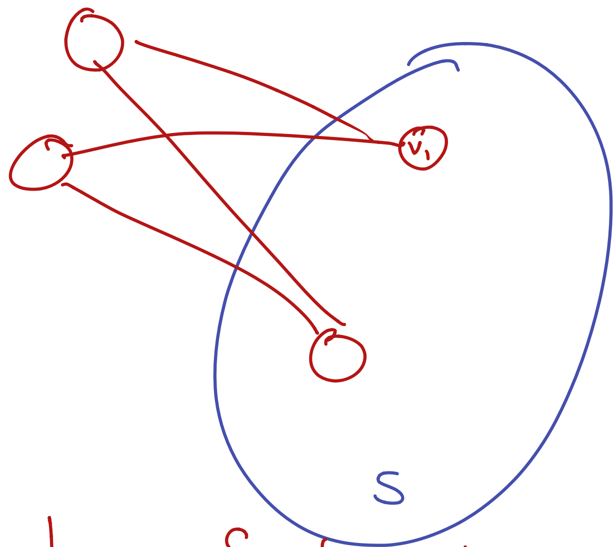


Cut S = {4, 5, 8}
Cutset = (5,6), (5,7), (3,4), (3,5), (7,8)

Cycles and Cuts

- Fact: a cycle and a cutset intersect in an even number of edges

$v_1 - v_2 - v_3 - v_4 - v_1$



Every time the cycle leaves S, it must come back to S.

Properties of MSTs

Assuming we are distinct.

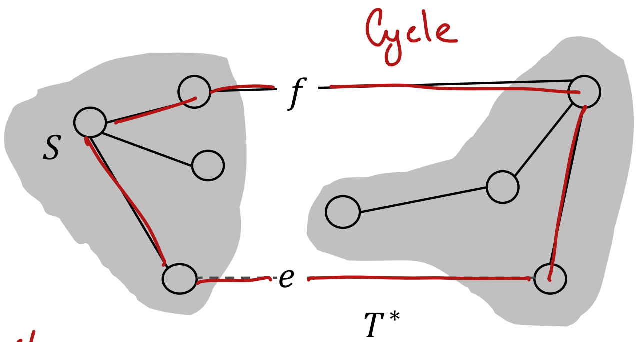
$e \in \text{Cutset}(S)$

- Cut Property: Let S be a cut. Let e be the minimum weight edge cut by S. Then the MST T^* contains e
  - We call such an e a safe edge

- Cycle Property: Let C be a cycle. Let f be the maximum weight edge in C. Then the MST T^* does not contain ~~e~~. f
  - We call such an ~~e~~ f a useless edge

Proof of Cut Property

- Cut Property: Let S be a cut. Let e be the minimum weight edge cut by S. Then the MST T^* contains e



Proof:

- Let T^* be the MST
- Assume $e \notin T^*$
- Consider adding e to T^* (now there is a cycle containing e)
- There must be some $f \neq e$ in Cutset$(S) \cap$ Cycle
  - $w_f > w_e$
- Let $T = T^* + \{e\} - \{f\}$

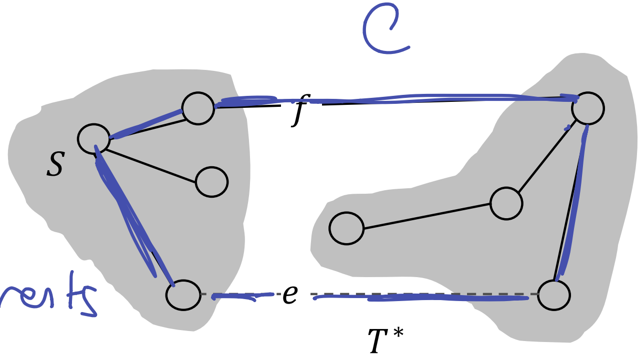
T is still a spanning tree

$cost(T) = cost(T^*) + w_e - w_f < cost(T^*)$

contradiction!

Proof of Cycle Property

- Cycle Property: Let C be a cycle. Let f be the maximum weight edge in C. Then the MST T^* does not contain ~~e~~ f.

