Lecture 17:
More Applications of Network Flow

March 25, 2020
Image Segmentation

• Separate image into foreground and background

• We have some idea of:
  → • whether pixel $i$ is in the foreground or background
  → • whether pair $(i,j)$ are likely to go together
Image Segmentation

- **Input:**
  - an undirected graph \( G = (V, E) \); \( V = \text{“pixels”}, \ E = \text{“pairs”} \)
  - likelihoods \( a_i, b_i \geq 0 \) for every \( i \in V \)
  - separation penalty \( p_{ij} \geq 0 \) for every \( (i, j) \in E \)

- **Output:**
  - a partition of \( V \) into \((A, B)\) that maximizes

\[
q(A, B) = \sum_{i \in A} a_i + \sum_{j \in B} b_j - \sum_{(i, j) \in E} p_{ij}
\]

Assume all values in the graph were given externally.
Reduction to MinCut

• Differences between SEG and MINCUT:
  • SEG asks us to maximize, MINCUT asks us to minimize

\[
\max_x f(x) \quad \text{vs.} \quad \min_x -f(x)
\]

\[
\max_{A,B} \sum_{i \in A} a_i + \sum_{j \in B} b_j - \sum_{(i,j) \in E} p_{ij}
\]

\[
\min_{A,B} \sum_{i \in A} b_i + \sum_{j \in B} a_j + \sum_{(i,j) \in E} p_{ij}
\]
Reduction to MinCut

• Differences between SEG and MINCUT:
  • SEG allows any partition, MINCUT requires $s \in A, t \in B$

Solution: Add “dummy nodes” s and t to the graph
Reduction to MinCut

- Differences between SEG and MINCUT:
  - SEG has edges **between A and B**, MINCUT considers edges **from A to B**

\[
\min_{A,B} \left( \sum_{i \in A} b_i + \sum_{j \in B} a_j + \sum_{(i,j) \in E} p_{ij} \right)
\]

**Solution:**
- Replace undirected edge \((i,j)\) \(uv\)
- \(i \rightarrow j\) and \(j \rightarrow i\)
- both with capacity \(p_{ij}\)

\[
\min_{A,B} \sum_{(i,j) \in E} p_{ij}
\]

Capacity \(p_{ij}\) in both directions
Reduction to MinCut

- Differences between SEG and MINCUT:
  - SEG has terms for each node in A,B, MINCUT only has terms for edges from A to B

\[
\min_{A,B} \sum_{i \in A} b_i + \sum_{j \in B} a_j + \sum_{(i,j) \in E} p_{ij}
\]

\[
\min_{A,B} \sum_{(i,j) \in E} p_{ij}
\]

Solution:
Use "dummy" edges from s and t

capacity: \( p_{xy} + p_{xz} \)
capacity we want:
\( p_{xy} + p_{xz} + b_x \)

+ \( a_y + a_z \)
Reduction to MinCut

• How should the reduction work?
  • capacity of the cut should correspond to the function we’re trying to minimize

\[
\min_{A,B} \sum_{i \in A} b_i + \sum_{j \in B} a_j + \sum_{(i,j) \in E \text{ from } A \text{ to } B} p_{ij}
\]

1. Replace \( \max \) with \( \min \)
2. Replace undirected edges with pairs of directed edges
3. Add dummy nodes \( s, t \)
4. Add dummy edges \( s \leadsto x \leadsto t \)
Step 1: Transform the Input

1. Replace max with min
2. Replace undirected edges with pairs of directed edges
3. Add dummy nodes $s, t$
4. Add dummy edges $s \rightarrow x \rightarrow t$

Total Time: $O(m+n)$
Step 2: Receive the Output

\[ s, u, v, x, t \] were the original graph

Input \( G' \) for MINCUT

Output \((A,B)\) for MINCUT

Solve

\((A,B)\) is a minimum \(s,t\) cut in \( G' \)

Running Time:
Solve mincut on a graph with \( n+2 \) nodes and \( 2M + 2n \) edges
\[ \sim O(mn) \text{ time} \]
Step 3: Transform the Output

**Output (A,B) for SEG**

Return partition

\[ A = \{ u, v \} \]
\[ B = \{ w, x \} \]

**Output (A,B) for MINCUT**

![Graph](image)

Time: \( O(n) \)
Reduction to MinCut

• correctness?
  • Every partition \((A, B)\) of the original nodes corresponds to an \(s, t\) cut \((AuSs?, BuSs?)\)
  • For every \(s, t\) cut \((AuSs?, BuSs?)\) the capacity is
    \[
    \sum_{i \in A} b_i + \sum_{i \in B} a_i + \sum_{(i,j) \in E} P_{ij} + \sum_{i \in A} \sum_{j \in B} P_{ij}
    \]

• running time?
  
