CS 4800: Algorithms & Data

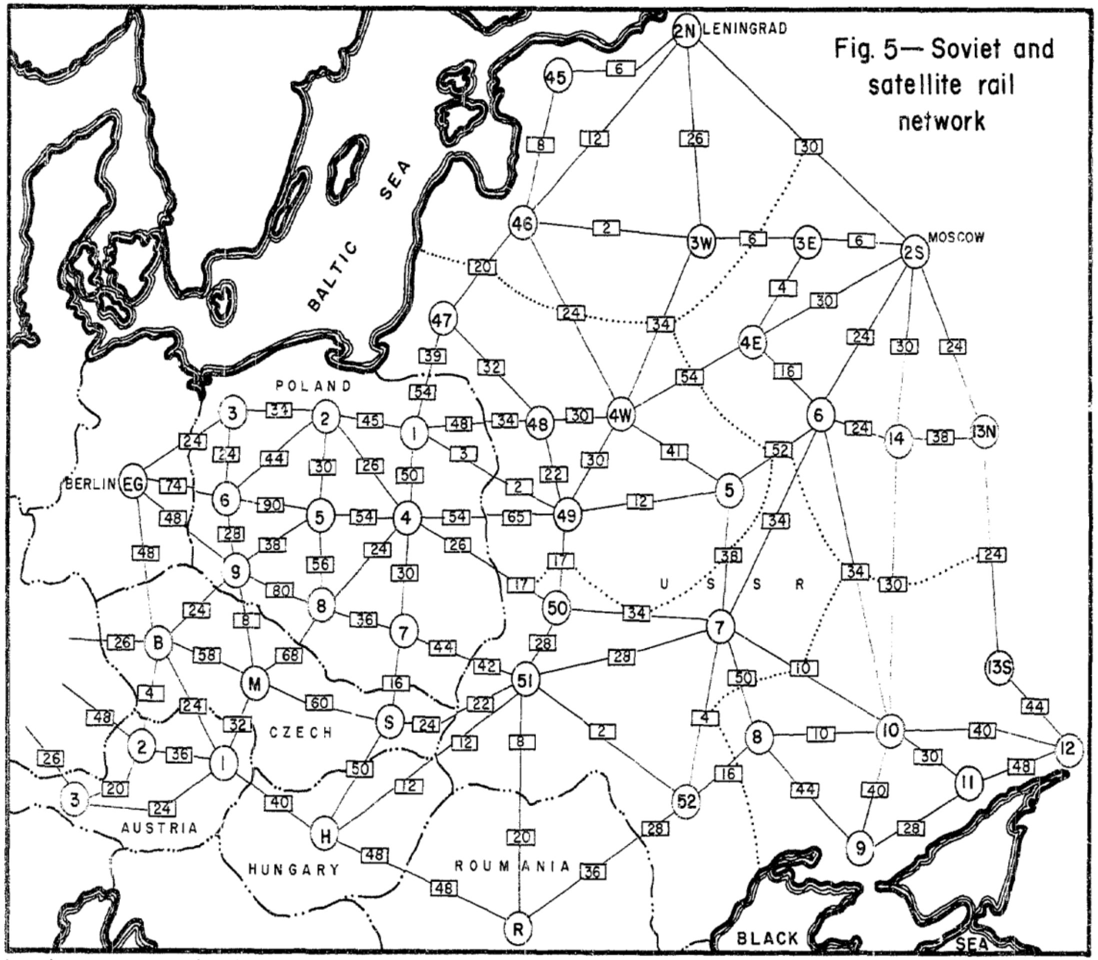
Lecture 19

March 30, 2018

Max flow, min cut

“Consider a rail network connecting two cities by way of a number of intermediate cities, where each link of the network has a number assigned to it representing its capacity. Assuming a steady state condition, find a maximal flow from one given city to the other.”

