Lecture 22:
• Online Learning

Dec 4, 2018
Picking Good Experts

• Suppose you have $N$ “experts” making predictions
  • Weather forecasters
  • Financial advisors
  • Recommender systems
  • ...

• Most of them are bad, but one might be good!
• Who’s predictions should we trust?
• There are $T$ time periods, in each:
  • experts make predictions $f_{t,1}, \ldots, f_{t,N} \in \{0,1\}$
  • you have to make a decision $p_t \in \{0,1\}$
  • some outcome $q_t \in \{0,1\}$ is revealed

  • a "mistake" is when $p_t \neq q_t$

• **Goal:** minimize mistakes

• The # of mistakes we make is $M$
• The best expert in hindsight makes $M^*$ mistakes
• Want to ensure that $M \approx M^*$
Level I: The Halving Algorithm (HA)

• Assumption: some expert makes 0 mistakes

Let $C_1 \leftarrow \{1, \ldots, n\}$
For $t = 1, \ldots, T$:
   Let $p_t$ be the majority vote of experts in $C_t$
   Let $C_{t+1}$ be the experts with no mistakes so far
Level I: The Halving Algorithm

\[ M^* = 0 \]

• **Thm:** If some expert makes 0 mistakes then HA makes \( \leq \log_2 N \) mistakes.

Measure of progress: \( |C_t| \)

Fact: \( N = |C_1| > |C_2| > \ldots > |C_T| > 1 \)

If we make a mistake on day \( t \), then

\[ |C_{t+1}| \leq \frac{|C_t|}{2} \]

\[ \therefore |C_T| \leq N \cdot 2^{-M} \]
Key Equation:

\[ |C_T| \leq N \cdot 2^{-M} \]

\[ \Rightarrow | \leq N \cdot 2^{-M} \]

\[ 2^M \leq N \]

\[ M \leq \log_2 N \]
Level II: Repeated Halving (RHA)

Let $C_1 \leftarrow \{1, \ldots, n\}$

For $t = 1, \ldots, T$:

Let $p_t$ be the majority vote of experts in $C_t$

Let $C_{t+1}$ be the experts with no mistakes so far

If $C_{t+1} = \emptyset$, let $C_{t+1} = \{1, \ldots, n\}$

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<th>$\log_2 N$</th>
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<td>$t=1$</td>
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<td>$T$</td>
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Best expert makes $M^*$ mistakes

$\sum (M^* + 1)$ intervals

$\times \log_2 N$ mistakes / interval

$M \leq (M^* + 1) \log_2 N$
Level II: Repeated Halving

- **Thm:** If some expert makes $\leq M$ mistakes then RHA makes $\leq (M + 1)(\log_2 N + 1)$ mistakes.

- Worse than the best expert by a factor of $\log_2 N$
Level III: Weighted Majority (WM)

Give each expert a weight $w_{t,i} \leftarrow 1$

For $t = 1, \ldots, T$:

- Let $p_t$ be the weighted majority vote of experts for $i = 1, \ldots, N$:
  - If (expert $i$ made a mistake): $w_{t+1,i} \leftarrow \frac{w_{t,i}}{2}$
  - Else: $w_{t+1,i} \leftarrow w_{t,i}$

$$W_t = \sum_{i=1}^{N} \omega_{t,i}$$
Level III: Weighted Majority

- **Thm:** If some expert makes $\leq M^*$ mistakes then WM makes $\leq 2.4(M^* + \log_2 N)$ mistakes.