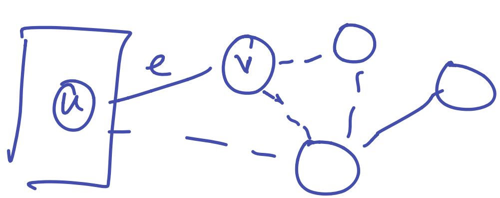


- Suppose T^* contains f
- Suppose we delete f from T^*
  - now T^* has 2 connected components
  - let S be one of those components
- $f \in C \cap \text{Cutset}(S)$, so there is $e \neq f \in C \cap \text{Cutset}(S)$, $w_e < w_f$
- Let $T = T^* - \{f\} + \{e\}$
  - T is spanning tree
  - $\text{cost}(T) < \text{cost}(T^*)$ contradiction!

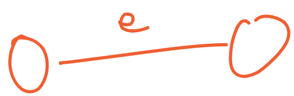
Ask the Audience

- Assume G has distinct edge weights
- True/False? If e is the edge with the smallest weight, then e is always in the MST T^*
- True/False? If e is the edge with the largest weight, then e is never in the MST T^*

Cut Property

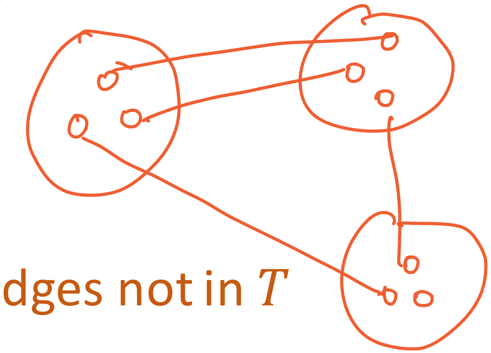


e is the min wt. edge in Cutset($\{u\}$)



The "Only" MST Algorithm

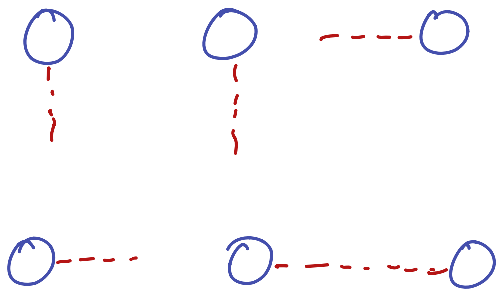
suppose T is not connected



- GenericMST:
  - Let $T = \emptyset$
  - Until T is connected:
    - Find one or more safe edges not in T
    - Add safe edges to T

- Theorem: GenericMST outputs an MST
  - $T \subseteq T^*$ (b/c T only contains safe edges)
  - $T = T^*$ (if T were not connected, there would be ≥ 2 connected components $\Rightarrow \exists$ a new safe edge)

