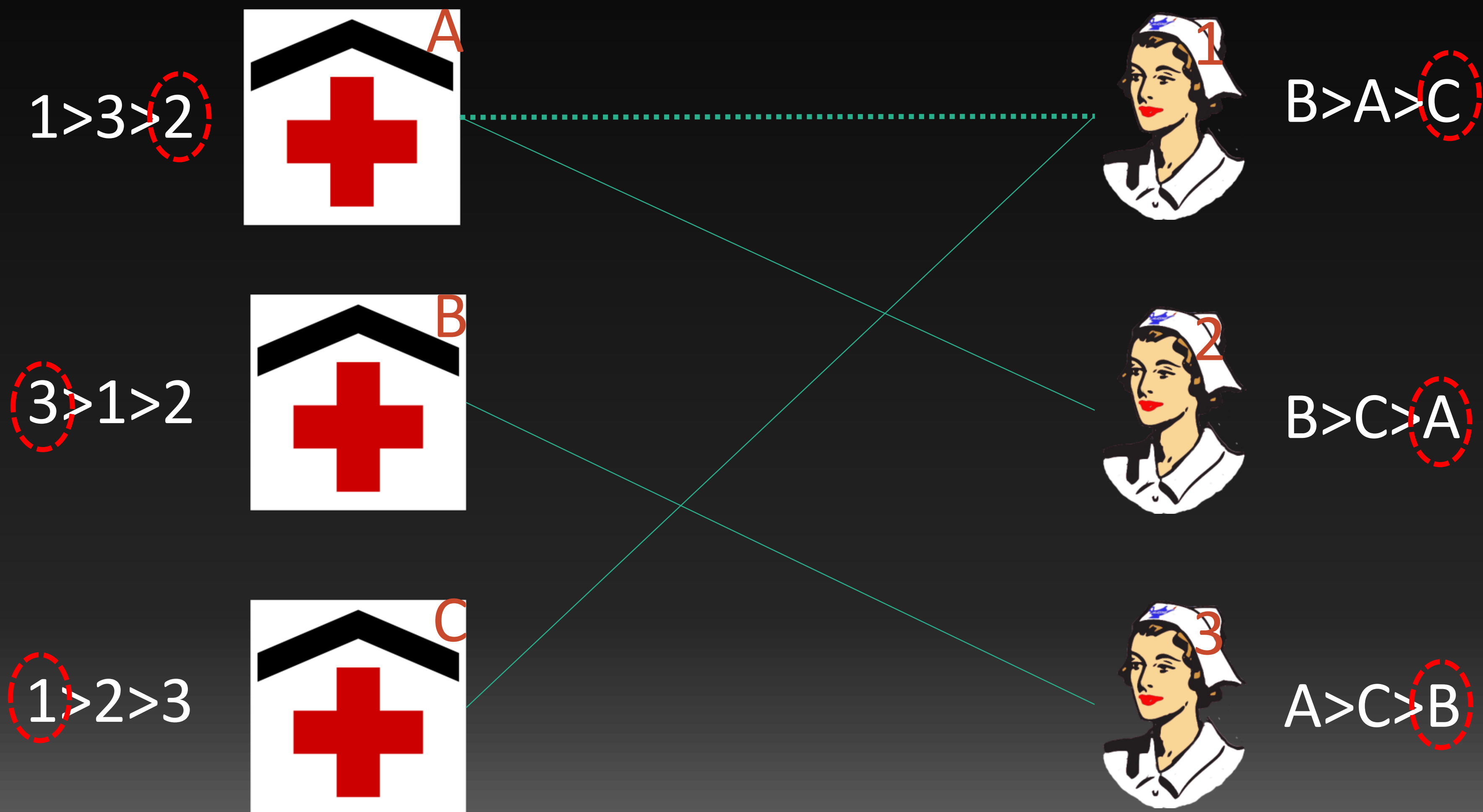
# CS 4800: Algorithms & Data

Lecture 22

April 10, 2018

# Stable matching

# Unstable matching



# Stable matching

# Economics Nobel prize 2012

**Stable matching: Theory, evidence, and practical design**

This year's Prize to Lloyd Shapley and Alvin Roth extends from abstract theory developed in the 1960s, over empirical work in the 1980s, to ongoing efforts to find practical solutions to real-world problems. Examples include the assignment of new doctors to hospitals, students to schools, and human organs for transplant to recipients. Lloyd Shapley made the early theoretical contributions, which were unexpectedly adopted two decades later when Alvin Roth investigated the market for U.S. doctors. His findings generated further analytical developments, as well as practical design of market institutions.

http://www.nobelprize.org/nobel_prizes/economic-sciences/laureates/2012/popular-economicsciences2012.pdf

