Lecture 22:
• Online Learning
Picking Good Experts

• Suppose you have $N$ “experts” making predictions
  • Weather forecasters
  • Financial advisors
  • Recommender systems
  • ...

• Most of them are bad, but one might be good!
• Who’s predictions should we trust?
<table>
<thead>
<tr>
<th>$f_{i,n}$</th>
<th>$P_i$</th>
<th>$q_i$</th>
<th>$P_{z_2}$</th>
<th>$q_2$</th>
<th>$P_3$</th>
<th>$q_3$</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_{i,n}$</td>
<td>$t=1$</td>
<td>2</td>
<td>3</td>
<td>...</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- There are $T$ time periods, in each:
  - experts make predictions $f_{t,1}, \ldots, f_{t,N} \in \{0,1\}$
  - you have to make a decision $p_t \in \{0,1\}$
  - some outcome $q_t \in \{0,1\}$ is revealed (may be adversarial)

- **Goal:** minimize mistakes $(M \approx M^*)$
  - You have a mistake on day $t$ if $P_t \neq q_t$
  - Want # of mistakes $\ll T$
  - $M^*$ is the # of mistakes made by the best expert in hindsight. $M$ is # of mistakes we make
Level I: The Halving Algorithm (HA)

- Assumption: some expert makes 0 mistakes

\[ C_t \text{ is the set of experts that have made no mistakes up to day } t \]

Let \( C_1 \leftarrow \{1, \ldots, N\} \)

For \( t = 1, \ldots, T \):

- Let \( p_t \) be the majority vote of experts in \( C_t \)
- Let \( C_{t+1} \) be the experts with no mistakes so far
Level I: The Halving Algorithm

\[ m^* = 0 \implies M \leq \log_2 N \]

- **Thm:** If some expert makes 0 mistakes then HA makes \( \leq \log_2 N \) mistakes.

- **Key Idea:** Use \( |C_t| \) as a "measure of progress".

\[-\]

\[ N = |C_1| \geq |C_2| \geq |C_3| \geq \ldots \geq |C_T| \geq 1 \]

1. \( |C_T| \leq N \cdot 2^{-m} \)
   (because if we make a mistake on day \( t \) then \( |C_{t+1}| \leq |C_t|/2 \))

2. \( |C_T| \geq 1 \)
\[ \lvert C_T \rvert \leq N \cdot 2^{-M} \]

\[ \Rightarrow \quad \lvert C_T \rvert \leq N \cdot 2^{-M} \]

\[ \Rightarrow \quad 0 \leq \log_2 N - M \]

\[ \Rightarrow \quad M \leq \log_2 N \quad \square \]
Level II: Repeated Halving (RHA)

Let \( C_1 \leftarrow \{1, \ldots, n\} \)

For \( t = 1, \ldots, T \):
- Let \( p_t \) be the majority vote of experts in \( C_t \)
- Let \( C_{t+1} \) be the experts with no mistakes so far
- If \( C_{t+1} = \emptyset \), let \( C_{t+1} = \{1, \ldots, n\} \)

Suppose some expert makes \( M^* \) mistakes

\[
\begin{array}{ccccccc}
\log_2 N + 1 & : & \log_2 N + 1 & : & \log_2 N + 1 & : & \cdots & : & \log_2 N + 1 \\
t = 1 & \uparrow & \uparrow & \uparrow & \uparrow & \cdots & \uparrow & \uparrow & t = T \\
\text{reset} & \text{reset} & \text{reset} & \text{reset} & \text{reset} & \cdots & \text{reset} & \text{reset} & \text{# of levels} = M^* + 1
\end{array}
\]
Level II: Repeated Halving

- **Thm:** If some expert makes $\leq M^*$ mistakes then RHA makes $\leq (M^* + 1)(\log_2 N + 1)$ mistakes.

  \[ M \approx M^* \cdot \log_2 N \]

  * Meaningless unless some expert is correct a \( 1 - \frac{1}{\log_2 N} \) fraction of the time
Level III: Weighted Majority (WM)

Give each expert a weight $w_{t,i} \leftarrow 1$

For $t = 1, \ldots, T$:

Let $p_t$ be the weighted majority vote of experts

For $i = 1, \ldots, N$:

If (expert $i$ made a mistake): $w_{t+1,i} \leftarrow \frac{w_{t,i}}{2}$

Else: $w_{t+1,i} \leftarrow w_{t,i}$

$W_t = \sum_{i=1}^{N} w_{t,i}$

$\sum_{i: f_t,i=1} w_{t,i} \text{ vs. } \sum_{i: f_t,i=0} w_{t,i}$

One of these is at least $\frac{W_t}{2}$

If we make a mistake on day $t$, $W_{t+1} \leq \left(\frac{3}{4}\right)W_t$
Level III: Weighted Majority

• **Thm:** If some expert makes $\leq M^*$ mistakes then WM makes $\leq 2.4(M^* + \log_2 N)$ mistakes.