Borůvka's Algorithm

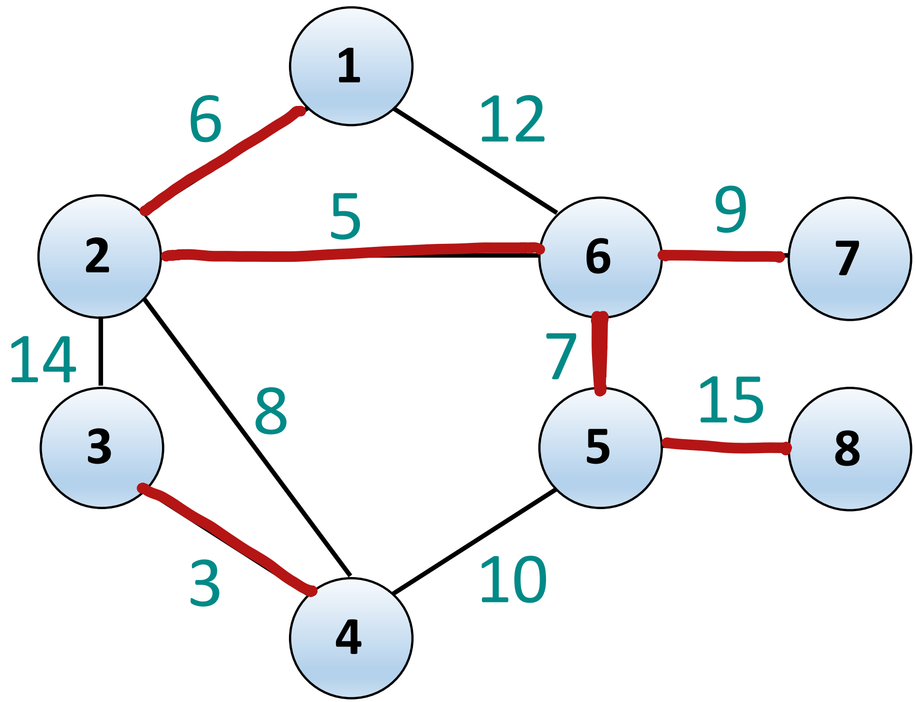


- Borůvka:
  - Let $T = \emptyset$
  - Until T is connected:
    - Let $C_1, \dots, C_k$ be the connected components of (V, T)
    - Let $e_1, \dots, e_k$ be the safe edge for the cuts $C_1, \dots, C_m$
    - Add $e_1, \dots, e_k$ to T