Ford-Fulkerson attributed to T. Harris



Legend: —·—·— International boundary ·········· Regional boundaries of the USSR (they are included as a matter of general information)

(7) Operating divisions. Those located in Russia are believed to be accurately located. Some Russian divisions (2, 3, 4 and 13) are located in two regions and are so indicated. Divisions shown in the satellites are indicated according to the authors' best judgment, since intelligence reports are unavailable. Train capacities in Russia are for 1000-net-ton trains or their equivalent. Train capacities in Poland are for 666 net tons (or the equivalent). Train capacities in all other satellites are for 400 net tons (or the equivalent) except in East Germany. In East Germany, train capacities are those of entering lines. The numbers shown in boxes are total interdivisional capacities.

RM 1573-5

SECRET

Harris-Ross '55

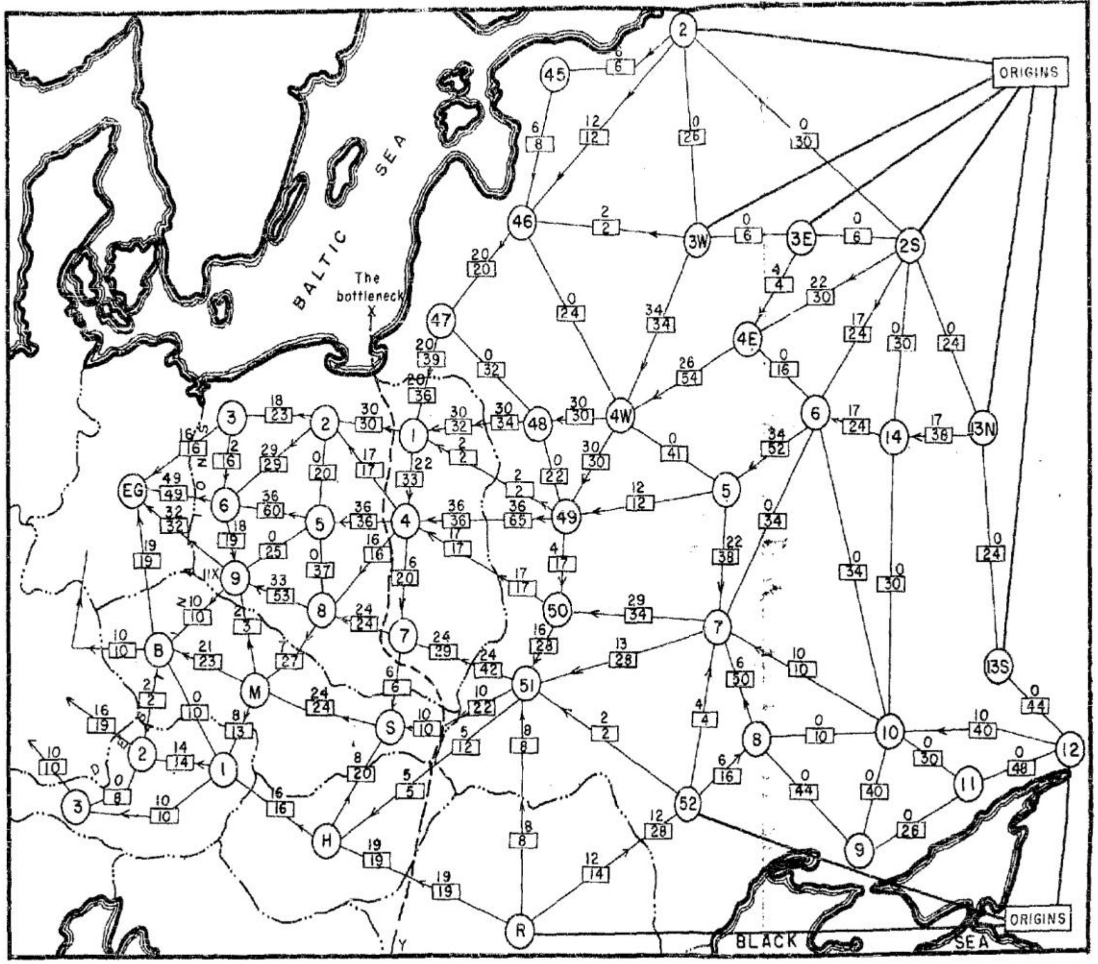


Fig. 7 — Traffic pattern: entire network available

Legend:

—··— International boundary

(8) Railway operating division

9 / [12] Capacity: 12 each way per day. Required flow of 9 per day toward destinations (in direction of arrow) with equivalent number of returning trains in opposite direction

All capacities in trains / √1000's of tons } each way per day

Origins: Divisions 2, 3W, 3E, 2S, 13N, 13S, 12, 52 (USSR), and Roumania

Destinations: Divisions 3, 6, 9 (Poland); B (Czechoslovavakia); and 2, 3 (Austria)

Alternative destinations: Germany or East Germany

Note IIX at Division 9, Poland

Assumption:

Entire network available for east—west traffic (no allowance for civilian or economic traffic)

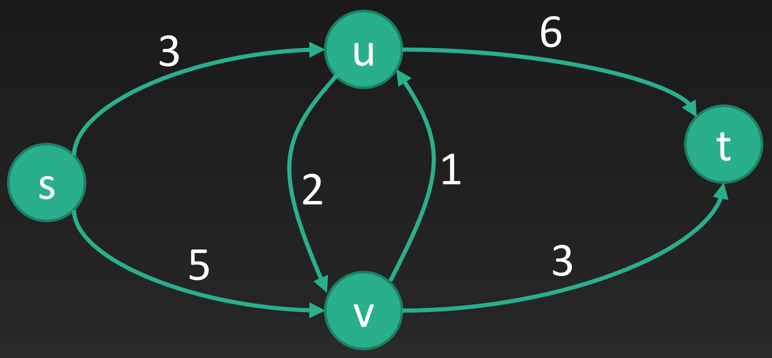
Results:

(a) 163,000 tons per day can be delivered from points of origin to destinations.
(b) 147,000 tons per day can be delivered without using Austrian lines.
(c) 152,000 tons per day can be delivered into Germany by all lines.
(d) 126,000 tons per day can be delivered into East Germany without using Austrian lines.

Harris-Ross '55

Flow network

- G = (V, E) directed
- Source vertex s, sink vertex t
- Each edge e has a capacity $c(e) \geq 0$



Flow

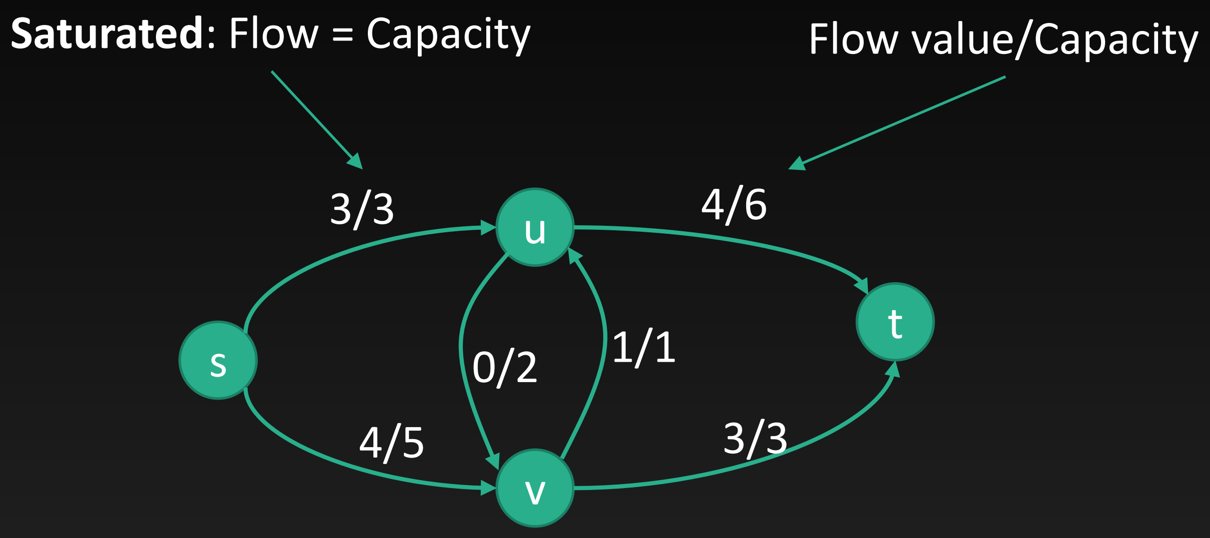
- Each edge e has a flow amount $f(e) \geq 0$
- Capacity constraints: $f(e) \leq c(e)\ \forall e \in E$
- Flow conservation: for every node $u \neq s, t$