- **Proof:**
  - measure of progress is $W_t = \sum_{i=1}^{N} \omega_{t,i}$
  - if we make $M$ mistakes, then $W_T \leq N \cdot \left( \frac{3}{4} \right)^M$
  - if the best expert $i^*$ makes $M^*$ mistakes, then
    $$W_T = \sum_{i=1}^{N} \omega_{T,i} \geq \omega_{T,i^*} = \left( \frac{1}{2} \right)^{M^*}$$
  $$\Rightarrow \quad \left( \frac{1}{2} \right)^{M^*} \leq W_T \leq N \cdot \left( \frac{3}{4} \right)^M$$
\[
\left(\frac{1}{2}\right)^{M^*} \leq N \cdot \left(\frac{3}{4}\right)^M \quad \text{(take } \log_2 \text{ of both sides)}
\]

\[
M^* \log_2 \left(\frac{1}{2}\right) \leq M \log_2 \left(\frac{3}{4}\right) + \log_2 N
\]

\[
- M^* \log_2 2 \leq - M \log_2 \left(\frac{4}{3}\right) + \log_2 N
\]

\[
M \leq \frac{M^* \log_2 2 \cdot \log_2 \left(\frac{4}{3}\right)}{\log_2 \left(\frac{4}{3}\right)}
\]

\[
M \leq 2.4 \left( M^* + \log_2 N \right)
\]
Some expert has used

Every mistake decreases total weight by a factor of \( Y \).

No\( E \)\( M \)\( E \)\( M \)\( log \)\( N \)\( t \)\( M \)\( log \)\( E \)\( \cdot 10^4 \)\( \cdot 44 \)\( I \)\( 43J \)\( 2.4 \)
Level III: Weighted Majority

- **Thm:** If some expert makes $\leq M^*$ mistakes then WM makes $\leq 2.4(M^* + \log_2 N)$ mistakes.

Some expert has $w_i > 2^{-M^*}$

Every mistake decreases total $w_i$ by a factor of $(\frac{1}{4})$

$$2^{-M^*} \leq w_i \leq N \cdot (\frac{3}{4})^M$$

$$-M^* \leq \log_2 N + M \log_2 (\frac{3}{4})$$

$$M \leq (M^* + \log_2 N) \cdot \left(\frac{-1}{\log_2 (3/4)}\right)$$

$$= \frac{1}{\log_2 (4/3)} \approx 2.4$$
Level III: Weighted Majority

- **Thm:** If some expert makes $\leq M^*$ mistakes then WM makes $\leq 2.4(M + \log_2 N)$ mistakes.

Some expert has ut $\geq (1-\varepsilon)^{M^*}$
Every mistake decreases total ut by a factor of $(1-\frac{\varepsilon}{2})$

\[(1-\varepsilon)^{M^*} \leq N \cdot (1-\frac{\varepsilon}{2})^M\]

\[M^* \cdot \ln (1-\varepsilon) \leq \ln N + M \cdot \ln (1-\frac{\varepsilon}{2})\]

\[-M^* \cdot \ln \left(\frac{1}{1-\varepsilon}\right) \leq \ln N - M \cdot \ln \left(\frac{1}{1-\frac{\varepsilon}{2}}\right)\]

\[M \leq M^* \cdot \left(\frac{\ln (\frac{1}{1-\varepsilon})}{\ln (\frac{1}{1-\frac{\varepsilon}{2}})}\right) + \frac{\ln N}{\ln (\frac{1}{1-\varepsilon})}\]
Level III: Weighted Majority

• **Thm:** If some expert makes \( \leq M^* \) mistakes then WM makes \( \leq 2.4(M^* + \log_2 N) \) mistakes.

• **Thm:** Any deterministic strategy can be forced to make at least \( 2M^* \) mistakes.
Level IV: Randomized Weighted Majority

Give each expert a weight $w_{t,i} \leftarrow 1$, $W_t \leftarrow \sum_i w_{t,i}$

For $t = 1, \ldots, T$:

Choose $i$ with probability $w_{t,i}/W_t$ (Splitting a dollar across the experts)

For $i = 1, \ldots, N$:

If (expert $i$ made a mistake): $w_{t+1,i} \leftarrow (1 - \varepsilon) \cdot w_{t,i}$

Else: $w_{t+1,i} \leftarrow w_{t,i}$, $W_{t+1} \leftarrow \sum_i w_{t+1,i}$
Level IV: Randomized Weighted Majority

- **Thm:** If some expert makes \( \leq M^* \) mistakes then RWM makes \( \leq (1 + \varepsilon) \cdot M^* + \frac{\log_2 N}{\varepsilon} \) mistakes.