- Correctness: every edge we add is safe

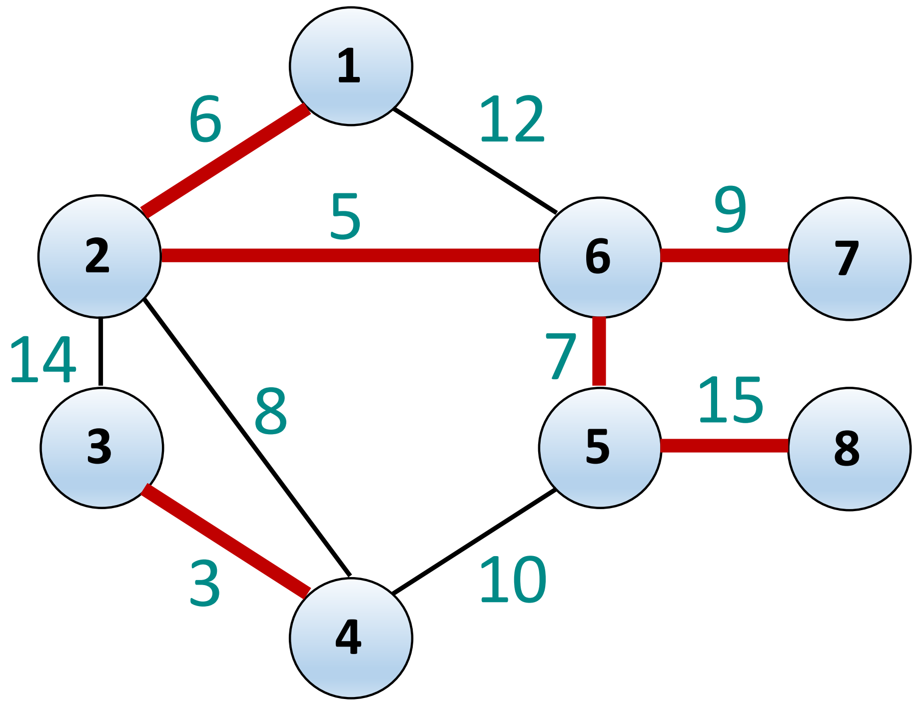
Borůvka's Algorithm

Label Connected Components



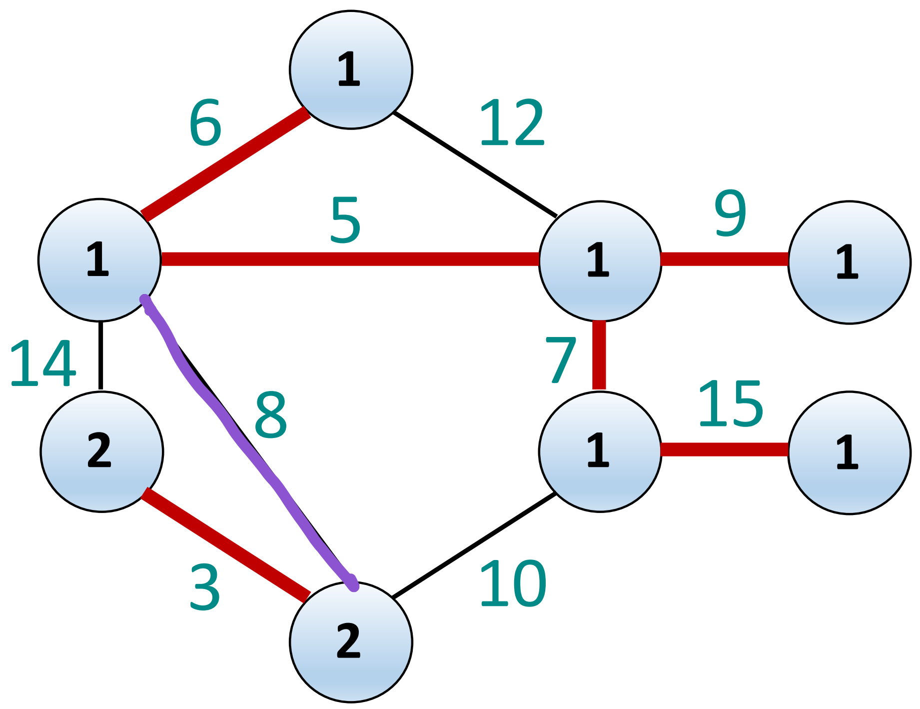
Borůvka's Algorithm

Add Safe Edges



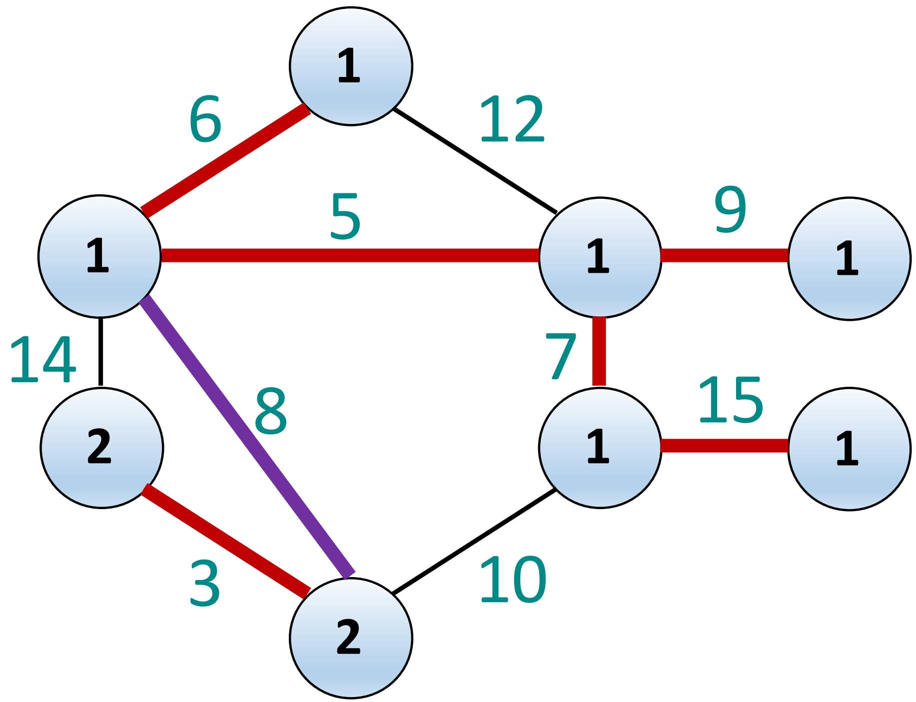
Borůvka's Algorithm

Label Connected Components



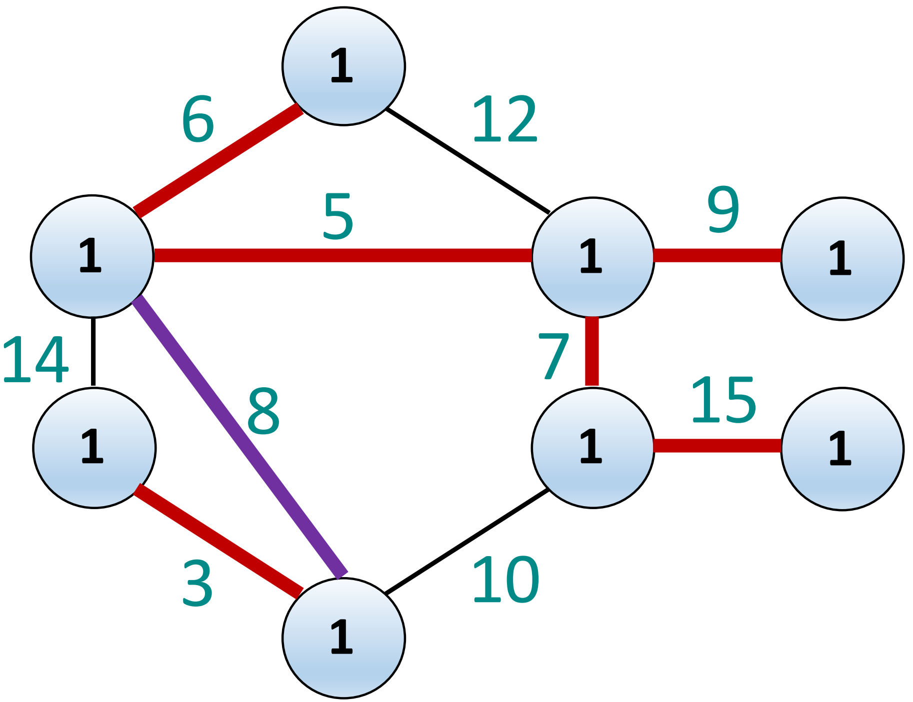
Borůvka's Algorithm

Add Safe Edges



Borůvka's Algorithm

Done!



Borůvka's Algorithm (Running Time)

- Borůvka
  - Let $T = \emptyset$
  - Until T is connected:
    - Let $C_1, \ldots, C_k$ be the connected components of (V, T)
    - Let $e_1, \ldots, e_k$ be the safe edge for the cuts $C_1, \ldots, C_m$
    - Add $e_1, \ldots, e_k$ to T

- How long to find safe edges?
- How many times through the main loop?

Borůvka's Algorithm (Running Time)

FindSafeEdges:

BFS $O(n+m)$ time

Find connected components $C_1, \dots, C_k$