$$\sum_{v \in V} f(u,v) = \sum_{v \in V} f(v,u)$$

Total out flow　　Total in flow

- Value |f| = net out flow of s = net in flow of t
  = $\sum_{v \in V} f(s,v) - \sum_{v \in V} f(v,s)$

Example



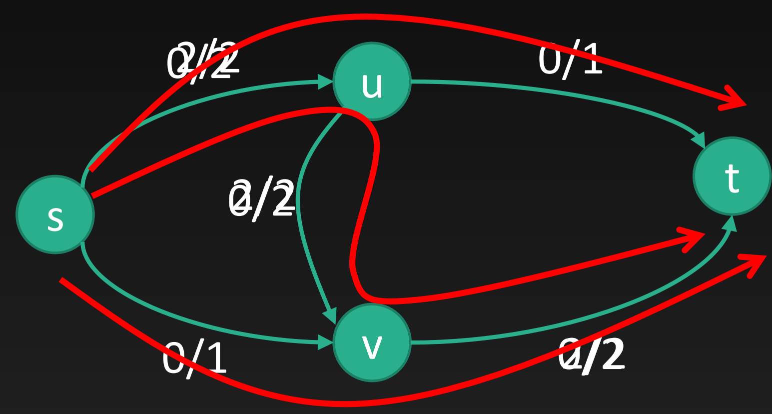
Total out flow of s: 7

Total in flow of t: 7

Max flow

- Given a graph $G=(V,E)$ and capacities c
- Find flow f maximizing value $|f|$

Greedy?



