CS3000: Algorithms & Data
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Lecture 21:
- Stable Matching: the Gale-Shapley Algorithm

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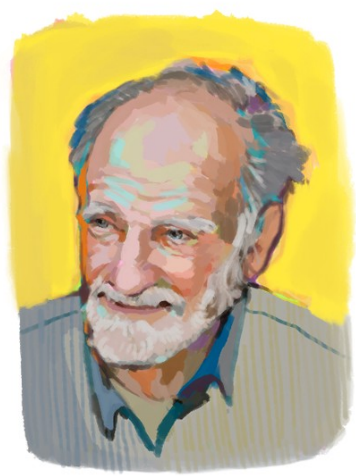
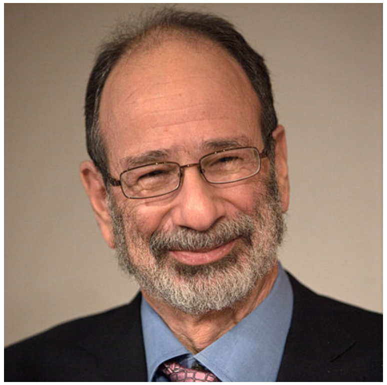
National Residency Matching Program

- National system for matching US medical school graduates to medical residencies
  - Roughly 40,000 doctors per year
  - Assignment is almost entirely algorithmic

David Gale (1921-2008)
PROFESSOR, UC BERKELEY



Lloyd Shapley
PROFESSOR EMERITUS, UCLA



Alvin Roth
PROFESSOR, STANFORD

Labor Markets

- Most labor markets are frustrating
  - Not everyone can get their favorite job
  - The market is decentralized

- Decentralized labor markets are frustrating

Centralized Labor Markets

- What if we could just assign jobs?
  - What information would we want?
  - How would we choose the assignment?

Matchings

- We are given the following information
  - n doctors $d_1 \ldots d_n$
  - n hospitals $h_1 \ldots h_n$
  - each doctor's ranking of hospitals $d_1 : h_2 \succ h_3 \succ h_1$
  - each hospital's ranking of doctors $h_1 : d_1 \succ d_3 \succ d_2$

	1st	2nd	3rd	4th	5th
MGH	Bob	Alice	Dorit	Ernie	Clara
BW	Dorit	Bob	Alice	Clara	Ernie
BID	Bob	Ernie	Clara	Dorit	Alice
MTA	Alice	Dorit	Clara	Bob	Ernie
CH	Bob	Dorit	Alice	Ernie	Clara

	1st	2nd	3rd	4th	5th
Alice	CH	MGH	BW	MTA	BID
Bob	BID	BW	MTA	MGH	CH
Clara	BW	BID	MTA	CH	MGH
Dorit	MGH	CH	MTA	BID	BW
Ernie	MTA	BW	CH	BID	MGH

Matchings

- A matching M is a set of doctor-hospital pairs
  - $M = \{ (d_1, h_2), (d_2, h_3), \ldots \}$
  - matching: no doctor/hospital appears twice
  - perfect matching: every doctor/hospital appears once
  - "d is matched to h": $(d, h) \in M$

Stable Matchings

- A matching M is unstable if some doctor-hospital pair prefer one another to their mate in M

- Instabilities
  - d, h such that d is matched to h', h is unmatched, but $d : h \succ h'$

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Ask the Audience

- Either find a stable matching or convince yourself that there is no stable matching

	1st	2nd	3rd
MGH	Alice	Bob	Clara
BW	Bob	Clara	Alice
BID	Alice	Clara	Bob

	1st	2nd	3rd
Alice	BW	BID	MGH
Bob	BW	MGH	BID
Clara	MGH	BID	BW

Gale-Shapley Algorithm

- **Let M be empty**
- **While (some hospital h is unmatched):**
  - **If (h has offered a job to everyone): break**
  - **Else: let d be the highest-ranked doctor to which h has not yet offered a job**
  - **Have h make an offer to d:**
    - **If (d is unmatched):**
      - **d accepts, add (d,h) to M**
    - **ElseIf (d is matched to h′ & d: h′ > h):**
      - **d rejects, do nothing**
    - **ElseIf (d is matched to h′ & d: h > h′):**
      - **d accepts, remove (d,h′) from M and add (d,h) to M**
- **Output M**