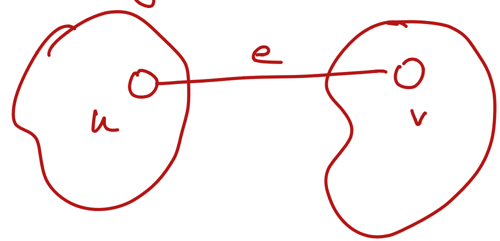
Let L[v] be the connected component of node v

Let S[i] be the safe edge of C_i (initially S[i] ← NULL)

$O(m)$ For each edge (u,v):

$O(1)$
- If L[u] $\neq$ L[v]:
  - If w(u,v) < w(S[L[u]]):
    - S[L[u]] = (u,v)
  - If w(u,v) < w(S[L[v]]):
    - S[L[v]] = (u,v)

Return {S[1],…,S[k]}



Total Time: $O(m)$

Borůvka's Algorithm (Running Time)

k components $\rightarrow \frac{k}{2}$ components

- Claim: every iteration of the main loop halves the number of connected components.
  - Every time we add a safe edge, # of components decreases by 1.
  - Suppose there are k components $C_1, \dots, C_k$
  - $\{S[1], \dots, S[k]\}$ contains at least $k/2$ distinct safe edges
    - any edge only is eligible for two components
  - $\# CC \le k - |\{S[1], \dots, S[k]\}| \le k - \frac{k}{2} = \frac{k}{2}$.