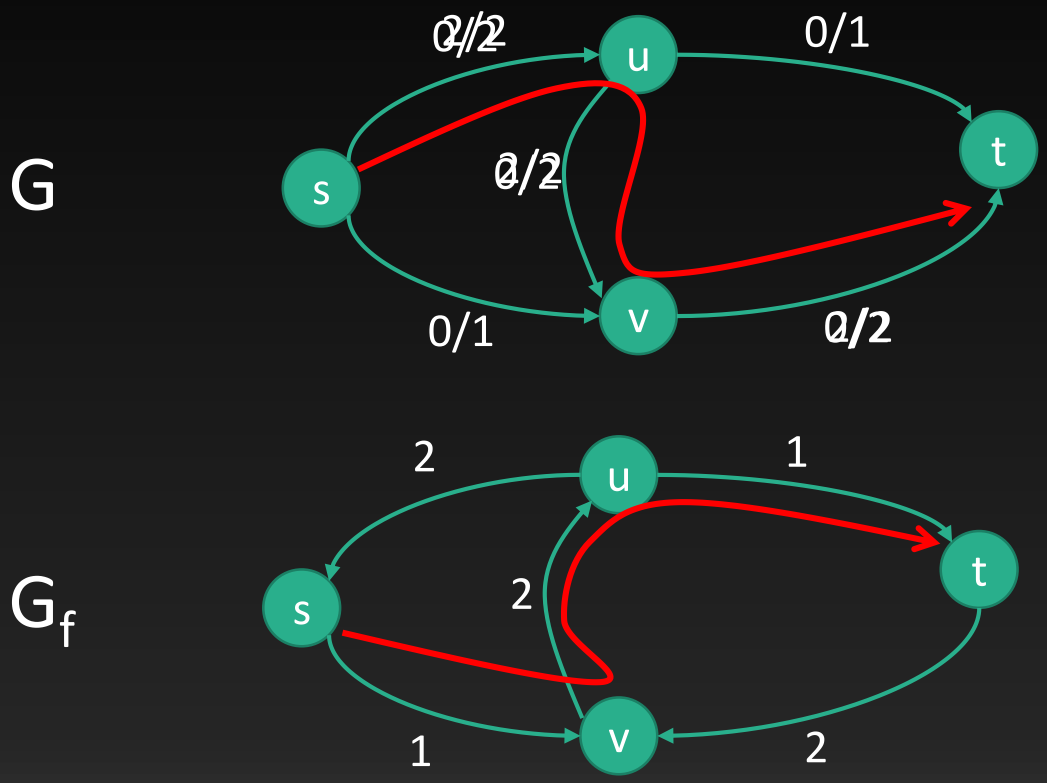
Need to allow for correction

- Residual graph G_f = (V, E_f) based on G = (V,E) and f
- For each $(u, v) \in E$
  - Edge (u, v) in E_f with capacity c(u,v) – f(u,v)
  - Edge (v, u) in E_f with capacity f(u,v)

How much more flow can be sent forward?

How much flow can be sent backward/cancelled?

Greedy with correction



Augmenting paths

- Path from s to t in G_f consisting of only edges with positive residual capacities

Ford-Fulkerson algorithm

- Initialize f(u,v) = 0 for all (u, v)
- While there exists an augmenting path p in G_f
  - Let D be minimum capacity of edges in p
  - For all forward edges e in p
    - Increase f(e) by D
  - For all backward edges e in p
    - Decrease f(e) by D

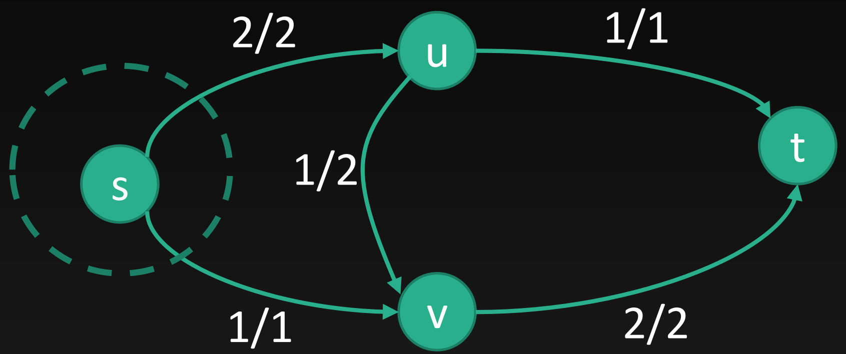
Termination

- f is always a valid flow
- Initially, f=0 is valid
- When flow is augmented along path p,
  - Capacities are not violated
    - Augment by D = minimum residual capacity
  - Flow conservation is preserved
    - Flow is pushed along a path so at any intermediate vertex, flow in = flow out
- If all c(e) are integers then all f(e) are always integers
  - Inductively, all residual capacities are integers so D is an integer and the flow stays integral
  - Flow increases by ≥ 1 every time so algo eventually finishes

Is final flow f any good?