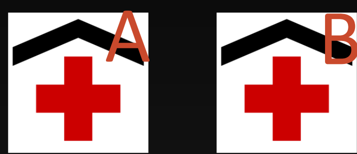
# Stable matching

- Given sets
  - $M = \{m_1, \ldots, m_n\}$ of jobs
  - $W = \{w_1, \ldots, w_n\}$ of candidates
- Each job has a preference ranking of candidates
- Each candidate has a preference ranking of jobs
- Want a collection of pairings (job, candidate) s.t.
  - Each job is paired with exactly one candidate
  - Each candidate is paired with exactly one job
- Stability: there are no 2 pairs (m, w), (m’,w’) s.t.
  - m prefers w’ to w
  - w’ prefers m to m’
  - Instability: m & w’ abandon their partners and form (m,w’)

# Questions

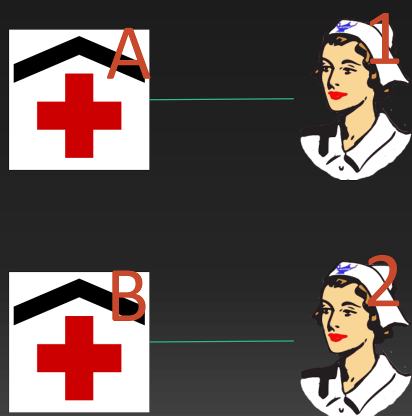
- Does there always exist a stable matching?
- Given preference rankings, can we find a stable matching?