Gale-Shapley Demo

	1st	2nd	3rd	4th	5th
MGH	Bob	Alice	Dorit	Ernie	Clara
BW	Dorit	Bob	Alice	Clara	Ernie
BID	Bob	Ernie	Clara	Dorit	Alice
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Bob	BID	BW	MTA	MGH	CH
Clara	BW	BID	MTA	CH	MGH
Dorit	MGH	CH	MTA	BID	BW
Ernie	MTA	BW	CH	BID	MGH

Observations

- Hospitals make offers in descending order
- Doctors that get a job never become unemployed
- Doctors accept offers in ascending order

Gale-Shapley Algorithm

- Questions about the Gale-Shapley Algorithm:
  - Will this algorithm terminate?
  - Does it output a perfect matching?
  - Does it output a stable matching?
  - How do we implement this algorithm efficiently?

GS Algorithm: Termination

- **Claim:** The GS algorithm terminates after n^2 iterations of the main loop (offers)

GS Algorithm: Perfect Matching

- **Claim:** The GS algorithm returns a perfect matching (all doctors/hospitals are matched)

GS Algorithm: Stable Matching

- **Stability:** GS algorithm outputs a stable matching
- Proof by contradiction:
  - Suppose there is an instability d, d', h, h'

GS Algorithm: Running Time

- **Running Time:**
  - A straightforward implementation is $O(n^3)$ time

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- **Running Time:**
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Clara	BW	BID	MTA	CH	MGH
Dorit	MGH	CH	MTA	BID	BW
Ernie	MTA	BW	CH	BID	MGH

	MGH	BW	BID	MTA	CH
Alice	2nd	3rd	5th	4th	1st
Bob	4th	2nd	1st	3rd	5th
Clara	5th	1st	2nd	3rd	4th
Dorit	1st	5th	4th	3rd	2nd
Ernie	5th	2nd	4th	1st	3rd

Real World Impact

TABLE I

STABLE AND UNSTABLE (CENTRALIZED) MECHANISMS

Market	Stable	Still in use (halted unraveling)
American medical markets		
NRMP	yes	yes (new design in '98)
Medical Specialties	yes	yes (about 30 markets)
British Regional Medical Markets		
Edinburgh ('69)	yes	yes
Cardiff	yes	yes
Birmingham	no	no
Edinburgh ('67)	no	no
Newcastle	no	no
Sheffield	no	no
Cambridge	no	yes
London Hospital	no	yes
Other healthcare markets		
Dental Residencies	yes	yes
Osteopaths (<'94)	no	no
Osteopaths (≥'94)	yes	yes
Pharmacists	yes	yes
Other markets and matching processes		
Canadian Lawyers	yes	yes (except in British Columbia since 1996)
Sororities	yes (at equilibrium)	yes

Table 1. Reproduced from Roth (2002, Table 1).

Real World Impact

- **Doctors ↔ Hospitals**
  - Have to deal with two-body problems
  - Have to make sure doctors do not game the system
- **Kidneys ↔ Patients**
  - Not all matches are feasible (blood types)
  - Certain pairs must be matched
- **Students ↔ Public Schools**
  - Siblings, walking zones, diversity
- **Reform Rabbis ↔ Synagogues**
  - No idea, just a fun example

2012 Nobel Prize to Lloyd Shapley and Al Roth!

Course Wrapup

- What did we study?
  - A toolkit for solving challenging computational problems
  - A way of thinking about computational problems
  - A way of talking about computational problems
  - How to ace your interviews

Course Wrapup

- What we didn't study?
  - Algorithms has become a more unified, and highly mathematically sophisticated field
  - Many kinds of algorithms and applications
    - Randomized algorithms
    - Linear and convex optimization
    - Numerical algorithms
    - Algorithms for number theory
    - Machine learning algorithms
    - Algorithms for strategic agents
    - Fairness, privacy, ethics in algorithms
    - Algorithms for quantum computers
    - Distributed / parallel algorithms
    - Naturally occurring algorithms

I hope you'll choose to learn about (some of) these topics!