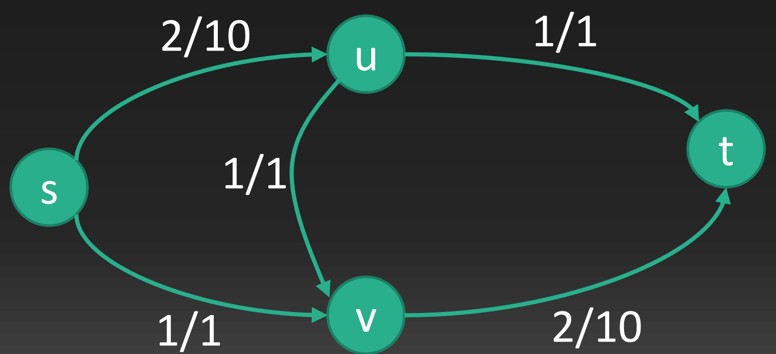
Optimality

G



Optimal since flow value = 3 and capacities out of s is 3

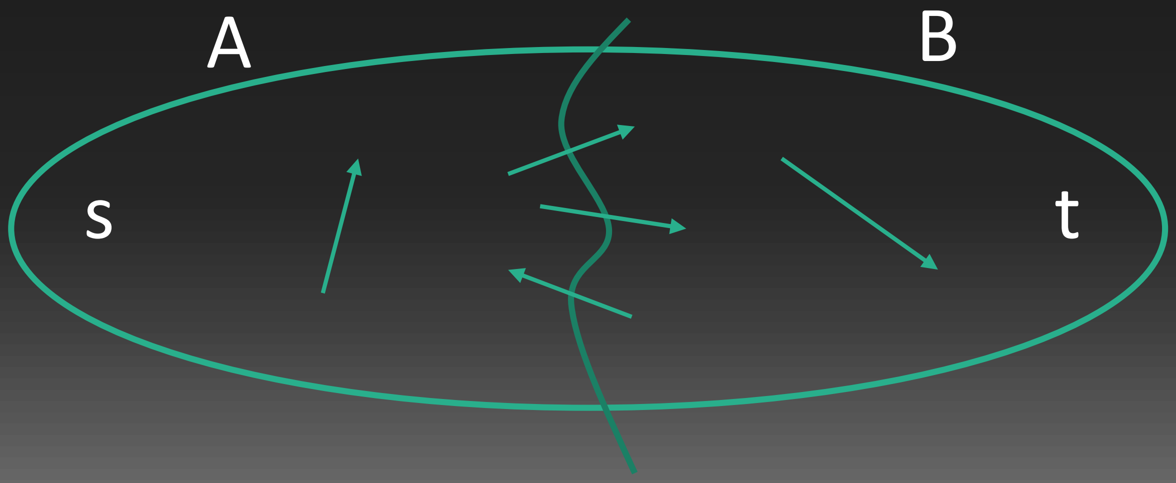
What about



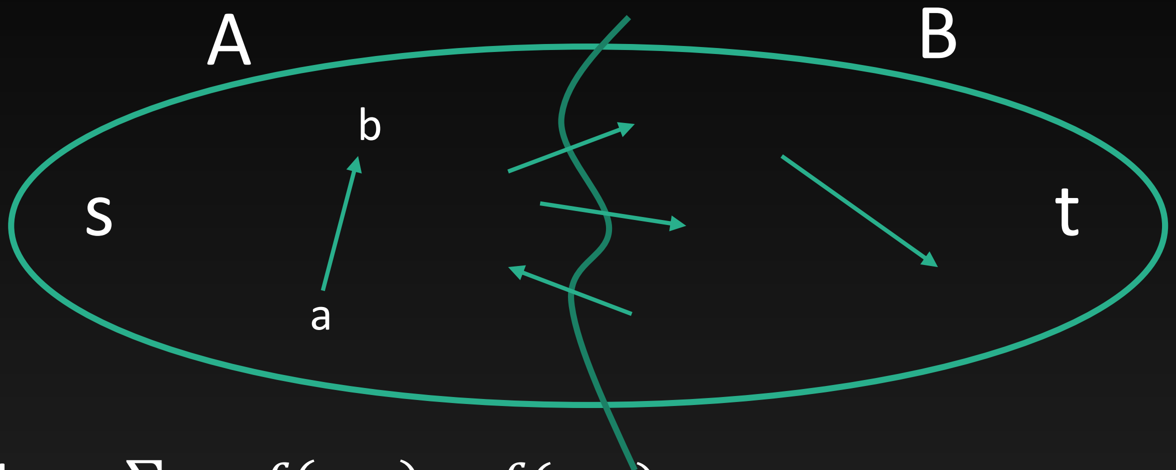
s-t cut

- A partition of vertices into two sets A, B with $s \in A$ and $t \in B$
- Capacity of cut A,B is

$$\mathrm{cap}(\mathrm{A},\mathrm{B}) = \sum_{u \in A, v \in B} c(u,v)$$



$Flow\ value\ \leq cut\ capacity$



- Flow value = $\sum_{v \in V} f(s,v) - f(v,s)$ ← Net out flow from s
- By flow conservation, $0 = \sum_{v \in V} f(u,v) - f(v,u)$ for all $u \neq s, t$
- Thus, flow value = $\sum_{u \in A, v \in V} f(u,v) - f(v,u)$ ← Net out flow from A