• **Proof:** “Measure of progress” $W_t$

  • $W_T \leq N \cdot \left(\frac{3}{4}\right)^M$
  
  • $W_T \geq w_{t,i^*} \geq \left(\frac{1}{2}\right)^{M^*}$
    
      ($i^*$ is the best expert)

  • $\left(\frac{1}{2}\right)^{M^*} \leq N \cdot \left(\frac{3}{4}\right)^M$
\[ M^* \cdot \log_2 \left( \frac{1}{x} \right) \leq \log_2 N + M \cdot \log_2 \left( \frac{3}{u} \right) \]

\[ = \log_2 \left( \frac{N}{43} \right) \]

\[- M^* \cdot \log_2 (x) \leq \log_2 N - M \cdot \log_2 \left( \frac{u}{3} \right) \]

\[ M \leq \frac{M^* + \log_2 N}{\log_2 \left( \frac{u}{3} \right)} \]
Level III: Weighted Majority

• **Thm:** If some expert makes \( \leq M^* \) mistakes then WM makes \( \leq 2.4(M + \log_2 N) \) mistakes.

Some expert has \( w_t \geq 2^{-M^*} \)

Every mistake decreases total weight by a factor of \( \frac{1}{4} \)

\[
2^{-M^*} \leq W_t \leq N \cdot \left(\frac{3}{4}\right)^M
\]

\(-M^* \leq \log_2 N + M \log_2 \left(\frac{3}{4}\right)\)

\[
M \leq (M^* + \log_2 N) \frac{-1}{\log_2 \left(\frac{3}{4}\right)} \approx 2.4
\]
Level III: Weighted Majority

• **Thm**: If some expert makes \( \leq M^* \) mistakes then WM makes \( \leq 2.4(M + \log_2 N) \) mistakes.

\[
\text{Some expert has } \omega \geq (1 - \epsilon)^{M^*} \\
\text{Every mistake decreases total } \omega \text{ by a factor of } (1 - \frac{\epsilon}{2}) \\
(1 - \epsilon)^{M^*} \leq N \cdot (1 - \frac{\epsilon}{2})^M \\
M^* \cdot \ln(1 - \epsilon) \leq \ln N + M \cdot \ln (1 - \frac{\epsilon}{2}) \\
- M^* \cdot \ln \left( \frac{1}{1 - \epsilon} \right) \leq \ln N - M \cdot \ln \left( \frac{1}{1 - \frac{\epsilon}{2}} \right) \\
M \leq M^* \cdot \left( \frac{\ln(1 - \epsilon)}{\ln \left( \frac{1}{1 - \frac{\epsilon}{2}} \right)} \right) + \frac{\ln N}{\ln \left( \frac{1}{1 - \epsilon} \right)}
\]
Level III: Weighted Majority

- **Thm:** If some expert makes \( \leq M \) mistakes then WM makes \( \leq 2.4(M + \log_2 N) \) mistakes.

- **Thm:** Any **deterministic** strategy can be forced to make at least \( 2M^* \) mistakes

  1. We have to put the whole dollar on one expert, would like to split the dollar on many experts.

  2. Have to make a single prediction, would like to randomize.
Level IV: Randomized Weighted Majority

Give each expert a weight $w_{t,i} \leftarrow 1$, $W_t \leftarrow \sum_i w_{t,i}$

For $t = 1, \ldots, T$:

Choose $i$ with probability $w_{t,i} / W_t$

For $i = 1, \ldots, N$:

If (expert $i$ made a mistake): $w_{t+1,i} \leftarrow (1 - \varepsilon) \cdot w_{t,i}$