  Total Time: \(O(mn)\)

Bottleneck is solving minimum cut
• Want to identify communities in a network.
  • “Community”: a set of nodes that have a lot of connections inside and few outside.
Densest Subgraph

• **Input:**
  • an undirected graph $G = (V, E)$

• **Output:**
  • a subset of nodes $A \subseteq V$ that maximizes $\frac{2|E(A,A)|}{|A|}$

- $E(A,A)$ = set of edges with both endpoints in $A$
- $E(A,B)$ = set of edges with one endpoint in $A$ and one in $B$
1. DS uses an undirected graph
2. DS lets us choose any set A
   \[ M\text{IN}\text{CUT} \text{ uses a directed edge} \]
   \[ M\text{IN}\text{CUT} \text{ lets choose } s, t \text{ cut} \]

3. Add "dummy" nodes s, t

Same transformations as SEG

Need to transform the objective function:

DS
\[
\frac{2 \cdot |E(A,A)|}{|A|}
\]

MINCUT
\[
\sum\limits_{(i,j) \in E} c_{i,j}
\]
\[
\sum\limits_{i \in A} a_i + \sum\limits_{i \in B} b_i + \sum\limits_{(i,j) \in E \text{ btw } A, B} c_{i,j}
\]
using "dummy" edges
Reduction to MinCut

• Different objectives
  • maximize $\frac{2|E(A,A)|}{|A|}$ vs. minimize $|E(A,B)|$

• Suppose $\left[\frac{2|E(A,A)|}{|A|}\right] \geq \delta$ and see what that implies

  $\Leftrightarrow 2|E(A,A)| \geq \delta|A|$

  $\Leftrightarrow \Sigma_{v \in A} \deg(v) - |E(A,B)| \geq \delta|A|$

  $\Leftrightarrow \Sigma_{v \in V} \deg(v) - \Sigma_{v \in B} \deg(v) - |E(A,B)| \geq \delta|A|$

  $\Leftrightarrow 2|E| - \Sigma_{v \in B} \deg(v) - |E(A,B)| \geq \delta|A|$

  $\Leftrightarrow \Sigma_{v \in B} \deg(v) + \delta|A| + |E(A,B)| \leq 2|E|$

  $\Leftrightarrow \Sigma_{v \in B} \deg(v) + \Sigma_{v \in A} \delta + \Sigma_{e \text{ from } A \text{ to } B} 1 \leq 2|E|$

Claim: If I can ask yes/no questions “Is the DS denser than $\delta$ ?” then I can find the densest subgraph.
Reduction to MinCut

- Different objectives
  - maximize \( \frac{2|E(A,A)|}{|A|} \) vs. minimize \( |E(A, B)| \)

- Suppose \( \left\lceil \frac{2|E(A,A)|}{|A|} \right\rceil \geq \delta \) and see what that implies

\[
\iff 2|E(A, A)| \geq \delta |A|
\]

\[
\iff \sum_{v \in A} \deg(v) - |E(A, B)| \geq \delta |A|
\]

\[
\iff \sum_{v \in V} \deg(v) - \sum_{v \in B} \deg(v) - |E(A, B)| \geq \delta |A|
\]

\[
\iff 2|E| - \sum_{v \in B} \deg(v) - |E(A, B)| \geq \delta |A|
\]

\[
\iff \sum_{v \in B} \deg(v) + \delta |A| + |E(A, B)| \leq 2|E|
\]

\[
\iff \sum_{v \in B} \deg(v) + \sum_{v \in A} \delta + \sum_{e \text{ from } A \text{ to } B} \delta \leq 2|E|
\]

Claim: If I can ask yes/no questions “Is the DS drawn than \( \delta \)?” then I can find the densest subgraph.
\[ \sum_{v \in B} \deg(v) + \sum_{v \in A} s + \sum_{e \text{ from } A \rightarrow B} 1 \]

If the value is \( \leq 2|E| \)
then the subgraph \( A \) has
\[ \frac{2 \cdot |E(A,A)|}{|A|} \geq 5 \]
Reduction to MinCut

\[ \sum_{v \in B} \deg(v) + \sum_{v \in A} \delta + \sum_{e \text{ from } A \text{ to } B} 1 \leq 2|E| \]

This graph has \( mn \)-cut \( \leq 2|E| \) if and only if there exists a subgraph of density \( > \delta \).