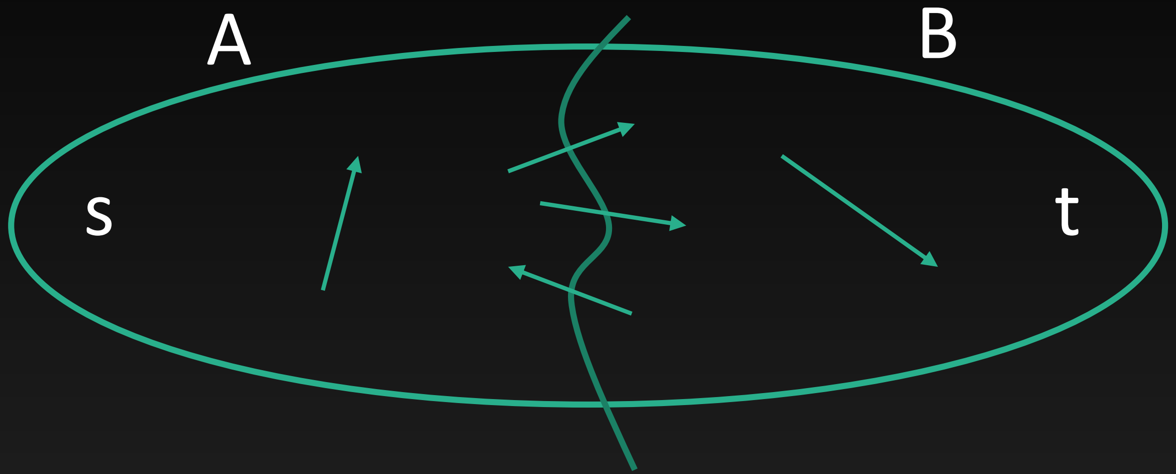
$$= \sum_{u \in A, v \in A} f(u,v) - f(v,u) + \sum_{u \in A, v \in B} f(u,v) - f(v,u)$$

$u = a, v = b$ $+f(a,b)$

$u = b, v = a$ $-f(a,b)$

Net flow from A to A = 0

$Flow\ value\ \le cut\ capacity$



- Flow value = $\sum_{u\in A, v\in B} f(u,v) - f(v,u)$
- By capacity constraints,

$$\sum_{u\in A, v\in B} f(u,v) - f(v,u) \le \sum_{u\in A, v\in B} f(u,v) \le \sum_{u\in A, v\in B} c(u,v)$$

Optimality

3 equivalent statements:

- f is maximum flow
- There is s-t cut (A,B) such that |f| = cap(A,B)
- There is no augmenting path in G_f