# Multiple stable matchings

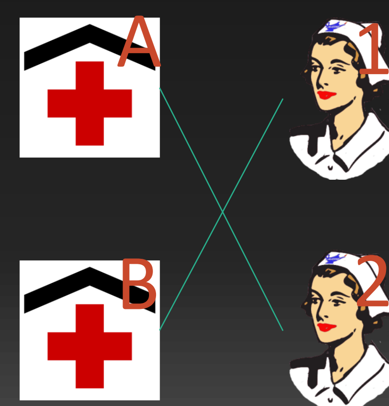




| | | | |
|---|---|---|---|
| 2 | 1 | A | B |
| 1 | 2 | B | A |




Doctors get their best choices

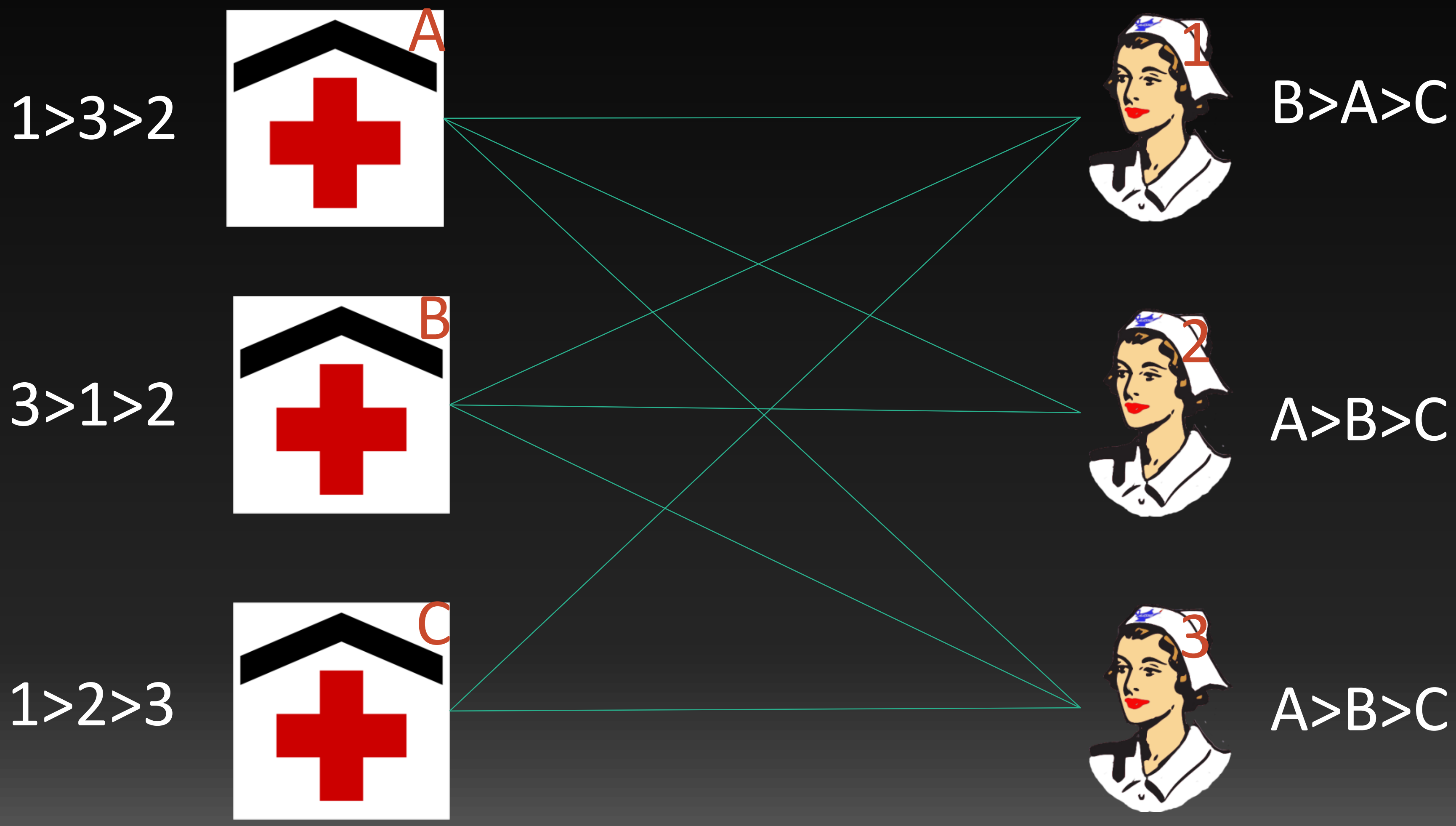



Hospitals get their best choices

# Local improvements

- As long as there exists 2 pairs (m, w), (m’,w’) s.t. m prefers w’ to w and w’ prefers m to m’
  - Swap partners and get new pairs (m,w’), (m’,w)

# Doesn't work, can loop



# Deferred acceptance algorithm (Gale-Shapley)

- Initially no jobs and no candidates are matched
- While there is an open job m that hasn't been offered to all candidates
  - Offer job m to w, the most preferred candidate for m that hasn't rejected m
  - If w is free then w holds onto the job offer
  - If w currently holds an offer from m'
    - If w prefers m' to m then
      - w rejects m and m remains free
    - Else w prefers m to m'
      - w rejects m' and holds onto the offer from m
      - m' becomes free
- All candidates accept the job offers they are holding


| A | B | C | D | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 3 | 1 | D | D | A | C |
| 2 | 3 | 4 | 2 | C | A | C | B |
| 3 | 4 | 2 | 4 | B | C | D | D |
| 4 | 2 | 1 | 3 | A | B | B | A |

- Doctor w always has an offer from the time w has her first offer
- w's offer gets better and better and w eventually accepts her best offer
- The sequence of doctors being offered the same job m gets worse and worse (in m's preference)

# Termination

Claim. The algorithm terminates after $O(n^2)$ iterations.

Proof.

In each iteration, job m is offered to doctor w that has not been offered job m before.

Each job is offered at most n times.

Thus, the algorithm terminates in $O(n^2)$ iterations.

# Perfect matching at termination

Claim. If job m is free at some point, then there is a doctor w that has not been offered job m.

Proof. Suppose for contradiction that job m is free but it has been offered to all doctors.

Each doctor w rejected m because w had a better offer.

By observation, w is still holding onto a better offer.

n doctors are holding offers for n jobs.

However, only n jobs in total and m is free (contradiction).

# Perfect matching at termination

Corollary. At termination, all jobs are filled and all doctors have jobs.

Proof. If there is still job m not filled then there is still some doctor w not getting offered job m.

Thus, the algorithm must continue and cannot terminate.

# Stable matching at termination

Assume for contradiction that at the end, there are two matched pairs (m, w), (m’, w’) where

- m prefers w’ to w
- w’ prefers m to m’

In algorithm, m’s last offer was to w.

Was m offered to w’ earlier?

- If m was not offered to w’ then m must prefer w to w’ (contradiction)
- If m was offered to w’ then w’ must have rejected m due to a better offer at the time for job m’’ (w’ prefers m’’ to m).

  m’ is final job of w’ so either m’=m’’ or w’ prefers m’ to m’’.

  In either case, w’ prefers m’ to m (contradiction).

# Deferred-acceptance is hospital optimal

- w is attainable for m if there is a stable matching S that contains (m, w)
- w is best(m) if w is attainable for m and all w’ that m prefers to w are not attainable for m.
- S* = {(m, best(m)} “Best matching for all m”

# Deferred-acceptance is hospital optimal

Claim. Every run of deferred-acceptance algorithm results in S*.

Proof. Suppose by contradiction that algorithm results in some job not matched to its best attainable choice.

Consider first moment in execution of algorithm when a job is rejected by an attainable candidate. Say the job is m.

m is offered in decreasing order of preference so it is rejected by w = best(m).

m is rejected because either

- m offered and w turned m down in favor of a current better offer, or
- w just received a better offer and rejected m.