- Set \( \varepsilon = \sqrt{\frac{\log_2 N}{t}} \):

\[
M \leq M^* + M^* \sqrt{\frac{\log_2 N}{t}} + \sqrt{t \log_2 N}
\]

\[
\leq M^* + 2 \sqrt{t \log_2 N}
\]

\[
\frac{M^*}{t} + 2 \sqrt{\frac{\log_2 N}{t}}
\]
Level IV: Randomized Weighted Majority

- **Thm:** If some expert makes \( \leq M \) mistakes then RWM makes \( \leq (1 + \varepsilon) \cdot M + \frac{\log_2 N}{\varepsilon} \) mistakes.

**Proof:**

- Measure of progress: \( W_t \)
- \( (1 - \varepsilon)^{M^*} \leq W_T \)
- Every day we lose \( F_t = \sum_{i: f_{t,i} \neq q_t} W_{t,i} \)
- \( W_{t+1} = W_t \left( 1 - \varepsilon F_t \right) \) \( \therefore W_T \leq N \cdot \prod_{t=1}^{T} (1 - \varepsilon F_t) \)
\[(1 - \varepsilon)^{M^*} \leq N \cdot \prod_{t=1}^{T} (1 - \varepsilon F_t)\]

\[M^* \cdot \ln (1 - \varepsilon) \leq \ln N + \sum_{t=1}^{T} \ln (1 - \varepsilon F_t)\]

\[\leq \ln N - e \sum_{t=1}^{T} F_t\]

\[\geq -\frac{M^* \ln (1 - \varepsilon) + \ln N}{\varepsilon}\]

\[= \frac{M^* \ln \left(\frac{1}{1 - \varepsilon}\right) + \ln N}{\varepsilon}\]

\[\approx \frac{M^* \ln (1 + \varepsilon) + \ln N}{\varepsilon}\]

\[\approx \frac{M^* (\varepsilon + \varepsilon^2) + \ln N}{\varepsilon}\]

\[= (1 + \varepsilon) M^* + \frac{\ln N}{\varepsilon}\]
Level IV: Randomized Weighted Majority

- **Thm:** If some expert makes $\leq M^*$ mistakes then RWM makes $\leq (1 + \varepsilon) \cdot M^* + \frac{\log_2 N}{\varepsilon}$ mistakes

Let $F_t$ be the fraction of mistakes we made on day $t$

Want to bound $\sum_t F_t \leq F_t \leq \sum_i w_t n_t$

\[
(1 - \varepsilon)^{M^*} \leq W_T \leq N \cdot \prod_{t=1}^{T} (1 - \varepsilon F_t)
\]

\[
M^* \cdot \ln (1 - \varepsilon) \leq \ln N + \sum_{t=1}^{T} \ln (1 - \varepsilon F_t)
\]

\[
\leq \ln N - \sum_{t=1}^{T} \varepsilon F_t
\]

\[
\sum_{t=1}^{T} F_t \leq \frac{\ln N}{\varepsilon} + \frac{M^*}{\varepsilon \ln (1/1 - \varepsilon)}
\]
Why I love this algorithm

• Endless applications:
  • continuous optimization / linear programming
    • including maximum flow!
  • machine learning
    • training machine learning models
    • combining weak models into strong models
    • online learning: updating models with more data
  • probability theory
  • game theory
    • how to play zero-sum games
  • theory of computation
Why I care so much about this

• **We often teach algorithms as a set of *ad hoc* tricks**
  • These algorithms are easier to deploy
  • These algorithms are used often
  • These algorithms came first historically
  • These algorithms require less mathematical background

• **Algorithms research today is much more systematic**
  • More powerful and unified techniques
  • But requires more mathematical sophistication
    • Randomization / Probability / Statistics
    • Continuous Mathematics / Linear Algebra
  • **But beautiful and worth studying!**