|f| = cap(A,B) implies f maximum

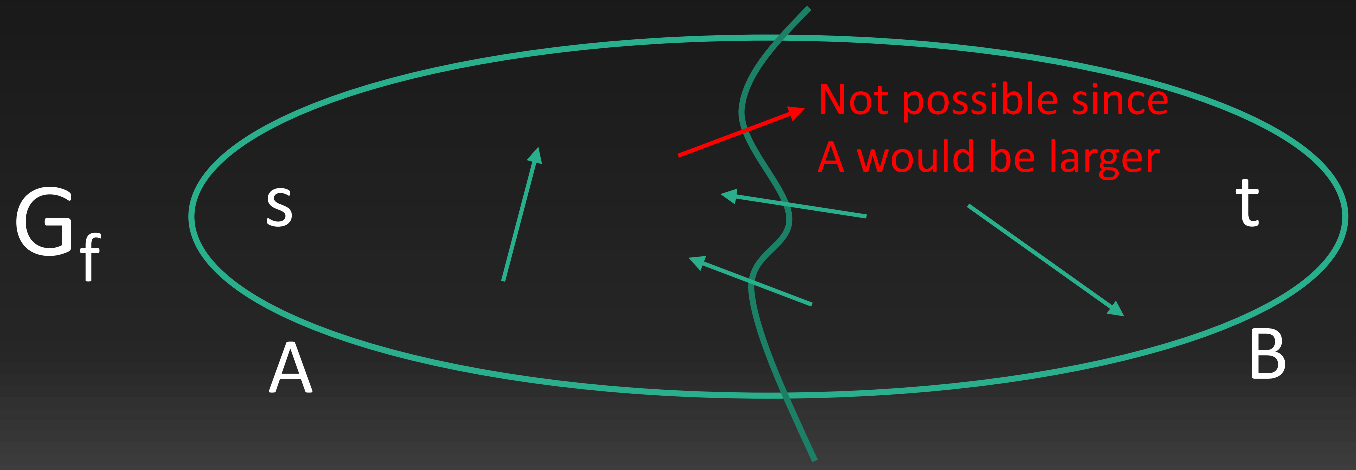
- $|f'| \leq cap(A, B)$ for all flow f’
- Thus f is optimal

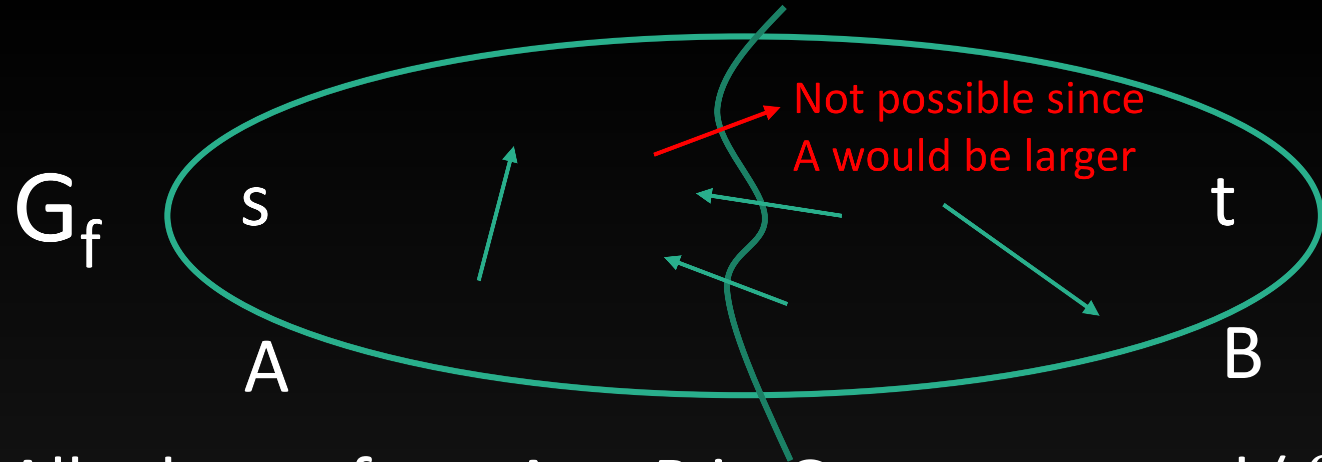
f maximum implies no augmenting path

- Prove the contrapositive
- If there were augmenting path p then Ford-Fulkerson can improve value of f
- Thus f is not maximum

No augmenting path implies |f|=cap(A,B) for some A,B

- Define A = {v reachable from s in G_f}, $B = V \setminus A$
- s is reachable from s to $s \in A$
- t is not reachable from s so $t \notin A$





- All edges e from A to B in G are saturated ($f(e) = c(e)$) since e goes backward in G_f
- All edges e from B to A in G are not used since there is no backward edge from A to B ($f(e) = 0$)
- Thus,

$$|f| = \sum_{u \in A, v \in B} f(u,v) - f(v,u)$$

- |f|=cap(A,B)