Else: $w_{t+1,i} \leftarrow w_{t,i}$, $W_{t+1} \leftarrow \sum_i w_{t+1,i}$

On day $t$, I lose

$F_t = \sum_{i: f_{t,i} \neq q_t} w_{t,i} / W_t$

In total, I lose

$M = \sum_{t=1}^T F_t$
• **Thm:** If some expert makes $\leq M^*$ mistakes then RWM makes $\leq (1 + \varepsilon) \cdot M^* + \frac{\log_2 N}{\varepsilon}$ mistakes

Set $\varepsilon = \sqrt{\frac{\log_2 N}{T}}$

Then $M \leq M^* + M^* \cdot \sqrt{\frac{\log_2 N}{T}} + \sqrt{T \log_2 N}$

$\leq M^* + \sqrt{T \log_2 N} + \sqrt{T \log_2 N}$

$= M^* + 2 \sqrt{T \log_2 N}$
Level IV: Randomized Weighted Majority

- **Thm**: If some expert makes $\leq M$ mistakes then RWM makes $\leq (1 + \varepsilon) \cdot M + \frac{\log_2 N}{\varepsilon}$ mistakes.

  **Proof**: measure of progress is $W_t$

  goal is to bound $\sum_{t=1}^{T} F_t$

  - $W_T \geq (1 - \varepsilon)^{M^*}$
  - If I make a mistake on day $t$
    then $W_{t+1} = (1 - \varepsilon F_t) \cdot W_t$
  - $W_T \leq N \cdot \prod_{t=1}^{T} (1 - \varepsilon F_t)$
\[(1 - \varepsilon)^{M^*} \leq N \cdot \prod_{t=1}^{T} (1 - \varepsilon F_t)\]

\[M^* \cdot \ln(1 - \varepsilon) \leq \ln(N) + \sum_{t=1}^{T} \ln(1 - \varepsilon F_t)\]

\[M^* \cdot \ln(1 - \varepsilon) \leq \ln(N) - \varepsilon \sum_{t=1}^{T} F_t\]

\[\sum_{t=1}^{T} F_t \leq \frac{-M^* \cdot \ln(1 - \varepsilon) + \ln(N)}{\varepsilon}\]

\[= M^* \left( -\frac{\ln(1 - \varepsilon)}{\varepsilon} \right) + \frac{\ln(N)}{\varepsilon}\]

\[= M^* \left( \frac{\varepsilon + \varepsilon^2}{\varepsilon} \right) + \frac{\ln(N)}{\varepsilon}\]

\[= M^* \left( 1 + \varepsilon \right) + \frac{\ln(N)}{\varepsilon}\]
Level IV: Randomized Weighted Majority

• **Thm:** If some expert makes $\leq M$ mistakes then RWM makes $\leq (1 + \varepsilon) \cdot M + \frac{\log_2 N}{\varepsilon}$ mistakes.

Let $F_t$ be the fraction of mistakes we made on day $t$.

Want to bound $\sum_{t} F_t \approx F_t \approx \sum_{i} w_t n_t$.

\[
(1 - \varepsilon)^{M_t} \leq W_T \leq N \cdot \prod_{t=1}^{T} (1 - \varepsilon F_t)
\]
\[
M_t \cdot \ln (1 - \varepsilon) \leq \ln N + \sum_{t=1}^{T} \ln (1 - \varepsilon F_t)
\]
\[
\leq \ln N - \sum_{t=1}^{T} \varepsilon F_t
\]
\[
\sum_{t=1}^{T} F_t \leq \frac{\ln N}{\varepsilon} + \frac{M_t}{\varepsilon \ln (1/1 - \varepsilon)}
\]
Why I love this algorithm

• **Endless applications:**
  • continuous optimization / linear programming
    • including maximum flow!
  • machine learning
    • training machine learning models
    • combining weak models into strong models
    • online learning: updating models with more data
  • probability theory
  • game theory
    • how to play zero-sum games
  • theory of computation
Why I care so much about this

• We often teach algorithms as a set of *ad hoc* tricks
  • These algorithms are easier to deploy
  • These algorithms are used often
  • These algorithms came first historically
  • These algorithms require less mathematical background

• Algorithms research today is much more systematic
  • More powerful and unified techniques
  • But requires more mathematical sophistication
    • Randomization / Probability / Statistics
    • Continuous Mathematics / Linear Algebra
  • But beautiful and worth studying!