In either case, w has an offer from m' that she prefers to m.

w is attainable for m so there exists stable matching S containing (m, w)

Suppose in S, m' is matched to w'. Thus, w' is attainable for m'.

Since we consider first rejection by attainable doctor, w' hasn't rejected m'

Since m' is offered in decreasing order of preference, m' prefers w to w'.

(m',w) is instability in S (contradiction).

# Deferred-acceptance is worst for doctors

Claim. In S*, each doctor gets her worst attainable job.

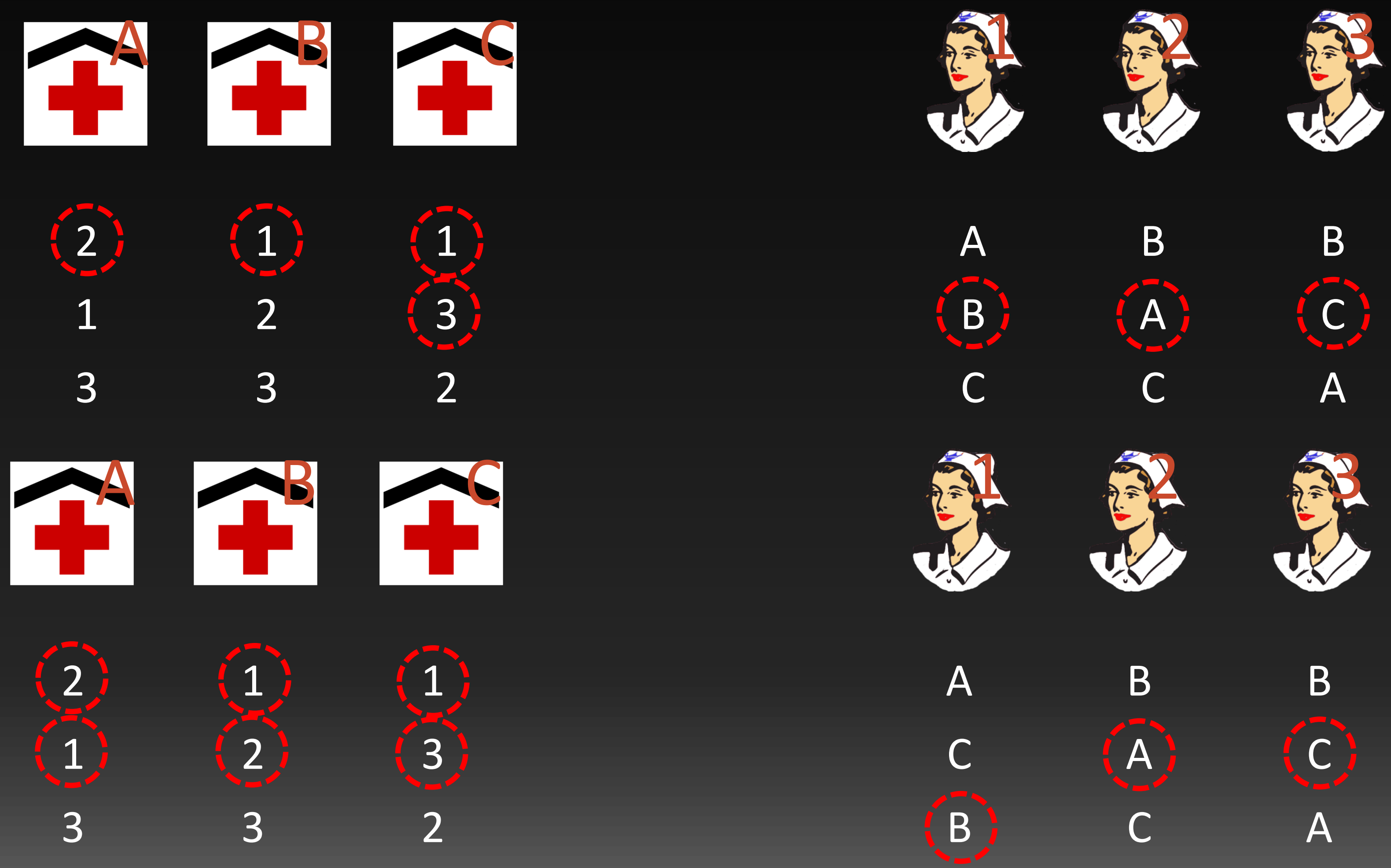
Proof. Suppose there is some pair (m,w) in S* such that m is not the worst attainable job for w.
There is stable matching S containing (m',w) and w prefers m to m'.

In S, m is paired with w'.

Since w is best attainable for m, m prefers w to w'.

(m,w) is instability in S (contradiction).

# Incentive for doctors to cheat



# 1970+ couple constraints

- There might not be a stable matching