Borůvka's Algorithm (Running Time)

- Borůvka
  - Let $T = \emptyset$
  - Until T is connected:
    - Let $C_1, \ldots, C_k$ be the connected components of (V, T)
    - Let $e_1, \ldots, e_k$ be the safe edge for the cuts $C_1, \ldots, C_m$
    - Add $e_1, \ldots, e_k$ to T

- How long to find safe edges? $O(m)$
- How many times through the main loop? $O(\log n)$

Running time: $O(m \log n)$

Prim's Algorithm

assuming $u \in S$ $v \notin S$

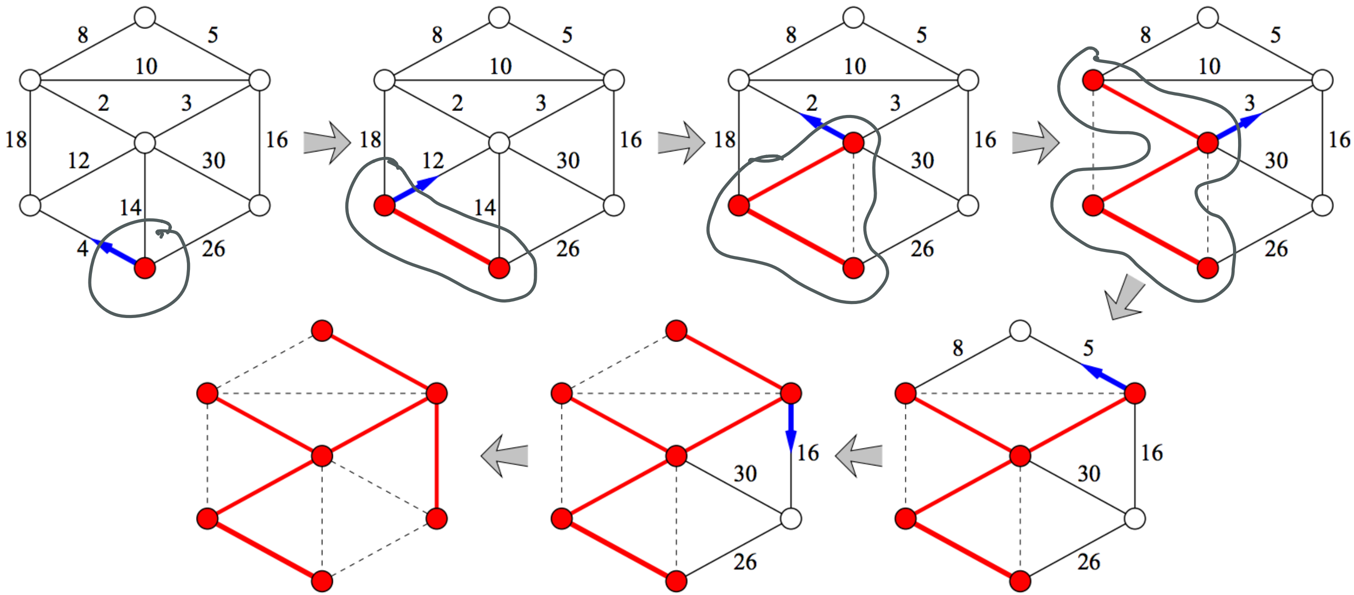
- Prim Informal
  - Let $T = \emptyset$. Let s be some arbitrary node and $S = \{s\}$.
  - Find the cheapest edge $e = (u, v)$ cut by S. Add e to T and add v to S

safe edge

- Correctness: every edge we add is safe

implements generic MST

Prim's Algorithm



Prim's Algorithm

PQ (Heap): Stores key-value pairs
- ExtractMin
- Decrease Key

in O(log n) time

Prim O(n) time

Let Q be a priority queue storing V

key[v] ← ∞, last[v] ← ⊥

key[s] ← 0 for some arbitrary s

While Q ≠ Ø:

n times u ← ExtractMin(Q) (assume u ~~has~~ hasnt been found already)

O(nlogn) For each edge (u,v):

If v ∈ Q and w(u,v) < key[v]:

DecreaseKey(v,w(u,v)) m times O(m log n)

last[v] ← u

Output T = {(1,last[1]),...,(n,last[n])} (excluding s)

set of blue edges that we used to explore each node.

Invariant:
- Q holds S^c
- value[v]: min wt. edge from S to v.

Running time is O(mlogn)

Kruskal's Algorithm

Running: $O(m \log m)$

- Kruskal's Informal
  - Let $T = \emptyset$
  - Consider edges e in ascending order of weight:
    - If adding e would merge two connected components
      - Add e to T

m times

$O(m \log m)$ to sort

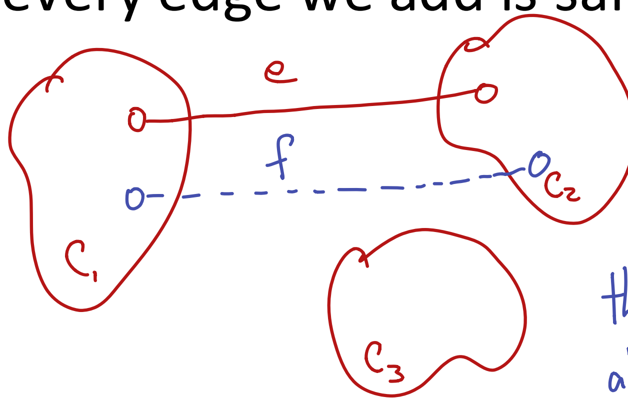
$O(m)$ time to check

$O(n \log n)$ time to merge

- Correctness: every edge we add is safe

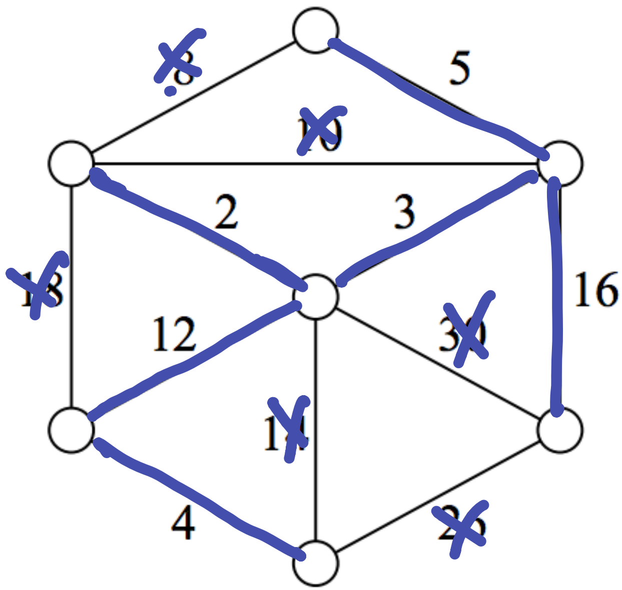
Suppose e connects C_1 to C_2

$e \in \text{Cutset}(C_1)$



suppose $w_f < w_e$, then would have already added f to T

Kruskal's Algorithm



Implementing Kruskal's Algorithm

- Union-Find Data Structure
- Need to store the set of connected components in such a way that we can efficiently:
  - Check if u,v are in the same component (Find(u), Find(v))
  - Merge the connected components of u,v (Union(u,v))

- Can implement Union-Find so that
  - Find(u) takes $O(1)$ time
  - Any k operations Union(u,v) take $O(k \log k)$ time
    - "Amortized Analysis"

- Lots of fancier versions of this

Kruskal's Algorithm (Running Time)

- Kruskal's Informal
  - Let $T = \emptyset$
  - Consider edges e in ascending order of weight:
    - If adding e would merge two connected components
      - Add e to T

- Time to sort:
- Time to test edges:
- Time to add edges:

Comparison

- Boruvka's Algorithm:
  - Only algorithm worth implementing
  - Low overhead, can be easily parallelized
  - Each iteration takes $O(m)$, very few iterations in practice

- Prim's/Kruskal's Algorithms:
  - Reveal useful structure of MSTs
  - Running time dominated by a single sort